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ON THE PARALLEL HYPERSURFACES WITH CONSTANT CURVATURE

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SUMMARY

Gaussian and mean curvatures, K_r and H_r, for parallel surfaces in E^3 are given in [2]. In the present note, by means of higher order Gaussian and mean curvatures, we calculate the generalized the curvatures K_r and H_r in E^{n+1}, $n > 2$.

I. BASIC CONCEPTS

DEFINITION I.1: Let M_1 and M_2 are two hypersurfaces in E^{n+1}, with unit normal vector N_1 of M_1,

$$N_1 = \sum_{i=1}^{n+1} a_i \frac{\partial}{\partial x_i},$$

where each a_i is a C^∞ function on M_1. If there is a function f, from M_1 to M_2 such that

$$f: M_1 \longrightarrow M_2$$
$$P \longrightarrow f(P) = (p_1 + ra_1(P), \ldots, p_{n+1} + ra_{n+1}(P)),$$

then M_2 is called a parallel hypersurface of M_1, where $r \in R$ [1].

THEOREM I.1: Let M_r be a parallel surface of the surface $M \subset E^3$. Let the Gaussian curvature and mean curvature of M be denoted by K and H at the point $P \in M$, respectively, and the Gaussian curvature and mean curvature of M_r be denoted by K_r and H_r at the point $f(P) \in M_r$, respectively. Then we know[1] that

$$K_r = \frac{K}{1 + r H + r^2 K}$$

and

$$H_r = \frac{H + 2\, rK}{1 + r\, H + r^2\, K}.$$

THEOREM I.2: Let M be a hypersurface of E^{n+1} and $K_1, K_2, \ldots, K_n$ are the higher order Gaussian curvatures and $k_1, k_2, \ldots, k_n$ are the principal curvatures at the point $P \in M$.

Let define of function

$$\Phi : M \longrightarrow R$$
$$P \longrightarrow \Phi(P) = \Phi(r, k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n} (1 + rk_i).$$

Then we have that

$$\Phi(r, k_1, k_2, \ldots, k_n) = 1 + r \sum_{i=1}^{n} k_i + r^2 \sum_{i<j} k_i k_j + \ldots + r^n \prod_{i=1}^{n} k_i$$

or

$$\Phi(r, k_1, k_2, \ldots, k_n) = 1 + r\, K_1 + r^2 K_2 + \ldots + r^n K_n,$$

where $r \in R$ is given in definition I.1 [3].

THEOREM I.3: Let M_r be a parallel hypersurface of the hypersurface M in E^{n+1}, $K_1, K_2, \ldots, K_n$ denote the higher order Gaussian curvatures of M, at the point $P \in M$. K_r and H_r are the generalized Gaussian and mean curvatures of M_r, respectively, at the point $f(P) \in M_r$

Suppose the function

$$\Phi : M \longrightarrow R$$
$$P \longrightarrow \Phi(P) = \Phi(r, k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n} (1 + rk_i).$$

Then we have

$$K_r = \frac{\dfrac{\partial^n \Phi(r, k_1, k_2, \ldots, k_n)}{(\partial_r)^n}}{(n!)\, \Phi(r, k_1, k_2, \ldots, k_n)}$$

and

$$H_r = \frac{\dfrac{\partial \Phi(r, k_1, k_2, \ldots, k_n)}{\partial_r}}{\Phi(r, k_1, k_2, \ldots, k_n)}$$

[3].

THEOREM I.4: Let M be a surface of constant positive Gaussian curvature K with no umbilics. Let $r_1=1/\sqrt{K}$ and $r_2=-1/\sqrt{K}$ define parallel sets M_1 and M_2, respectively. Then M_1 and M_2 are immersed surfaces of M which have constant mean curvatures $\sqrt{K}$ and $-\sqrt{K}$, respectively. If M′ is a surface with constant mean curvature H (non zero) and non zero Gaussian curvature, let $r=-1/H$ yields a parallel set that is an immersed surface of M′ with constant positive Gaussian curvature H^2 [2].

