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SURFACES WHICH CONTAIN INCLINED CURVES AS GEODESICS

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ABSTRACT

Let M be a complete differentiable surface in E^3 whose one of the principal curvature is identically constant. If M has a geodesic γ on itself such that γ makes a fixed angle with one of the parameter curves of M. then M is either a plane, a sphere, or a circular cylinder.

1. INTRODUCTION

We mean by an inclined curve on a surface M a curve which makes a constant angle φ, $\varphi \neq 0$, $\varphi \neq \frac{\pi}{2}$, $\varphi < \pi$, with a constant direction on E^3, at each point of the curve. This definition gives us that the ratio of the curvatures of an inclined curve is constant at each point of the curve. We define this ratio as harmonic curvature of the curve.

Let us denote the first curvature by $\varkappa$ and the second curvature by τ (torsion) of a space curve, then we know that the harmonic curvature of the curve is $h = \frac{\varkappa}{\tau}$. If h is constant on the curve then $\varkappa = \varkappa(s)$, and $\tau = \tau(s)$, where s is the arc length parameter of the curve, but h = constant for an inclined curve. Then, it is clear that a circular helix (which has $\varkappa$ = constant and τ = constant) is a special inclined curve which lies on a circular cylinder [5], which is also a geodesic on the circular cylinder. Another example of the inclined curve is the ordinary helix on a helicoid in E^3. This curve is an intersection of a circular cylinder and the helicoid. This means that the ordinary helix on the helicoid is not different from the circular helix. But the ordinary helix is not a geodesic on the helicoid, and it is a geodesic on the circular cylinder. Ba-

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sed on this fact, an inclined curve on E^3 which is, at the same time, a geodesic on M.

M. Tamura [6] characterized surfaces in E^3 which contain helical geodesics. A helical geodesic on a surface M in E^3 is a curve which is a helix ($\varkappa$ = constant and τ = constant) as a curve in E^3 and a geodesic as a curve on M. On the other hand, we know that a circular helix must lie only on one kind of surface which is a circular cylinder [5].

Some surfaces, like circular cylinder, circular cone and sphere, contain inclined lines [3]. However the only inclined curve among these which is also a geodesic on the same surface is the circular helix (which lines on the circular cylinder):

In this paper, we shall characterize surface in E^3 which contain inclined geodesic. We find that, an inclined geodesic lie only on a circular cylinder.

2. PRELIMINARIES

Let M be a complete differentiable surface in Euclidean 3-space E^3 with standard metric g and χ (M) be the Lie algebra of all smooth tangent vector fields to M. Further, let D and $\bar{D}$ be the Levi-Civita connections of E^3 and M, with the metric induced by g, $g_{ij} = \delta_{ij}$, respectively. The second fundamental form II of M in E^3 is given by the Gauss equation:

$$\text{(2.1)} \qquad II(X, Y) = D_X Y - \bar{D}_X Y, \quad \forall X, Y \in \chi(M).$$

Let N be a unit normal vector field to M and then the shape operator S of M derived from N is a (1,1)-tensor field on M given by

$$g(S(X), Y) = g(II)(X. Y), N), \forall X, Y \in \chi(M)$$

and it is well known that

$$D_X N = S(X), \forall X \in \chi(M).$$

Let

$$\gamma : I \subset IR \to M \subset E^3$$
$$s \longrightarrow \gamma(s)$$

be an inclined curve parametrized by the arc length. Then. Frenet thrihedron fields be $\{V_1, V_2, V_3\}$ along γ and the curvature functions be

$$\varkappa : I \to R \quad \text{and} \quad \tau : I \to R$$
$$s \to \varkappa(s) \qquad\qquad s \to \tau(s)$$

such that

$$(2.2)\qquad \begin{bmatrix} D_{V_1}V_1 \\ D_{V_1}V_2 \\ D_{V_1}V_3 \end{bmatrix} = \begin{bmatrix} 0 & \varkappa & 0 \\ -\varkappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where V_1, V_2 and V_3 denote unit tangent, principal normal and binormal vector fields of γ.

The following fundamental theorem [4] will play an important role in this paper.

Theorem 1. If two families of geodesics intersect at a constant angle everywhere on M, then M is flat ($K \equiv 0$).

3. CHARACTERZIATION OF SURFACES WHICH CONTAIN INCLINED GEODESICS

In this part, we give two characterization theorems for inclined curves and a characterization theorem for surfaces which contain inclined geodesics.

Let M be a differentiable surface in E^3 and

$$\gamma : I \to M$$
$$s \to \gamma(s)$$

be a geodesic on M. Then $\bar{D}_{V_1} V_1 = 0$ and Eq. (2.1) implies that

$$(3.1)\qquad D_{V_1} V_1 = II\,(V_1, V_1) \in \chi\,(M)^{\perp}$$

where $\chi(M)^{\perp}$ denotes the set of all smooth normal fields to M. (2.2) and (3.1) give us that

$$(3.2)\qquad II\,(V_1, V_1) = \varkappa N \Rightarrow \varkappa = g\,(II\,(V_1, V_1), N).$$

This means that one has

$$V_2 = N,$$

and then (2.1) and (2.2) give us that

$$(3.3)\qquad II\,(V_1, V_3) = -\tau N \Rightarrow \tau = -g\,(II\,(V_1, V_3), N).$$

