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ON THE PAIR OF GENERALIZED RULED SURFACES UNDER THE SYMMETRIC HOMOTHETIC MOTION OF ORDER k IN THE EUCLIDEAN SPACE E^n

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ABSTRACT

In this paper, after presenting a summary of known results of the homothetic motions and the generalized ruled surfaces in the Euclidean space E^n, A symmetric homothetic motion of order k was defined and it was shown that a symmetric homothetic motion of order k of E^n is a reflection with respect to the (n-k-m)- dimensional subspace of E^n.

Moreover, the parameters of distribution, the pitches, and the apex angles of the moving and fixed ruled surfaces which are correspond to each other under the symmetric homothetic motion of order k were given.

1. HOMOTHETIC MOTIONS OF E^n

A homothetic motion of E^n is described in matrix notation in [4] by

$$x = S\bar{x} + C,\ S = hA,\ AA^t = I_n,$$

where A^t is transpozed of the proper orthogonal matrix A, and h is a homothetic scale and are functions of differentiability class C^r ($r \geq 3$) on

$$A{:}J \to O(n),\ C{:}\ J \to IR^n,\ h{:}J \to IR \tag{1.2}$$

real interval J. $\bar{x}$ and x correspond to the position vectors of the same point with respect to the rectangular coordinate systems of the moving space $\bar{E}$ and the fixed space E, respectively. At the initial time $t = t_0$, we consider the coordinate system of $\bar{E}$ and E are coincident.

To avoid the case of affine transformation we assume that $h = h(t) \neq$ constant and to avoid the case of pure translation and pure rotation we assume that

$$\dot{h}A + h\dot{A} \neq 0,\ C \neq 0,$$

where (.) indicates d / dt.

Let $\bar{x}$ be fixed in $\bar{E}$, then the equation (1.1) defines, by (1.2) a parametrized curve in E which we call the trajectory curve or path of

$\bar{x}$ under the motion. We get the (trajectory) velocity vector $\dot{x}$ in the path–point x from (1.1) by differentiation for $\bar{x} = 0$ in the form

$$\dot{x} = B\,(x-\dot{C}) + C,\ B = S\dot{S}^{-1}, \tag{1.4}$$

where S^{-1} is the inverse of S. Since $\dot{S}$ is a regular matrix, (See [3]), $|B(t)| \neq 0$ in $t \in J$. Therefore we get exactly one solution p(t) of the equation

$$B\,(p-C) + \dot{C} \neq 0 \tag{1.5}$$

p(t) is the center of the instantaneous rotation of the motion in $t \in J$ and called **pole of the motion.** At a pole p the velocity vector vanishes by the equation (1.4). Since $|B|$ does not vanish on J, by considering the regularity condition of the motion we get a differentiable curve $p: J \rightarrow E$ of poles in the fixed space E, called the **fixed pole curve.** By (1.1) there is uniquely determined the **moving pole curve** $\bar{p}: J \rightarrow \bar{E}$ from the fixed pole curve point to point on J: $p(t) = S(t)\bar{p}(t) + C(t)$.

II. GENERALIZED RULED SURFACES IN E^n

For the purpose of this paper we first summarize the basic properties of the generalized ruled surface from the paper [2] and [3]: In any k-dimensional generator $E_k(t)$ of a $(k+1)$- dimensional generalized ruled surface (axoid, in [2] "(k + 1)–Regelfläche) $\varnothing \subset E^n$ there exists a maximal linear subspace $K_{k-m}(t) \subset E_k(t)$ of dimension k–m with the property that in every point of $K_{k-m}(t)$ no tangent space of $\varnothing$ is determined ($K_{k-m}(t)$ contains all singularities of $\varnothing$ in $E_k(t)$) or there exists a maximal linear subspace $Z_{k-m}(t) \subset E_k(t)$ of dimension k–m with the property that in every point of Z_{k-m} the tangent space of $\varnothing$ is orthogonal to the asymptotic bundle of the tangent spaces in the points of infinity of $E_k(t)$ (all points of $Z_{k-m}(t)$ have the same tangent space of $\varnothing$). We call $K_{k-m}(t)$ the **edge space** in $E_k(t) \subset \varnothing$ and $Z_{k-m}(t)$ the **central space** in $E_k(t) \subset \varnothing$. A point of $Z_{k-m}(t)$ is called a **central point.** If $\varnothing$ possesses generators all of the same type the edge spaces resp. the central spaces generate a generalized ruled surface contained in $\varnothing$ which call the **edge ruled surface** resp. the **central ruled surface.** For $m = k$ the edge ruled surface degenerates in the edge of $\varnothing$, the central ruled surface in te **line of striction.** So the ruled surface with edge ruled generalize the tangent surfaces of E^3, the ruled surface with central ruled surface generalize the ruled surfaces with line of striction of E^3.

