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FIXED POINT THEOREMS FOR SOME DISCONTINOUS OPERATORS IN 2 - METRIC SPACES

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The purpose of this paper is to discuss the existence of fixed points for some discontinuous operators T on a 2-metric space and belonging to a class D (a, b) for which certain sequences are asymptotically regular.

1. INTRODUCTION

The well known Banach Contraction Principle states that a self mapping T of a complete metric space (X, d) that satisfies, for some λ, $0 \leq \lambda < 1$, the inequality

$$d(Tx, Ty) \leq \lambda\, d(x, y) \tag{1}$$

for all x, y in X, has a unique fixed points. J. Schauder [15], Tychonoff [16], S. Lefschetz [13], F. Browder [1], G. Hardy and T. Rogers [6], K. Goebel, W.A. Kirk and T.N. Shimi [5] and many others have extended and generalised this basic result.

Recently, Nova G. [14] proved some fixed point theorems for operators T defined on a closed subset K of a Banach space X that satisfy

$$\| Tx-Ty \| \leq a \| x-y \| + b [\| x-Tx \| + \| y-Ty \|] \tag{2}$$

for all x, y in K, where $0 \leq a, b < 1$. He calls that an operator satisfying (2) belongs to the class D (a, b). The contraction operator satisfying (1) is in the class D (λ, 0), $0 \leq \lambda < 1$.

Note that the condition (1) implies the continuity of the operator T, the condition (2) may hold even if the operator is discontinuous. In fact any operator is in class D (1, 1). Since by triangle inequality.

$$\| Tx-Ty \| \leq \| Tx-x \| + \| x-y \| + \| y-Ty \|$$

The concept of 2–metric space was initiated by Gayler [2] and subsequently enhanced by Gahler [3, 4], White [17] and many others. On the other hand, Iseki [7, 8, 9], Khan–Fisher [10], Khan [11, 12] and many others have studied the aspect of fixed point theory in the setting of 2–metric spaces.

In this paper, we have studied some fixed point theorems in 2–metric spaces for operators T belonging to the class D (a, b) for which certain sequences are asymptotically regular.

2. PRELIMINARIES

Following Gahler [2] and White [17], we have the following definitions.

Definition 2.1. A 2–metric space X is a space in which for each triple of points x, y, z, there exists real valued function d (x, y, z) such that

(i) for each pair of distinct points x, y in X, there exists a point z in X such shat $d(x, y, z) \neq 0$,

(ii) $d(x, y, z) = 0$, when at least two of x, y, z are equal,

(iii) $d(x, y, z) = d(y, z, x) = d(x, z, y)$,

(iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all w in X.

It is clearly seen that d is non–negative.

Definition 2.2. A 2–metric d on a set X is said to be sequentially continuous on X if it is **sequentially continuous** in two of its three arguments.

It follows that if d is sequentially continuous in two of its three arguments, it is continuous (sequentially) in all the three arguments.

Definition 2.3. A sequence $\{x_n\}$ in a 2–metric space (X, d) is said to be a **Cauchy sequence** if $\lim_{n,m\to\infty} d(x_m, x_n, p) = 0$ for all p in X.

Definition 2.4. A sequence $\{x_n\}$ in a 2–metric space (X, d) is said to be **convergent** with limit x in X if $\lim_{n\to\infty} d(x_n, x, p) = 0$ for all p in X.

It follows that if the sequence $\{x_n\}$ converges to x in X then $\lim_{n\to\infty} d(x_n, p, q) = d(x, p, q)$ for all p, q in X.

Definition 2.5. A 2–metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges.

Definition 2.6. Let $T: Y \to Y$, $Y \subset X$ and $x \in Y$. Then T is said to **asymptotically regular at** x if for all natural numbers n, $T^n(x) \in Y$ and $\lim_{n\to\infty} d(T^n(x), T^{n+1}(x), p) = 0$ for all p in x.

Definition 2.7. A sequence $\{x_n\}$ of elements of $Y \subset X$ is said to **asymptotically T-regular** if $\lim_{n\to\infty} d(x_n, T(x_n), p) = 0$ for all p in X.

Remark 2.1. It is obvious that T is asymptotically regular at some $x \in Y$ if and only if for all natural numbers n, $T^n(x) \in Y$ and $\{T^n(x)\}$ is asymptotically T–regular.

