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ON THE BLASCHKE INVARIANTS OF THE AXOIDS OF HELICAL MOTIONS

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ABSTRACT

In this paper the relations between the invariants of the moving axoid $\bar{\varnothing}$ and the fixed axoid $\varnothing$ under the helical motions of order k in E^n are discussed. Moreover we have the statement (17) for the pair of the 2-ruled surfaces $\bar{\Psi} \subset \bar{\varnothing}$ and $\Psi \subset \varnothing$ which correspond to each other under the helical motion of order k in E^n.

1. HELICAL MOTIONS OF ORDER k

A motion of E^n is described in matrix notation by

(1) $$x = A\bar{x} + c, \qquad AA^T = A^TA = I$$

where A^T is the transposed of the orthogonal matrix A and

$$A{:}J \to O(n), \qquad c{:}J \to \mathbb{R}^n$$

are functions of differentiability class $C^r (r > 3)$ on a real interval J. Considering a motion as a movement of the space $\bar{E}$ against the space E the co-ordinate vector $\bar{x}$ in (1) describes a point of so-called moving space $\bar{E}$ and x a point of the so-called fixed space E.

Let $\bar{x}$ be fixed point in $\bar{E}$ then (1) defines a parametrized curve in E which is called the trajectory curve of x under the motion. From (1) by differetiating with respect to t we get

(2) $$\dot{x} = B(x - c) + \dot{c}, \qquad B = \dot{A}A^T$$

where $B + B^T = O$, since the matrix A is orthogonal. Therefore in the case of even dimension it is possible that the determinant $|B|$ may not vanish. If $|B(t)| \neq 0$ for all $t \in J$, we get exactly one solution P(t) of the equation

(3) $$B(t)(P - c(t)) + \dot{c} = O\ .$$

The point P (t) is called the pole of the motion at the instant t which is the center of the instantaneous rotation of the motion for $t \in J$. If

$|B|$ dose'nt vanish on J, by cosidering the regularity condition of the motion we get a differentiable curve $P:J \to E$ of poles in the fixed space E, called the fixed pole curve. By (1) there is uniquely determined a moving pole curve $\bar{P}:J \to \bar{E}$ from the fixed pole curve point to point on J.

H.R. Müller proved in [4]; under the motions the fixed pole curve and the moving pole curve are rolling on each other without sliding. Merely in the case $n = 2$ the motion is determined by the pair of rolling pole curves.

In all other cases (that means $|B| = 0$), especially for odd n, we obtain by the rules of Linear Algebra: that for every $t \in J$ there exists a unit vector $e(t) \in \text{kern}B(t)$ and $\lambda(t) \in R$ so that the solutions y of the equation

$$(4) \qquad B(t)(y - c(t)) + \acute{c}(t) = \lambda(t)\, e(t)$$

fill a uniquely determined linear subspace $E_k(t) \subset E^n$ with the dimension $k = n - \text{rank}B$. $E_k(t)$ is the axis of the instantaneous screw ($\lambda \neq 0$) of the motion or the axis of the instantaneous rotation ($\lambda = 0$) and will be called the instantaneous axis of the motion in $t \in J$ [1].

If $|B| = 0$ on the whole interval J under the regularity conditions we obtain a generalized ruled surface of dimension $k + 1$ in the fixed space E generated by the instantaneous axes $E_k(t)$, $t \in J$, which we call the fixed axoid $\varnothing$ of the motion. The fixed axoid $\varnothing$ determines the moving axoid $\bar{\Psi}$ in the moving space $\bar{E}$ generator to generator by (1). $\bar{\varnothing}$ and $\varnothing$ are mapped upon each other by the same values of parameter. In this second case Müller proved in [4]: The axoids $\bar{\varnothing}$, $\varnothing$ of a motion in E^n touch each other along every common pair $E_k(t) \subset \varnothing$, $\bar{E}_k(t) \subset \bar{\varnothing}$ for all $t \in J$ by rolling and sliding upon each other under the motion. Such a motion is called an (instantaneous) **helical** motion of order k in E^n [1]. A helical motion of order k is a pure rolling for $\lambda = 0$.