For the analytical represention of a $(k+1)-$ dimensional ruled surface $\varnothing$ we choose a leading curve α in the edge resp. central ruled surface $\Omega \subset \varnothing$ transversal to the generators. In [2] it is sohwn that there exists a distinguished moving orthonormal frame (ONF) of $\varnothing$ $\{e_1, \ldots, e_k\}$ with the properties:

(i) $\{e_1, \ldots, e_k\}$ is an ONF of the $E_k(t) \subset \varnothing$

(ii) $\{e_{m+1}, \ldots, e_k\}$ is an ONF of the $K_{k-m}(t)$ resp. $Z_{k-m}(t) \subset E_k(t)$

(iii) $\dot{e}_i = \sum_{j=1}^{m} \alpha_{ij} e_j + \chi_i a_{k+i}, \; 1 \leq i \leq m,$

$$\dot{e}_{m+P} = \sum_{l=1}^{m} \alpha_{(m+p)} e_1, \text{ with } K_i > 0, \; \alpha_{ij} = -\alpha_{ji}, \tag{2.1}$$

$\alpha_{(m+P)\;(m+\chi)} = 0, \; 1 \leq p, \; K_i \leq k-m$

(iv) $\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}\}$ is an ONF.

A moving ONF of $\varnothing$ with the properties (i) – (iv) is called **a principal frame** of $\varnothing$. If $K_1 > \ldots > K_k > 0$, the principal frame of $\varnothing$ is determined up to the signs. By a given principal frame vectors $a_{k+1}, \ldots, a_{k+m}$ are well defined.

A leading curve α of $(k+1)-$ dimensinoal ruled rurface $\varnothing$ is a leading curve of the edge resp. central surface $\Omega \subset \varnothing$ too iff its tangent vector has the form

$$\dot{\alpha}(t) = \sum_{i=1}^{k} \zeta_i e_i + \eta_{m+1} a_{k+m+1}, \tag{2.2}$$

where $\eta_{m+1} \neq 0$, a_{k+m+1} is a unit vector well defined up to the sign with the property that $\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1}\}$ is an ONF of the tangential bundle of $\varnothing$. One shows: $\eta_{m+1}(t) = 0$, in $t \in J$ iff the genarator $E_k(t) \subset \varnothing$ contains the edge space $K_{k-m}(t)$. If $\eta_{m+1}(t) \neq 0$, we call the m–magnitudes

$$P_i = \frac{\eta_{m+1}}{K_i}, \; 1 \leq i \leq m \tag{2.3}$$

the **principal parameters of distribution.** These parameters are direct generalizations of the parameter of distribution of the ruled surface in E^3 (See [2]). A $(k+1)-$ dimensional ruled surface with central ruled surface and no principal parameter of distribution $(m = 0)$ is a $(k+1)-$ dimensional cylinder.

Moreover the paramcter of distribution of a generalized ruled surface $\varnothing$ given in [3] by

$$P = \sqrt[m]{|P_1 P_2 \ldots P_m|} \tag{2.4}$$

and the **total parameter of distribution** of $\varnothing$ can be defined in [5] by

$$D = \prod_{i=1}^{m} P_i \quad . \tag{2.5}$$

Suppose that $\varnothing_i$, $1 \leq i \leq k$, are 2–dimensional closed principal ruled surfaces such that the generators of $\varnothing_i$ have the direction of the unit vectors $e_i(t)$, $1 \leq i \leq k$. Then, in the case $m = k$, there exist **k-pitches** given by

$$L_i = - \int_0^p \zeta_i(t)dt, \ 1 \leq i \leq k, \tag{2.6}$$

where $p \in IN$ denotes a period of the motion.

Let $\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1}\}$ be ONF of the tangential bundle $T(t)$ of $\varnothing$. If we complete this ONF by an arbitrary $\{a_{k+m+2}, \ldots, a_n\}$ of the orthogonal complement, called a complementary ONF, from the orthogonality conditions, then we obtain by differentiation, [3]:

$$\dot{a}_{k+i} = -K_i e_i + \sum_{j=1}^{m} \tau_{ij} a_{k+j} + w_i a_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \gamma_{i\lambda}\, a_{k+m+\lambda}, 1 \leq i \leq m. \tag{2.7}$$

