Commun. Fac. Sci. Univ. Ank. Series A_1
V. 38, pp 7-13 (1989)

ON THE LINE CLASSES IN SOME FINITE HYPERBOLIC PLANES

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ABSTRACT

Determination of the line classes (and the number of lines in each class) in some hyperbolic planes of type π_m occurs as an open problem. In this paper we give a partial answer to the problem for the special hyperbolic planes π_3, π_4 π_5, π_6, π_7 and $\pi^o{}_{n-2}$, $\pi^o{}_{n-1}$.

INTRODUCTION

It is well known that if a line is deleted from a projective plane then the remaining substructure forms an affine plane. Graves [1962], Ostrom [1962] and Bumcrot [1971] have given examples of hyperbolic planes obtained by deletion from projective planes. Graves [1962] also asked for additional constructions of such planes. Sandler [1963] has shown that if three non-concurrent lines are deleted from a projective plane then the remaining incidence structure forms a hyperbolic plane in the sense of Graves [1962]. Kaya-Özcan [1984] has extended the Sandler's construction as follows: Let π be a finite projective plane of order n and m a positive integer provided that $m \leq n+2$. Let $\iota_1, \iota_2, \ldots, \iota_m$ denote distinct m lines of π such that no three are concurrent. Let π_m be the substructure obtained by deleting from π all of the lines ι_i, $i=1,2,\ldots,m$, and all points on these m lines. A point of π is called corner point if it is intersection of any two lines in the set $\{\iota_1, \iota_2, \ldots, \iota_m\}$. Let r denote the minimum number of corner points on a line of π_m as a line of π. In [Kaya, R.-Özcan, E., (1984)] it has been shown that if $3 \leq m \leq n+r+\frac{1}{2}(1-\sqrt{4n+5})$ then π_m is a hyperbolic plane.

The lines of π_m are classified according to the number of points which are contained in each line of a class. Let C_s denote the set of all

ISSN 02571-081 A.Ü. Basımevi

lines of π_m such that each line in it contains exactly s corner points in π. Each line in C_s contains exactly $n+1-(m-s)$ points. There exist $\frac{1}{2}m-r+1$ or $\frac{1}{2}(m+1)-r$ classes of lines in π_m according as m is an even or an odd positive integer, respectively. The line classes are C_r, $C_{r+1},\ldots, C_{m/2}$ or $C_r, C_{r+1},\ldots, C_{(m-1)/2}$ according as m is even or odd, respectively. It follows that if m is even then there exist exactly $\frac{1}{2}m-r+1$ classes of lines in π_m and π_{m+1} obtained from a projective plane, namely $C_r, C_{r+1}\ldots, C_{m/2}$. Furthermore, if q_s denote the number of all lines in C_s then one has the following:

$$\text{(I)} \quad \sum_{s=r}^{t} q_s = n^2+n+1-m$$

$$\text{(II)} \quad \sum_{s=r}^{t} s q_s = (n-1)\binom{m}{2}$$

$$\text{(III)} \quad \sum_{s=r}^{t} s^2 q_s = \left[n-1+\binom{m-2}{2}\right]\binom{m}{2}$$

Where t is $\frac{m}{2}$ or $\frac{1}{2}(m-1)$ according as m is even or odd, respectively. (In what follows t will be used in that sense). One of the unsolved problems related to these hyperbolic planes is to determine the number of lines in each class C_s of π_m. A partial answer to the problem is given in [Olgun, (1986)]. In the first part of this paper, we formulate the answer to the question for any finite planes of type π_3, π_4, π_5, and determine the required numbers for a π_6 and π_7 in terms of the number of lines in C_3. In the second part, the problem is solved for some special hyperbolic planes of type π_{n-1} and π_{n-2}. It would be very interesting to find the full answer to the above problem for any m and n.

THE LINE CLASSES IN π_3, π_4 π_5, π_6, AND π_7.