For the analytical represetation of an axoid $\varnothing$ we choose a leading curve y in the central (resp. edge) ruled surface $\Omega \subset \varnothing$ transversal to the generators. In [2] it is shown that there exists a distinguished moving orthonormal frame (ONF) $\{e_1, e_2, \dots, e_k\}$ of $\varnothing$ with the properties:

(i) $\{e_1, e_2, \dots, e_k\}$ is an ONF of the $E_k(t)$,

(ii) $\{e_{m+1}, e_{m+2}, \dots, e_k\}$ is an ONF of the central space.

z^{k-m} (resp. the edge space $K^{k-m} \subset E_k(t)$)

$$\text{(iii)}\quad \dot{e}_\sigma = \sum_{\nu=1}^{k} \alpha_{\sigma\nu}\, e_\nu + \kappa_\sigma\, a_{k+\sigma}, \qquad 1 \leq \sigma \leq m,$$

$$\dot{e}_{m+\rho} = \sum_{i=1}^{m} \alpha_{(m+\rho)i} e_i, \qquad 1 \leq \rho, \chi \leq k-m,$$

(5) $$\text{with } \kappa_\sigma > 0,\ \alpha_{\mu\nu} = -\alpha_{\nu\mu},\quad \alpha_{(m+\rho)\,(m+\chi)} = 0$$

(iv) $\{e_1, \dots, e_k, a_{k+1}, \dots, a_{k+m}\}$ is an ONF.

A leading curve y of an axoid $\varnothing$ is a leading curve of the edge (resp. central) ruled surface $\Omega \subset \varnothing$ too iff its tangent vector has the form

(6) $$\dot{y} = \sum_{\nu=1}^{k} \zeta_\nu e_\nu + \eta_{m+1}\, a_{k+m+1}$$

for $\eta_{m+1} \neq 0$, a_{k+m+1} is a unit vector well defined up to the sign with the property that $\{e_1, \dots, e_k, a_{k+1}, \dots, a_{k+m}, a_{k+m+1}\}$ is an ONF of the tangent bundle of $\varnothing$. One shows: $\eta_{m+1}(t) = 0$ in $t \in J$ iff the generator $E_k(t) \subset \varnothing$ contain the edge space $K^{k-m}(t)$.

Let $\bar{\varnothing}$ and $\varnothing$ be the corresponding axoids of the given helical motion of order k in E^n and $\{\bar{e}_1, \dots, \bar{e}_k\}$ is a principal ONF of the moving axoid $\bar{\varnothing}$. Then the equations (iii) hold for $\bar{e}_i$ with barred coefficients, $\bar{\varnothing}$ has the parameter representation on the interval J by

$$\bar{z}(t, u_1, \dots, u_k) = \bar{y}(t) + \sum_{\nu=1}^{k} u_\nu \bar{e}_\nu(t),\ \ t \in J,\ \ u_\nu \in \mathbb{R}$$

where $\bar{y}$ is a leading curve of the edge (resp. central) ruled surface $\bar{\Omega} \subset \bar{\varnothing}$. If we set

(7) $$A\bar{e}_\nu = e_\nu, \qquad 1 \leq \nu \leq k,$$

then we have the following results [1]:

$$Be_\nu = 0\ ,\quad 1 \leq \nu \leq k,$$

$$A\,\dot{\bar{e}}_\nu = \dot{e}_\nu$$

$$A\bar{a}_{k+\sigma} = a_{k+\sigma}\ ,\quad 1 \leq \sigma \leq m,$$

$$\alpha_{\mu\nu} = \bar{\alpha}_{\mu\nu}\ ,\quad \kappa_\sigma = \bar{\kappa}_\sigma > 0,\ \ 1 \leq \mu, \nu \leq k,\ 1 \leq \sigma \leq m$$