Suppose that $\dim T(t) = k + m + 1$. If $\varnothing$ is a closed ruled surface, the **m-apex angles** of $\varnothing$ can be defined by

$$\lambda_i = \int_0^p w_i(t)dt, \ 1 \leq i \leq m, \tag{2.8}$$

and also the **apex angle** of $\varnothing$ is defined, in [5], by

$$\lambda = \sqrt[m]{|\lambda_1 \ldots \lambda_m|} \ . \tag{2.9}$$

III. THE PAIR OF GENERALIZED RULED SURFACES UNDER THE SYMMETRIC HOMOTHETIC MOTION IN E^n

Let $\bar{\alpha} \subset \bar{E}$ and $\alpha \subset E$ be moving and fixed pole curves, resp. Suppose that $\{\bar{e}_1(t), \ldots, \bar{e}_k(t)\}$ is an orthonormal vector field system at $\alpha(t)$ and let $\bar{E}_k(t) = Sp\{\bar{e}_1(t), \ldots, \bar{e}_k(t)\}$. Then $\bar{E}_k(t)$ generates a $(k+1)$– dimensional ruled surface with the leading curve α in the moving space $\bar{E}$ which called the **moving ruled surface** $\bar{\varnothing}$, $\varnothing$ has the following parameter representation

$$\bar{\varnothing}(t, \bar{u}_1, \ldots, \bar{u}_k) = \bar{\alpha}(t) + \sum_{i=1}^{k} \bar{u}_i \bar{e}(t), \; \bar{u}_i \in \mathbb{R}, \; t \in J. \tag{3.1}$$

Let $\{\varepsilon_1(t), \ldots, \varepsilon_k(t)\}$ be an orthogonal vector field system satisfaying the following equations at the point $\alpha(t)$in the fixed space E:

$$S(\bar{e}_i) = -\varepsilon_i \tag{3.2}$$

and

$$(\dot{S}S^{-1})\, \varepsilon_i = 0, \; 1 \leq i \leq k. \tag{3.3}$$

If we denote $E_k(t) = Sp\{\varepsilon_1(t), \ldots, \varepsilon_k(t)\}$, $E_k(t)$ generates a $(k+1)$– dimensional ruled surface with leading curve α given by (1.1) in the fixed space E which is called the **fixed ruled surface** and denoted by $\varnothing$.

Definition 3.1. If a homothetic motion given by (1.1) satisfies the equations (3.2) and (3.3), then this homothetic motion is called a **symmetric homothetic motion** of order k.

We will assume throughout this paper that the homothetic scale h is a possitive real number. Let $\bar{\varnothing}$ and $\varnothing$ be the $(k+1)$– dimensional moving and fixed generalized ruled surfaces with the leading curves $\bar{\alpha}$ and α resp. If α and $\bar{\alpha}$ are pole curves, then we have following equations, [1].

$$\dot{\alpha} = S\dot{\bar{\alpha}} \tag{3.4}$$

and

$$ds = h d\bar{s} \tag{3.5}$$

where $\bar{s}$ and s are the are lengthes of $\bar{\alpha}$ and α resp. From (3.4) and (3.5) we get the following theorem.

Theorem 3.2. Under the symmetric homothetic motion of order k, the moving and the fixed generalized ruled surfaces touch each other along every common pair $\bar{\alpha}$ and α for all $t \in J$ by rolling and sliding upon each other.

Let $\bar{E}_k(t)$ and $E_k(t)$ be the generator spaces of the generalized ruled surfaces $\bar{\varnothing}$ and $\varnothing$ resp. Then we have,

$$\bar{E}_k(t) = Sp\{\bar{e}_1, \ldots, \bar{e}_k\} \tag{3.6}$$

and

$$E_k(t) = S_p \{\varepsilon_1, \ldots, \varepsilon_k\}. \tag{3.7}$$

We obtain from (3.2) by differentiation:

$$\dot{S}\bar{e}_i + S\dot{\bar{e}}_i = -\dot{\varepsilon}_i,\ 1 \leq i \leq k. \tag{3.8}$$

From this equation and (3.3) we get

$$S\dot{\bar{e}}_i = -\dot{\varepsilon}_i,\ 1 \leq i \leq k. \tag{3.9}$$

Then we immediately read off from the equation (3.2) and (3.9):

Theorem 3.3. Under the symmetric homothetic motion of order k, the generator spaces $\overline{E}_k(t)$ and $E_k(t)$ correspond to each other by the equations (3.2) and (3.9).