PROPOSITION 1. For any hyperbolic plane π_m

$$\text{(i)} \quad q_2 = \frac{1}{2}\binom{m}{2}\binom{m-2}{2} - \sum_{s=3}^{t}\binom{s}{2} q_s$$

$$\text{(ii)} \quad q_1 = \binom{m}{2}\left[n-1-\binom{m-2}{2}\right] + \sum_{s=3}^{t} s(s-2) q_s$$

(iii) $q_0=n^2+n\left[1-\binom{m}{2}\right]+\binom{m-1}{2}+\frac{1}{2}\binom{m}{2}\cdot\binom{m-2}{2}$

$$-\sum_{s=3}^{t}\binom{s-1}{2}q_s.$$

PROOF: Equality (i) can be obtained substructing the equalities II and III side by side. Similarly, (ii) can be obtained from II using (i), and (iii) from I using (i) and (ii).

COROLLARY 1. For any hyperbolic plane π_m with $m \in \{3, 4, 5\}$

(i) $q_2 = \frac{1}{2}\binom{m}{2}\binom{m-2}{2}$

(ii) $q_1 = \binom{m}{2}\left[n-1-\binom{m-2}{2}\right]$

(iii) $q_0=n^2+n\left[1-\binom{m}{2}\right]+\binom{m-1}{2}+\frac{1}{2}\binom{m}{2}\binom{m-2}{2}$.

Proof follows from proposition 1 since $q_i=0$, $i\geq 3$, for $m=4$ or 5 and, also $q_i=0$ $i\geq 2$ for $m=3$.

Notice that, if $m=3$ then $q_1=3(n-1)$, $q_0=(n-1)^2$. Similarly if $m=4$ then $q_2=3$, $q_1=6(n-2)$, $q_0=n^2-5n+6$, and if $m=5$ then $q_2=15$, $q_1=10(n-4)$, $q_0=n^2-9n+21$.

The following corollaries are immediate:

COROLLARY 2. Number of lines in C_0, C_1, C_2 of any hyperbolic plane of type π_6 and π_7 can be determined in terms of the number of lines in C_3 as follows:

$q_2=45-3q_3$		$q_2=105-3q_3$
$q_1=15(n-7)+3q_3$	and	$q_1=21(n-11)+3q_3$
$q_0=n^2-14n+55-q_3$		$q_0=n^2-20n+120-q_3$

respectively.

COROLLARY 3. Total number of lines of C_1 and C_2 in any hyperbolic plane π_m can be determined independently from the number of lines of C_0 and C_3, and vice versa. That is,

$$q_1+q_2 = \binom{m}{2} \left[n-1 - \tfrac{1}{2} \binom{m-2}{2}\right] + \tfrac{1}{2} \left(\sum_{s=3}^{t} s(s-3)q_s \right)$$

$$q_0+q_3=n^2+n \left[1-\binom{m}{2} + \binom{m-1}{2}\right] + \tfrac{1}{2} \binom{m}{2} \binom{m-2}{2} - \sum_{s=4}^{t} \binom{s-1}{2} q_s.$$