(8) $$\eta_{m+1} a_{k+m+1} = \bar{\eta}_{m+1} A\bar{a}_{k+m+1}, \text{ and } |\eta_{m+1}| = |\bar{\eta}_{m+1}|$$

$$\dot{y} = A\dot{\bar{y}} + \lambda e,$$

$$\zeta_\nu = \bar{\zeta}_\nu + \lambda\lambda_\nu, \qquad e = \sum_{\nu=1}^{k} \lambda_\nu e_\nu,\ \|e\| = 1.$$

Let a 2-ruled surface (not cylinder) ψ in E^n be given by

$\psi(t, u) = y(t) + ue(t)$.

Then the magnitude $b = \zeta / \kappa$ is called the Blaschke invariant of ψ where ζ and κ are given by (5) and (6) [3].

Let $\varnothing$ be a $(k+1)$ —ruled surface. The dimension of the asymptotic bundle of $\varnothing$ being $k + m$, $m > 0$, the magnitudes

(9) $$b_i = \zeta_i / \kappa_i, \qquad 1 \leq i \leq m,$$

are called the principal Blaschke invariants of $\varnothing$ and

(10) $$B = \sqrt[m]{|b_1 \dots b_m|}$$

is called the Blaschke invariant of $\varnothing$ [5].

In the case $m = k$ the central ruled surfaces $\Omega \subset \varnothing$ degenerate in the line of striction. Thus, the Blaschke invariant b of the 2-ruled surface ψ generated by the 1-dimensional subspace $E(t) = Sp\{e(t)\} \subset E_k(t)$ can be given by

(11) $$b = \frac{\sum_{\nu=1}^{k} \zeta_\nu \cos\theta_\nu}{\sqrt{\sum_{\mu=1}^{k} \left[\left(\sum_{\nu=1}^{k} \cos\theta_\nu \alpha_{\nu\mu} \right)^2 + (\cos\theta_\mu \kappa_\mu)^2 \right]}}$$

where $e(t) = \sum_{\nu=1}^{k} \cos\theta_\nu e_\nu(t)$, $\theta_\nu = \text{constant}$, $\|e\| = 1$. [5]

2. ON THE BLASCHKE INVARIANTS OF THE AXOIDS UNDER THE HELICAL MOTIONS OF ORDER k IN THE EUCLIDEAN n-SPACE E^n

In this section we will discuss the relation between the Blaschke invariants of the moving and fixed axoids $(m > 0)$ under the helical motions of order k in E^n. From (6) we obtain

(12) $$\xi_i = < \dot{y}, e_i >, \qquad 1 \leq i \leq k.$$

If (8) is considered together with (12) we have

(13) $$\xi_i = < \dot{\bar{y}}, \dot{e}_i > + \lambda\lambda_i, \qquad e = \sum_{i=1}^{k} \lambda_i e_i.$$

Thus, From (8), (9) and (13) we get

$$b_i = \frac{\bar{\varsigma}_i}{\bar{\kappa}_i} + \lambda\lambda_i / \bar{\kappa}_i$$

or

(14) $\quad b_i = \bar{b}_i + \lambda\lambda_i / \bar{\kappa}_i$.

Hence we have the following results:

COROLLARY 1. Let $\bar{\varnothing}$ an $\varnothing$ be the moving and fixed axoids (not cylinder) of a helical motion of order k in E^n and $\bar{b}_i$ and b_i, $1 \leq i \leq m$, be principal Blaschke invariants of $\bar{\varnothing}$ and $\varnothing$, respectively. Then $\bar{b}_i$ the and b_i are generally different and the relation between them is given by (14).

For $\lambda = 0$ which means that the motion is a pure rolling and the Blaschke invariants are $\bar{b}_i$ and b_i agree.