Let $\bar{A}(t)$ and $A(t)$ be the astmptotic bundles, with respect to the $\overline{E}_k(t)$ and $E_k(t)$, of the generalized ruled surfaces $\bar{\varnothing}$ and $\varnothing$ resp. Then $\bar{A}(t)$ and $A(t)$ can be given resp. by

$$\bar{A}(t) = Sp\ \{\bar{e}_1, \ldots, \bar{e}_k, \dot{\bar{e}}_1, \ldots, \dot{\bar{e}}_k\} \tag{3.10}$$

and

$$A(t) = Sp\{\varepsilon_1, \ldots, \varepsilon_k, \dot{\varepsilon}_1, \ldots, \dot{\varepsilon}_k\}. \tag{3.11}$$

Suppose that $\dim A(t)$ ($= \dim \bar{A}(t)$) $= k + m$, $0 \leq m \leq k$, then m vectors of $\dot{\bar{e}}_1, \dot{\bar{e}}_2, \ldots, \dot{\bar{e}}_k$ are linearly independent. Let the linearly independent vectors are renumbered as $\dot{\bar{e}}_{k+1}, \dot{\bar{e}}_{k+2}, \ldots, \dot{\bar{e}}_{k+m}$. Then the set

$$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{e}_{k+1}, \ldots, \bar{e}_{k+m}\}. \tag{3.12}$$

is a basis of the asymptotic bundle $\bar{A}$ (t). Similarly, we get a basis for the asymptotic bundle $A(t)$ as follows.

$$\{\varepsilon_1, \ldots, \varepsilon_k, \dot{\varepsilon}_{k+1}, \ldots, \dot{\varepsilon}_{k+m}\}. \tag{3.13}$$

By the Gram–Schmidt process from (3.12) and (3.13) we get the following orthogonal bases for $\bar{A}(t)$ and $A(t)$ resp.

$$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{y}_{k+1}, \ldots, \bar{y}_{k+m}\} \tag{3.14}$$

and

$$\{\varepsilon_1, \ldots, \varepsilon_k, y_{k+1}, \ldots, y_{k+m}\}. \tag{3.15}$$

Under the symmetric homothetic motion of order k, the above orthogonal systems correspond to each other by the equation

$$S\bar{y}_{k+j} = -y_{k+j},\ 1 \leq j \leq m. \tag{3.16}$$

If we set

$$\bar{a}_{k+j} = \frac{\bar{y}_{k+j}}{\|\bar{y}_{k+j}\|},\ e_i = \frac{\varepsilon_i}{h},\ a_{k+j} = \frac{y_{k+j}}{\|y_{k+j}\|},\ 1 \leq i \leq k,\ 1 \leq j \leq m,$$

then we get the following ONFs for $\bar{A}(t)$ and $A(t)$ resp. (3.17)

$$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{a}_{k+1}, \ldots, \bar{a}_{k+m}\} \tag{3.18}$$

and

$$\{e_1, \ldots, e_k, a_{+1}, \ldots, a_{k+m}\}. \tag{3.19}$$

We have the following theorem.

Theorem 3.4. Under the symmetric homothetic motion of order k, the asymptotic bundles $\bar{A}(t)$ and $A(t)$ correspond to each other by the following equations.

$$S\bar{e}_i = -he_i \tag{3.20}$$

$$S\bar{a}_{k+j} = -ha_{k+j},\ 1 \leq i \leq k,\ 1 \leq j \leq m. \tag{3.21}$$

Let $\bar{T}(t)$ and $T(t)$ be the tangential bundles of $\bar{\varnothing}$ and $\varnothing$ resp. If

$$\dim \bar{T}(t)\ (= \dim T(t)) = k + m + 1 \tag{3.22}$$

then

$$\{\bar{e}_1, \ldots, \dot{\bar{e}}_k, \bar{e}_{k+1}, \ldots, \dot{\bar{e}}_{k+m}, \dot{\bar{\alpha}}\} \tag{3.23}$$

is a basis for $\bar{T}(t)$ and

$$\{\varepsilon_1, \ldots, \dot{\varepsilon}_k, \varepsilon_{k+1}, \ldots, \dot{\varepsilon}_{k+m}, \dot{\alpha}\} \tag{3.24}$$

is a basis for $T(t)$.

Using the Gram–Schmidt process we get the following ONFs for $\bar{T}(t)$ and $T(t)$ resp.

$$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{a}_{k+1}, \ldots, \bar{a}_{k+m}, \bar{a}_{k+m+1}\} \tag{3.25}$$

and

$$\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1}\} . \tag{3.26}$$

From (3.25), (3.26) and Theorem 3.4 we have:

Theorem 3.5. Under the symmetric homothetic motion of order k, the tangential bundles $\overline{T}(t)$ and $T(t)$ correspond to each other by the following equations.

$$\begin{aligned} S\bar{e}_i &= -he_i \\ S\bar{a}_{k+j} &= -ha_{k+j} \\ S\bar{a}_{k+m+1} &= ha_{k+m+1}, \ 1 \ i\leq \ \leq k, \ 1 \leq j \leq m. \end{aligned} \tag{3.27}$$

