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SOME METRICS DEFINED ON TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD

ISMAIL KARAKUYU

Akdeniz University, Eng. Faculty, ISPARTA

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ABSTRACT

Some Riemannian metrics on tangent and cotangent bundles of a Riemannian manifold endowed with a Riemannian metric g were defined and studied by S. Sasaki [2], S. Ishakawa [3], K. Sato [6] and others. In this paper, we will define Riemannian metrics T_g, F_g, K_g on TM and state some properties related to these metrics.

1. INTRODUCTION

Let M be an n-dimensional Riemannian C^∞ manifold endowed with the Riemannian metric g and TM be its tangent bundle. Let $\{u^i\}$ be a local coordinate system for a chart $(U, \varnothing)$ and $\{\upsilon^i, \upsilon^{n+i}\}$ be the induced coordinate system for the chart $(TU, T\varnothing)$ on TM. We put for simplicity $\partial_i = \partial / \partial u^i$ and $\mu_i = \partial / \partial \upsilon^i$. Throughout the paper we will use Einstein's summation. Using local coordinates on TM, the Sasaki metric g^s on TM was defined by S. Sasaki. And S. Ishikawa [3] defined the Riemannian metric on T_2M as follows:

$$d\sigma^2 = g_{ij}\, dx^i dx^j + g_{ij}\, Dy^i Dy^j + g_{ij}\, D\xi^i D\xi^j$$

where $Dy^i = dy^i + \Gamma^i{}_{jk}\, y^i dx^k$, $\Gamma^i{}_{jk} = z^i + \Gamma_{jk} y^i y^k$, $D\xi^i = dz^i + \Gamma_{jk}\ \xi^i dx^k$ and (x^i, y^i, z^i) is a local coordinate system on T_2M.

Let Π and B denote the natural projection of TM onto M and the binding map associated with the linear connection ∇ of M respectively. Let $\varkappa^v(TM)$ be the set of the vertical vectorfields on TM and $\varkappa^h(TM)$ be the set of the horizontal vectorfields on TM.

II. SOME METRICS ON TM FOR A RIEMANNIAN MANIFOLD AND SOME PROPERTIES RELATED TO THESE METRICS

Using the pair (g, ∇) for the Riemannian manifold, we will define metrics T_g, F_g and K_g on TM.

Definition 2.1. For the pair (g, ∇) of the Riemannian manifold, we will define the first Riemannian metric T_g on TM by

$$T_g(Z, W) = g(\Pi_* Z, \Pi_* W) + g(BZ, BW) \tag{2.1}$$

where Z, W are vectorfields on TM.

The components of the metric tensor T_g of TM with respect to $(\upsilon^i, \upsilon^{n+i})$ can be written in the form

$$T_g = G_{rs}\, d\upsilon^r dv^s$$

where $G_{rs} = T_g(\mu_r, \mu_s)$, and $r, s = 1, 2, \ldots, 2n$.

If $Z = (z^i, z^{n+i}: 1, 2, \ldots, n)$, $W = (w^i, w^{n+i}: 1, 2, \ldots, n)$ and Γ^i_{jk} are the Christoffel symbols for the connection ∇, then we have by Π_* and B $T_g(Z, W) = g(z^i(\partial_i \circ \Pi), w^i(\partial_j \circ \Pi)) +$

$+ g((z^{n+i} + (\Gamma^i_{jk} \circ \Pi) z^j \upsilon^{n+k})(\partial_i \circ \Pi), (w^{n+p} + (\Gamma^k_{sr} \circ \Pi) w^s \upsilon^{n+r})(\partial_p \circ \Pi)$.

Then we can easily obtain that

$$\begin{aligned} G_{ij} &= g_{ij} \circ \Pi + (g_{pk} \circ \Pi)(\Gamma_{ri} \circ \Pi)(\Gamma_{sj} \circ \Pi) v^{n+r} \upsilon^{n+s} \\ G_{i(n+j)} &= (g_{kj} \circ \Pi)(\Gamma^k_{ri} \circ \Pi) \upsilon^{n+r} \\ G_{(n+i)j} &= (g_{jk} \circ \Pi)(\Gamma^k_{ir} \circ \Pi) \upsilon^{n+r} \\ G_{(n+i)(n+j)} &= g_{ij} \circ \Pi \end{aligned} \tag{2.2}$$

where $g_{ij} = g(\partial_i, \partial_j)$ $i, j = 1, 2, \ldots, n$.

Theorem 2.1. The TG-transition structure for TM leaves invariant the metric T_g.

Proof: If Z is a vectorfield on TM, then (TG)Z is also a vectorfield on TM. For $(TG)\, Z = (\lambda^i, \lambda^{n+i}: 1, 2, \ldots, n)$, we can write by the natural projection Π and the binding map B associated with the linear connection ∇ of a manifold M,

$$\begin{aligned} \lambda^i &= z^{n+i} + (\Gamma^i_{jk} \circ \Pi) z^j \upsilon^{n+k} \\ \lambda^{n+i} &= z^i - (\Gamma^i_{jk} \circ \Pi) \upsilon^{n+j} \upsilon^{n+k} - (\Gamma^i_{jk} \circ \Pi)(\Gamma^i_{sr} \circ \Pi) z^s \upsilon^{n+r} \upsilon^{n+k}. \end{aligned} \tag{2.3}$$

Thus we obtain

$$(\Pi_*(TG)Z)^i = z^{n+i} + (\Gamma^i_{jk} \circ \Pi) z^j \upsilon^{n+k}$$

$$(B(TG)Z)^i = \lambda^{n+i} + (\Gamma^i_{jk} \circ \Pi) \lambda^j \upsilon^{n+k} = z^i.$$

For the vectorfields Z, W on TM, using the definition (2.1) we obtain

$$g((TG)Z, (TG)W) = g(\Pi_*(TG)Z), \Pi_*(TG)W) + g(B(TZ)Z, B(TG)W)$$
$$= g(BZ, BW) + g(\Pi_*Z, \Pi_*W).$$

Then the proof is completed.