Motivated by Iseki [8], we have the following

Definition 2.8. Let (X, d) be a 2–metric space and x_0 an arbitrary point in X. Then a mapping $T: X \to X$ is said to **x_0-Orbitally Continuous** if $\lim_{n\to\infty} d(T^n x_0, z, p) = 0$ for all p in X implies that $\lim_{n\to 0} d(TT^{n_i} x_0, Tz, p) = 0$ for all p in X.

Definition 2.9. Let (X, d) be a 2–metric space and $T: X \to X$. We say that $T \in D(a, b)$ if the inequality

$$d(Tx, Ty, p) \leq ad(x, y, p) + b\,[d(x, Tx, p) + d(y, Ty\, p)]$$

holds for all x, y, p in X, $0 \leq a, b < 1$.

3. RESULTS:

Now we present the main results

Theorem 3.1. Let (X, d) be a 2–metric space and $T: X \to X$. If $T \in D(a, b)$, $0 \leq a, b < 1$, $a + 2b < 1$. Then T is asymptotically regular at every point in X.

Proof. Let x_0 be an arbitrary point in X. Define $x_n = T^n x_0$. Then for all p in X, $n \geq 1$, we have $d(x_n, x_{n+1}, p) = d(Tx_{n-1}, Tx_n, p)$

$$\leq ad(x_{n-1}, x_n, p) + b\,[d(x_{n-1}, Tx_{n-1}, p) + d(x_n, Tx_n, p)]$$

$$= ad(x_{n-1}, x_n, p) + b\,[d(x_{n-1}, x_n, p) + d(x_n, x_{n+1}, p)]$$

so that

$$d(x_n, x_{n+1}, p) \leq \frac{a+b}{1-b} d(x_{n-1}, x_n, p)$$

Hence

$$d(x_n, x_{n+1}, p) \leq \left(\frac{a+b}{1-b}\right)^n d(x_0, x_1, p).$$

Since by hypothesis, $\frac{a+b}{1-b} < 1$, it follows that $d(x_n, x_{n+1}, p) = d(T^n x_0, T^{n+1} x_0, p) \to 0$ as $n \to \infty$. Since x_0 is arbitrary, T is asympcotically regular at every point in x. This completes the proof.

Theorem 3.2. Let X be complete 2-metric space and $T: X \to X$ be a mapping in D (a, b), $0 \leq a, b < 1$. Then a sequence $\{x_n\}$ in X is asymptotically T–regular if and only if it converges to a fixed point of T.

Proof: Suppose $\lim_{n\to\infty} x_n = z$ and $z = Tz$. Then for all p in X, we have

$$\begin{aligned} d(x_n, Tx_n, p) &\leq d(x_n, Tx_n, z) + d(x_n, z, p) + d(z, Tx_n, p) \\ &= d(x_n, Tx_n, z) + d(x_n, z, p) + d(Tz, Tx_n, p) \end{aligned}$$

Thus letting $n \to \infty$, we have $d(x_n, Tx_n, p) \to 0$ so that $\{x_n\}$ is asymptotically T–regular.

Conversely

$$\begin{aligned} d(Tx_n, Tx_m, p) &\leq ad(x_n, x_m, p) + b[d(x_n, Tx_n, p) + d(x_m, Tx_m, p)] \\ &\leq a[d(x_n, x_m, Tx_n) + d(x_n, Tx_n, p) + d(Tx_n, x_m, Tx_m) \\ &\quad + d(Tx_n, Tx_m, p) + d(Tx_m, x_m, p)] \\ &\quad + b[d(x_n, Tx_n, p) + d(x_m, Tx_m, p)] \end{aligned}$$

So that

$$\begin{aligned} (1-a)\, d(Tx_n, Tx_m, p) \leq\ & (a+b)\ [d(x_n, Tx_n, p) + d(x_m, Tx_m, p)] \\ & + b\ [d(x_n, x_m, Tx_m) + d(x_n, x_m, Tx_n)] \end{aligned}$$

Letting $m, n \to \infty$, we observe that $\{Tx_n\}$ is a Cauchy sequence. Since X is complete $\{Tx_2\}$ converges to, say, z in X. Since $\lim_{n\to\infty} d(x_n, Tx_n, p) \to 0$, $\{x_n\} \to z$ as $n \to \infty$.