II. GENERALIZED THEOREMS

THEOREM II.1: Let M_r be a parallel hypersurface of the hypersurface M in E^{n+1}. Let $K_1, K_2, \ldots, K_n$ denote the higher order Gaussian curvatures of M, at the point $P \in M$ and let

$$\sum_{i=1}^{n-1} r^i K_i = -1$$

then generalized Gaussian curvature of M_r is

$$K_r = \frac{1}{r^n} .$$

PROOF: It follows from Theorem I.3 that the generalized Gaussian curvature of a parallel hypersurface is given by

$$K_r = \frac{\dfrac{\partial^n \Phi(r, k_1, k_2, \ldots, k_n)}{(\partial r)^n}}{(n!)\, \Phi(r, k_1, k_2, \ldots, k_n)}$$

$$= \frac{\prod_{i=1}^{n} k_i}{\prod_{i=1}^{n} (1 + r k_i)}$$

$$= \frac{\prod_{i=1}^{n} k_i}{1 + rK_1 + r^2K_2 + \ldots + r^{n-1}K_{n-1} + r^nK_n}$$

since we have,

$$\sum_{i=1}^{n-1} r^i K_i = -1$$

then

$$K_r = \frac{\prod_{i=1}^{n} k_i}{r^n \prod_{i=1}^{n} k_i}$$

or

$$K_r = \frac{1}{r^n} .$$

Note that there exists a sphere in E^3 such that $\sum_{i=1}^{n-1} r^i K_i = -1.$

THEOREM II.2: Let M_r be a parallel hypersurface of the hypersurface M in E^{n+1}. Let $K_1, K_2, \ldots, K_n$ denote the higher order Gaussian curvatures of M, at the point $P \in M$ and let

$$\sum_{i=1}^{n} (i-1)\, r^i K_i = 1$$

then the generalized mean curvature of M_r is

$$H_r = \frac{1}{r} .$$

PROOF: Theorem I.3 gives us that the generalized mean curvature of a parallel hypersurface M_r is given by

$$H_r = \frac{\dfrac{\partial \Phi (r, k_1, k_2, \ldots, k_n)}{\partial r}}{\Phi (r, k_1, k_2, \ldots, k_n)}$$

$$= \frac{K_1 + 2rK_2 + \ldots + nr^{n-1}K_n}{1 + rK_1 + r^2K_2 + \ldots + r^{n-1}K_{n-1} + r^nK_n}$$

$$= \frac{1}{r} \left[\frac{rK_1 + 2r^2K_2 + \ldots + nr^nK_n}{1 + rK_1 + r^2K_2 + \ldots + r^{n-1}K_{n-1} + r^nK_n} \right]$$

$$= \frac{1}{r} \left[\frac{\sum_{i=1}^{n} i\, r^i K_i}{1 + \sum_{i=1}^{n} r^i K_i} \right]$$

$$= \frac{1}{r}\left[1 - \frac{1 - \sum_{i=1}^{n} (i-1)\, r^i K_i}{1 + \sum_{i=1}^{n} r^i K_i}\right]$$

since we have that

$$\sum_{i=1}^{n} (i-1)\, r^i K_i = 1$$

then we get that

$$H_r = \frac{1}{r} .$$

COROLLARY: In the case of $n = 2$, Theorem II.1 and Theorem II.2 reduce to the results of [2].

ÖZET

SABİT EĞRİLİKLİ PARALEL HİPERYÜZEYLER ÜZERİNE

[2] de verilen E^3 deki paralel yüzeylerin K_r ve H_r, Gauss ve ortalama eğrilikleri, bu çalışmada, $n > 2$ olmak üzere, E^{n+1} deki yüksek mertebeden Gauss ve ortalama eğrilikleri yardımıyla genelleştirilmiş ve hesaplanmıştır.

REFERENCES

1. HACISALIHOĞLU, H.H., 1982. Diferensiyel Geometri. İnönü Üniversitesi Fen-Ed. Fakültesi Yayınları Mat. No: 2.

2. HICKS, NOEL J., 1974. Notes on Differential Geometry. Von Nostrand Reinhold Company, London, pp:35–45.

3. SAĞEL, M. KEMAL–HACISALİHOĞLU, H. H., 1988. On the Gaussian and Mean Curvatures of Parallel Hypersurfaces. Comm. Fac. Sci. Univ. Ankara. Ser. A_1: Mathematics, Vol. 37. pp:9–15.