THE LINE CLASSES IN π^o_{n-1} AND π^o_{n-2}

Let π be a projective plane of order n. A set of $\mathcal{O}$ of $n+1$ points in π is called an oval if no three points of $\mathcal{O}$ are collinear. A line of π which contains exactly one point, two points and no points of $\mathcal{O}$ is called tangent line, secant line and exterior line, respectively. A point of π is called an exterior point and interior point if it lies on exactly two tangent lines and on no tangent lines, respectively. A secant line contains $\frac{1}{2}$ (n—1) exterior points and an exterior line contains $\frac{1}{2}$ (n+1) exterior points. Total number of the exterior points and interior points of π is $\frac{1}{2}$ n(n+1) and $\frac{1}{2}$ n(n—1), respectively. There are n+1 tangent lines of $\mathcal{O}$ and a tangent line contains n exterior points. Let π be a projective plane of odd order n, $n \geqslant 9$ and $\mathcal{O}$ an oval in π. Let β be the set of interior points of $\mathcal{O}$, and consider the restrictions of the secant and exterior lines of π to the interior points of $\mathcal{O}$. Hence the restrictions of these lines are the set theorical intersections of the secant and exterior lines of π with β. It has been shown by Ostrom [1962] that the geometric structure so obtained is a hyperbolic plane. Clearly the above model of the hyperbolic plane can be considered as a special hyperbolic plane of type π_m provided that $\iota_1, \iota_2, \ldots, \iota_{n+1}$ are the tangent lines of an oval $\mathcal{O}$. Therefore it will be convenient to use the notation π^o_{n+1} for the Ostrom's hyperbolic plane. Furthermore, in what follows we use π^o_m instead of π_m provided that the set of deleted lines, $\{\iota_1, \iota_2, \ldots, \iota_m\}$, with $3 \leq m \leq n$, is a subset of the set of all tangent lines of $\mathcal{O}$. It can easily be shown that each of $\pi^o_3, \pi^o_4, \ldots, \pi^o_{n-2}, \pi^o_{n-1}$ is a hyperbolic plane but not π^o_n since the non-deleted tangent line in π^o_n contains only one point. It is clear from the definitions of corner and exterior points that a corner point for π^o_m, $3 \leq m \leq n+1$, is also n exterior point which is deleted from π. It is known that the line classes of π^o_{n+1} are $C_{\frac{1}{2}(n+1)}$ and $C_{\frac{1}{2}(n-1)}$

and $q_{\frac{1}{2}(n+1)} = \frac{1}{2} n(n-1)$ and $q_{\frac{1}{2}(n-1)} = \frac{1}{2} n(n+1)$. We give line classes of $\pi^o{}_{n-1}$ and $\pi^o{}_{n-2}$ in the following propositions:

PROPOSITION 2. There exist four line classes in $\pi^o{}_{n-1}$, namely C_0, $C_{\frac{1}{2}(n-5)}$, $C_{\frac{1}{2}(n-3)}$, $C_{\frac{1}{2}(n-1)}$, and the number of lines in these classes are

$$q_0 = 2, q_{\frac{1}{2}(n-5)} = \frac{1}{2} (n-3)(n-1), q_{\frac{1}{2}(n-3)} = \frac{1}{2} (n-1)(n+4),$$

$q_{\frac{1}{2}(n-1)} = \frac{1}{2} (n+1)$, respectively.

PROOF. Let t_1, t_2 be tangent lines of $\pi^o{}_{n-1}$ and $P = t_1 \cap t_2$, $Q_i = \mathcal{O} \cap t_i$. And let Q be any point of $\mathcal{O}$ with $Q \neq Q_i$ $i=1,2$. Clearly none of the two lines $t_1 = PQ_1$ and $t_2 = PQ_2$ contains a corner point. Therefore t_1 and t_2 belong to C_0. Let ι be a secant line which passes through none of P, Q_1 and Q_2. All exterior points on ι except $\iota \cap t_1$ and $\iota \cap t_2$ are corner points. ι contains exactly $\frac{1}{2}(n-1)-2 = \frac{1}{2}(n-5)$ corner points since there exist $\frac{1}{2}(n-1)$ exterior points on ι. Thus ι belongs to $C_{\frac{1}{2}(n-5)}$. The secant line PQ contains exactly $\frac{1}{2}(n-1)-1 = \frac{1}{2}(n-3)$ corner points since all exterior points on PQ except P are corner points. Similarly all exterior points on Q_1Q (or Q_2Q), except $Q_1Q \cap t_2$ (or $Q_2Q \cap t_1$), are corner points. Hence each of the lines Q_iQ contains $\frac{1}{2}(n-1)-1 = \frac{1}{2}(n-3)$ corner points. Thus PQ, Q_1Q and Q_2Q belong to $C_{\frac{1}{2}(n-3)}$. Now let ι be any line not passing through P. All exterior points on ι except $\iota \cap t_1$ and $\iota \cap t_2$ are corner points. ι contains $\frac{1}{2}(n+1)-2 = \frac{1}{2}(n-3)$ corner points since there exist exactly $\frac{1}{2}(n+1)$ exterior points on ι in π. Thus ι belongs to $C_{\frac{1}{2}(n-3)}$. The secant line Q_1Q_2 belongs to $C_{\frac{1}{2}(n-1)}$ since all exterior points on Q_1Q_2 are corner points. Finally, an exterior line passing through the point P contains exactly $\frac{1}{2}(n+1)-1 = \frac{1}{2}(n-1)$ corner points since all exterior points on such a line except P are corner points. Thus, these lines belong to $C_{\frac{1}{2}(n-1)}$. Consequently, the line classes in $\pi^0{}_{n-1}$ are