Let k_1 and k_2 be the principal curvatures on M and E_1 and E_2 be the corresponding principal vector fields on M. Denote θ the angle between V_1 and E_1 then we have

$$(3.4)\qquad \begin{cases} V_1 = \cos\theta\, E_1 + \sin\theta\, E_2, \\ V_3 = -\sin\theta\, E_1 + \cos\theta\, E_2. \end{cases}$$

Since we know that

$$II\ (X, Y) = g\ (S(X), Y)\ N.$$

we can write

$$II\ (E_1, E_1) = k_1\ N \text{ and } II\ (E_2, E_2) = k_2\ N.$$

On the other hand, the linear operator S and Eq. (3.4) give us

(3.5) $$II\ (V_1, V_1) = (k_1 \cos^2 \theta + k_2 \sin^2 \theta)\ N,\ g\ (E_i, E_j) = \delta_{ij},$$

and

(3.6) $$II\ (V_3, V_3) = (k_1 \sin^2 \theta + k_2 \cos^2 \theta)\ N.$$

Eq. (3.2) and (3.5) give us

(3.7) $$\varkappa = g\ (II\ (V_1, V_1), N) \text{ or } \varkappa = k_1 \cos^2 \theta + k_2 \sin^2 \theta,$$

and from Eq. (3.4) we have

(3.8) $$II\ (V_3, V_3) = (2H - \varkappa)\ N.$$

where $H = (k_1 + k_2)/2$ denotes the mean curvatuıe of M. In a similar way from Eq. (3.3) we obtain

(3.9) $$\tau = (k_1 - k_2) \sin \theta \cos \theta.$$

From Eq. (3.7) and (3.9) we can write

(3.10) $$h = \frac{\varkappa}{\tau} = \frac{k_1 \cos^2 \theta + k_2 \sin^2 \theta}{(k_1 - k_2) \sin \theta \cos \theta}$$

$$= \frac{H + \sqrt{H^2 - K} \cos 2\theta}{\sqrt{H^2 - K} \sin 2\theta},$$

where $k_1 \neq k_2$, that is M has to umbilic points.

Theorem 2. Let M be a differentiable flat surface in E^3. Thus we have that, the goedesic curve

$$\gamma : I \to M$$

makes a fixed angle with one of the parameter curves $\Leftrightarrow$ γ is an inclined geodesic on M.

Proof: In this case, $K = 0$ and from Eq. (3.10) we have

$$h = \frac{\varkappa}{\tau} = \operatorname{cotg} \theta,$$

and since θ = constant then $h = \frac{\varkappa}{\tau}$ = constant and cive versa].

Theorem 3. Let M be a minimal surface in E^3. Then. we have that: the geodesic curve

$$\gamma : I \rightarrow M$$

makes a fixed angle with one of the parameter curves $\Leftrightarrow$ γ is an inclined curve on M.

Proof: In this case, H = 0 and from Eq. (3.10) we have

$$h = \frac{\varkappa}{\tau} = \text{cotg } 2\theta$$

and since θ = constant then $\frac{\varkappa}{\tau}$ = constant and vice versa].

Now, we are ready to give the characterization theorem for the surfaces which contain the inclined geodesics.

Theorem 4. Let M be a complete differentiable surface in E^3 whose one of the principal curvatures is identically constant. If M has a geodesic γ on itself such that γ makes a fixed angle with one of the parameter curves of M, then M is either a plane, a sphere, or a circular cylinder.

Proof: We can choose $\{E_1, E_2\}$ as canonical base for χ (M) such that

$$\bar{D}_{E_1}E_1 = \lambda\, E_2,\ \bar{D}_{E_2}E_2 = \mu\, E_1$$

Then we have

$$\bar{D}_{E_1}E_2 = \lambda\, E_1, \text{ and } \bar{D}_{E_2}E_1 = -\mu\, E_2$$

Since γ is a geodesic on M, we can write

$$\bar{D}_{V_1}\, V_1 = 0,\ V_1 = \gamma.$$

or

(3.11) $\lambda \cos \theta + \mu \sin \theta = 0$

On the other hand, using the Codazzi equation,

(3.12) $$\begin{cases} \bar{D}_{E2}(k_1) = \lambda(k_1 - k_2), \\ \bar{D}_{E1}(k_2) = -\mu\,(k_1 \Rightarrow k_2). \end{cases}$$

$$\bar{D}_{E2}(k_1) = \lambda\,(k_1 - k_2) \text{ and } k_1 = \text{constant}$$

give us

$$(3.13) \qquad \lambda\,(k_1 - k_2) = 0.$$

Then, the following two cases occur.

a) Let $k_1 \neq k_2$, then M has no umbilic points and (3.13) given us

$$\lambda = 0,$$

and from (3.11) we obtain

$$\mu = 0,$$

so we have

$$\bar{D}_{E_1}E_1 = 0 \text{ and } \bar{D}_{E_2}E_2 = 0$$

This means that the parameter curves γ_1 and γ_2 are geodesics on M. These two families of curves intersect at a constant $\left(\frac{\pi}{2}\right)$ angle, since $g\,(E_1, E_2) = 0$. Hence, by Theorem 1, M is flat. Therefore M is a circular cylinder (See Berger and Gostiaux [1, p. 425] or O'Neill [2, p. 263]). In this case, from Theorem 2 the geodesic γ is an inclined geodesic on circular cylinder.

b) Let $k_1 = k_2$, then M is umbilic and $k_1 = k_2 =$ constant, so M is either a plane or a sphere.

This completes the proof of Theorem 4.

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