COROLLARY 2. The Blaschke invariants $\bar{B}$ of the moving axoid $\bar{\varnothing}$ and B of the fixed axoid $\varnothing$ are generally different. If $\lambda = 0$ they agree.

Now that is the point to discuss the relation between the Blaschke invariants of the 2-ruled surfaces $\bar{\psi}$ and ψ which correspond to each other generator by generator under the helical motion of order k (k=m) such that the ruled surface ψ and the fixed axoid $\varnothing$ have the same leading curve y and ψ is generated by the 1-dimensional subspace $E(t) = Sp\,\{e(t)\} \subset E_k(t)$. $\bar{b}$ and b being the Blaschke invariants of $\bar{\psi}$ and ψ, respectively, as in [5].

(15) $$b = \frac{\sum_{\nu=1}^{k} \varsigma_\nu \cos\theta_\nu}{\sqrt{\sum_{\mu=1}^{k} \left[\left(\sum_{\nu=1}^{k} \cos\theta_\nu \alpha_{\nu\mu} \right)^2 + (\cos\theta_\mu \kappa_\mu)^2 \right]}}$$

where $\quad e(t) = \sum_{\nu=1}^{k} \cos\theta_\nu e_\nu, \qquad \theta_\nu = \text{const.} \qquad 1 \leq \upsilon \leq k.$

and $\quad \dot{e}_\nu = \sum_{\mu=1}^{k} \alpha_{\upsilon\mu} e_\mu, \qquad 1 \leq \nu,\mu \leq k$ [3].

for the helical motions we have

(16)
$$\langle e, e_\upsilon \rangle = \langle A\bar{e}, A\bar{e}_\upsilon \rangle = \langle \bar{e}, \bar{e}_\upsilon \rangle$$
$$\langle \dot{e}_\upsilon, e_\mu \rangle = \langle A\dot{\bar{e}}_\upsilon, A\bar{e}_\mu \rangle = \langle \dot{\bar{e}}_\upsilon, \bar{e}_\mu \rangle.$$

Joining (8), (15) and (16) we get

$$(17) \qquad b = \bar{b} + \frac{\sum_{\upsilon=1}^{k} \lambda\lambda_{\upsilon} \cos\bar{\theta}_{\upsilon}}{\sqrt{\sum_{\mu=1}^{k} \left[\left(\sum_{\upsilon=1}^{k} \cos\theta_{\nu}\alpha_{\nu\mu} \right)^2 + (\cos\theta_{\mu}\kappa_{\mu})^2 \right]}} .$$

If we take $e = e_i$, $1 \leq i \leq m$ we obtain (14) from (17). Thus (14) can be considered as a generalization of (17).

COROLLARY 3. The Blaschke invariants $\bar{b}$ of $\bar{\psi}$ and b of ψ are generally different for the helical motions. For $\lambda = 0$ (pure rolling) $\bar{b}$ and b agree. If $\bar{\varnothing}$ and $\varnothing$ are 2- dimensional axoids then $\bar{\psi}$ and ψ coincide with $\bar{\varnothing}$ and $\varnothing$, respectively. In this case, since $\nu = \mu = 1$, $\cos\theta_1 = 1$, $\alpha_{11} = 0$ we obtain $b = \bar{b} + \lambda / \kappa$.

This relation can be obtained from (14) since $\lambda_1 = 1$.

ÖZET:

Bu çalışmada E^n, n-boyutlu Öklid uzayında k-yıncı mertebeden helisel hareketler altında meydana gelen $\bar{\varnothing}$ ve $\varnothing$ hareketli ve sabit aksoidlerinin Blaschke invaryantları arasındaki ilişkiler incelendi. Ayrıca bu hareket altında birbirlerine karşılık gelen $\bar{\psi} \subset \varnothing$ ve $\psi \subset \varnothing$ 2-regle yüzey çiftlerinin Blaschke invaryantları arasında bir bağıntı bulundu.

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