Proposition 1.1. Let X^v, Y^v be vertical vectorfields on TM and X^h, Y^h be horizontal vector fields on TM. We can easily see that

(2.4) a) $T_g(X^v, Y^v) = T_g(X^h, Y^h) = g(X, Y) o\Pi$

b) $T_g(X^v, Y^h) = 0.$

Definition 2.2. For the pair (g, ∇) of the Riemannian manifold, we will define the second Riemannian metric F_g on TM by

(2.5) $F_g(Z, W) = g(\Pi_*Z, BW) + g(BZ, \Pi_*W).$

Proposition 2.2. Let TG be the transition structure for TM, X^v, Y^v vertical vector fields on TM and X^h, Y^h be horizontal vector fields on TM. We can easily see that

(2.6) a) $F_g(Z, W) = T_g(Z, (TG)W)$

b) $F_g(X^v, Y^v) = F_g(X^h, Y^h) = 0$

c) $F_g(X^v, Y^h) = g(X, Y) o\Pi.$

Definition 2.3. Let υ be a point in TM. We will define that the third Riemannian metric K_g on TM is a function which associates with each point υ in TM such that if $W \in T_\nu(TM)$, then

(2.7) $(K_g)_\nu(W_\nu) = g(\upsilon, \Pi_*W_\nu).$

Proposition 2.3. Let d be the exterior derivative operator. For vertical evecorfields and horizontal vector fields on TM, the following identity is satisfied

(2.8) $dK_g = F_g.$

Proof: For the vector fields Z, W on TM, we have by the definition of the exterior derivative

$$dK_g(Z, W) = ZK_g(W) - WK_g(Z) - K_g([Z, W]).$$

K_g vanishes for every vertical vector. For the vectorfields X, Y on M, X^v, Y^v are vertical vector fields on TM. Hence we obtain

$$dK_g(X^v, Y^v) = X^vK_g(Y^v) - Y^vK_g(X^v) - K_g([X^v, Y^v]).$$
It is well-known that

$$[X^v, Y^v]^i = (x^i o \Pi)\mu_{n+j} (y^j O\Pi) - (y^i o\Pi)\mu_{n+j} (x^j o\Pi) = 0$$

$$[X^h, Y^v]^i = (x^j \partial_j(y^i) + x^j\Gamma^i{}_{jk}y^k) o\Pi = ((\nabla_X Y)^v)^i.$$

For the horizontal vectorfields X^h, Y^h

$$dK_g (X^h, Y^v) = X^h K_g(Y^v) - K_g (D_X Y)^v.$$

Thus using $\Pi_*(X^v) = 0$, for the pair (g, ∇), we get

$$dK_g (X^v, Y^v) = 0 \quad \text{and} \quad dK_g (X^v, X^h) = g (X, Y)o\Pi.$$

In the same way, we see that $dK_g(X^h, Y^h) = 0$

Thus, using (2.5) b, c we complete the proof.

Theorem 2.2. Let S be geodesic flow of ∇ for (M, g) and Z be vectorfield on TM. Then the following identity is satisfied:

(2.9) $$T_g (S, Z) = K_g (Z).$$

Proof: If Y (I) is an integral curve of vectorfield S on TM and $\Pi o Y = \alpha$ is a geodesic with respect to ∇ in M, then the vectorfield S is geodesic flow for M. For $S = (S^i, S^{n+i}: 1, 2, ..., n)$ we can state

$$S^i = \upsilon^{n+i}, \qquad S^{n+i} = -(\Gamma_{jk} o\Pi)\, \upsilon^{n+j}\upsilon^{n+k}.$$

Thus the geodesic flow of ∇ at the point υ in TM can be expressed by

$$\Pi_* (S_\nu) = \upsilon \quad \text{and} \quad B (S_\nu) = 0.$$

We have by using (2.1) and (2.6)

$$T_g(S, Z)_\nu = g (\Pi_* S_\nu, Z_\nu) = g(\upsilon, \Pi_* Z_\nu) = (K_g)_\nu(Z_\nu).$$

This completes the proof.

The Lie derivative of the first metric tensor T_g on TM with respect to a vector field A on TM is two-form on TM. If the vector field A is the vectorfield which is obtained from $L_A(T_g) = O$, then the vectorfield A satirfies the equations

$$a^i\mu_i(G_{jk}) + G_{j(n+i)}\mu_k(a^i) + G_{(n+i)}\mu_j(a^i) = 0$$

where G_{ij} which are given in (2.2) are the components of the metric tensor T_g. It is easily seen that the vector field A satisfies $L_A(T_g) = 0$ if and only if T_g is constant on each element of one-parameter group of A.

RİEMANN MANİFOLDUNUN TANJANT DEMETLERİNDE TANIMLANMIŞ BAZI METRİKLER

ÖZET

Bir g Riemann metriğine sahip Riemann manifoldunun tanjant ve kotanjant demetlerinde bazı metrikler S. Sasaki [5], S. Ishakawa [3], K. Sato [6] ve daha başkaları tarafından tanımlandı ve çalışıldı. Bu çalışmada, TM üstünde T_g, F_g ve K_g Riemann metrikleri tanımlanmış ve bu metriklerle ilgili bazı özellikler verilmiştir.

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