We assert that $z = Tz$. For if $z \neq Tz$, then

$$\begin{aligned} d(z, Tz, p) &\leq d(z, Tz, Tx_n) + d(z, Tx_n, p) + d(Tx_n, Tz, p) \\ &\leq d(z, Tz, Tx_n) + d(z, Tx_n, p) \\ &+ ad(x_n, z, p) + b\,[d(x_n, Tx_n, p) + d(z, Tz, p)] \end{aligned}$$

Letting $n \to \infty$, we obtain

$$d(z, Tz, p) \leq bd(z, Tz, p)$$

a contradiction. Hence $z = Tz$. This completes the proof.

Theorem 3.3. Let X be a complete 2–metric space and $T:X \to X$ be a mapping in D (a, b), $a, b \geq 0$, $a + 2b < 1$. Then T has unique fixed point in X.

Proof: By Theorem 3.1, T is asymptotically regular at every point in X. Let x_0 be an arbitrary point in X. Define $x_n = T^n x_0$. Then the sequence $\{x_n\}$ is asymptotically T–regular (see Remark 2.1). Thus by Theorem 3.2. the sequence $\{x_n\}$ converges to a point z in X such that $z = Tz$.

To show that z is unique, suppose z and z_1 are two fixed points of T. Then for all p in X, we have

$$d(x, z_1, p) = d(Tz, Tz_1, p) \leq ad(z, z_1)$$

which is inadmissible. Hence $z = z_1$. This completes the proof.

Remark 3.1. Theorem 3.3. is a 2–metric analogue of Theorem 3 due to Nove G. [14]. It may be observed that for establishing the existence of fixed points in Theorem 3.3, we have used the asymptotic regularity of T at one point only. Keeping this in mind we obtain an extension of the above theorem in which the condition $a + 2b < 1$ may be relaxed. Thus we have the following theorem. Note that $a + 2b$ may exceed 1 in this case.

Theorem 3.4. Let X be a complete 2–metric space and $T:X \to X$ be a mapping in D (a, b), $a, b \geq 0$, $b < 1$. If T is asymptotically regular at some point in X, then T has a fixed point in X. Further if $a < 1$, then the fixed point is unique.

Proof: Let T be asymptotically regular at $x_0 \in X$. Define $x_n = T^n x_0$. Then the result immediately follows from Theorem 3.3. This completes the proof.

Finally, the following theorem is another extension of Theorem 3.3. in which by assuming T to be x_0–orbitally continuous, the condition of completeness of X has relaxed.

Theorem 3.5. Let X be a 2–metric space and $T:X \to X$ be a mapping in D (a, b), a, b ≥ 0, $a + 2b < 1$. If T is x_0–orbitally continuous at some point x_0 in X and the sequence $\{T^n x_0\}$ has a cluster point z in X, then z is a fixed point of T.

Proof: By Theorem 3.1, T is asymptotically regular at every point in X. Let $\{T^n x_0\} \supset \{T^n x_0\} \to z$. Then for all p in X, we have

$$d(z, Tz, p) \leq d(z, Tz, T^{n_i}x_0) + d(z, T^{n_i}x_0, p) + d(T^{n_i}x_0, Tz, p)$$

$$\leq d(z, Tz, T^{n_i}x_0) + d(z, T^{n_i}x_0, p) + d(T^{n_i}x_0, Tz, T^{n_i+1}x_0)$$

$$+ d(T^{n_i}x_0, T^{n_i+1}x_0, p) + d(T^{n_i+1}x_0, Tz, p)$$

Using asymptotic regularity of T and its x_0–orbital continuity, we find that $d(z, Tz, p) = 0$ as $n_i \to \infty$. Therefore $z = Tz$. This completes the proof.

It is worth noting that if T is asymptotically regular as well as x_0–orbitally continuous at some point $x_0 \in X$, then using Theorem 3.4 and Theorem 3.5, we have the following:

Corollary 3.1. Let X be a 2–metric space and $T:X \to X$ a mapping in D (a, b), a, b ≥ 0, $b < 1$. If T is asymptotically reqular and x_0–orbitally continuous at some point x_0 in X and the sequence $\{T^n x_0\}$ has a cluster point z in x, then z is the fixed point T. Moreover if $a < 1$, then the fixed point is unique.

The following is a direct consequence of Theorem 3.2.

Corollary 3.2. Let X be a 2–metric space and $T: X \to X$ be a mapping in D (a, b), $0 \leq a, b < 1$. If a sequence $\{x_n\}$ in X converges to a fixed z of T, then $\{x_n\}$ is asymptotically T–regular.

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