$C_0 = \{t_1, t_2\}$

$C_{\frac{1}{2}(n-5)} = \{\iota : \iota$ is a secant line passing through none of $P, Q_1, Q_2\}$

$C_{\frac{1}{2}(n-3)} = \{\iota : \iota = Q_iQ$, $i=1,2$, or ι is a secant line on P or an exterior line not on $P\}$

$C_{\frac{1}{2}(n-1)} = \{\iota : \iota = Q_1Q_2$ or ι is an exterior line on $P\}$.

Hence, it is clear that $q_0=2$. $q_{\frac{1}{2}(n-5)} = \frac{1}{2}(n-3)(n-1)$ since the number of secant lines on P is $\frac{1}{2}(n-1)$, the number of secant lines on Q_1 or Q_2 is $2(n-1)+1$, and the total number of secant lines of $\pi^0{}_{n-1}$ is $\frac{1}{2}n(n+1)$. $q_{\frac{1}{2}(n-3)} = \frac{1}{2}(n-1)(n+4)$ since the number of secant lines on P is $\frac{1}{2}(n-1)$, the number of secant lines on Q_1 or Q_2 except Q_1Q_2 is $2(n-1)$, and the total number of exterior lines not on P is $\frac{1}{2}(n-1)^2$. $q_{\frac{1}{2}(n-1)} = \frac{1}{2}(n+1)$ since the number of exterior lines on P is $\frac{1}{2}(n-1)$, and $Q_1Q_2 \in C_{\frac{1}{2}(n-1)}$.

PROPOSITION 3. There exist four line classes in $\pi^0{}_{n-2}$, namely C_0, $C_{\frac{1}{2}(n-7)}$, $C_{\frac{1}{2}(n-5)}$, $C_{\frac{1}{2}(n-3)}$, and the number of lines in these classes are

$q_0=3$, $q_{\frac{1}{2}(n-7)} = \frac{1}{2}(n-3)(n-5)$, $q_{\frac{1}{2}(n-5)} = \frac{1}{2}(n+8)(n-3)$,

$q_{\frac{1}{2}(n-3)} = \frac{3}{2}(n+3)$, respectively.

SKETCH OF PROOF. Let t_1, t_2, t_3 be non deleted tangent lines and $t_1 \cap t_2 = P_3$, $t_1 \cap t_2 = P_2$, $t_2 \cap t_3 = P_1$; and let $t_i \cap \mathcal{O} = Q_i$ with $i = 1,2,3$. One can find the line classes of $\pi^o{}_{n-2}$ as follows:

$C_0 = \{t_1, t_2, t_3)\}$

$C_{\frac{1}{2}(n-7)} = \{\iota$: ι is a secant line passing through none of P_i, Q_i with $i=1,2,3\}$

$C_{\frac{1}{2}(n-5)} = \{\iota$: ι is a secant line on P_i but not on Q_i or a secant line passing through only one Q_i but none of P_i or ι is an exterior line not on P_i, $i=1,2,3\}$

$C_{\frac{1}{2}(n-3)} = \{\iota$: $\iota = P_iQ_i$ or $\iota = Q_iQ_j$ with $i \neq j$ or ι is an exterior line on P_i, $i=1,2,3\}$.

Proof can be completed by a similar way in the proof of proposition 2.

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