Let $\dim T(t) = k + m + 1$. Then we can complete the ONF

$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{a}_{k+1}, \ldots, \bar{a}_{k+m+1}\}$ of $\overline{T}(t)$ to the ONF .

$$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{a}_{k+1}, \ldots, \bar{a}_{k+m+1}, \bar{a}_{k+m+2}, \ldots, \bar{a}_n\} \tag{3.28}$$

of E^n. The orthogonal complement

$$\{\bar{a}_{k+m+2}, \ldots, \bar{a}_n\} \tag{3.29}$$

is called **a complementary** ONF of $\bar{\varnothing}$.

If we set

$$S\bar{a}_{k+m+\lambda} = y_{k+m+\lambda}, \ 2 \leq \lambda \leq n-k-m \tag{3.30}$$

then we get an orthogonal complement $\{y_{k+m+2}, \ldots, y_n\}$ of $\varnothing$ under the symmetric homothetic motion of order k. If we set.

$$a_{k+m+1} = \frac{y_{k+m+1}}{\|y_{k+m+1}\|}, \ 2 \geq \lambda \geq n-k-m, \tag{3.31}$$

we get the following complementary ONF of $\varnothing$

$$\{a_{k+m+2}, \ldots, a_n\} . \tag{3.23}$$

Theorem 3.6. Under the symmetric homothetic motion of order k, the complementary ONFs (3.29) and (3.32) satisfy the following equation.

$$S\bar{a}_{k+m+1} = ha_{k+m+1}, \ 2 \leq \lambda \leq n-k-m. \tag{3.33}$$

From the Theorem 3.5 and Theorem 3.6 we can get the following corollaries.

Corollary 3.7. Let $\overline{T}(t)$ and $T(t)$ be two tangential bundles which are correspond to each other under the symmetric homothetic motion and Let

$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{a}_{k+1}, \ldots, \bar{a}_{k+m}, \bar{a}_{k+m+1}, \ldots, a_n\}$ and

$\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1}, \ldots, a_n\}$ be two ONFs of E^n with respect to the $\overline{T}(t)$ and $T(t)$ resp. Then we have the following equations.

$$\begin{aligned} S\bar{e}_i &= -he_i \quad , 1 \leq i \leq k \\ S\bar{a}_{k+j} &= -ha_{k+j} \quad , 1 \leq j \leq m \\ S\bar{a}_{k+m+1} &= ha_{k+m+1} \quad , 1 \leq r \leq n-k-m. \end{aligned} \tag{3.34}$$

Corollary 3.8. A symmetric homothetic motion of order k of E^n is a reflection with respect to the subspace Sp $\{a_{k+m+1}, \ldots, a_n\}$ of dimension n–k–m.

4. THE INTEGRAL INVARIANTS OF THE PAIR OF GENERALIZED RULED SURFACES WHICH CORRESPOND TO EACH OTHER UNDER THE SYMMETRIC HOMOTHETIC MOTION

Theorem 4.1. Let $\overline{\varnothing}$ and $\varnothing$ be the $(k+1)$– dimensional moving and fixed generalized ruled surfaces which correspond to each other under the symmetric homothetic motion with the leading curves $\bar{\alpha}$ and α resp. If $\{\bar{e}_1, \ldots, \bar{e}_k\}$ and $\{e_1, \ldots, e_k\}$ are the principal ONFs of $\overline{\varnothing}$ and $\varnothing$ resp., then we have the following results:

$$\begin{aligned} \bar{\alpha}_{ii} &= (\dot{h}/h) + \alpha_{ii} \quad , i = j,\ 1 \leq i \leq m, \\ \bar{\alpha}_{ij} &= \alpha_{ij} \quad , i \neq j,\ 1 \leq j \leq k \\ \bar{K}_i &= K_i \end{aligned} \tag{4.1}$$

Proof: Since $\{\bar{e}_1, \ldots, \bar{e}_k\}$ and $\{e_1, \ldots, e_k\}$ are the principal ONFs of $\overline{\varnothing}$ and $\varnothing$ resp., we have the equations (2.1). Therefore we have

$$\dot{\bar{e}}_i = \sum_{j=1}^{k} \alpha_{ij}\bar{e}_j + \bar{K}_i\bar{a}_{k+i} \tag{4.2}$$

and

$$\dot{e}_i = \sum_{j=1}^{k} \alpha_{ij}e_j + K_i a_{k+i},\ 1 \leq i \leq m. \tag{4.3}$$

From (3.17) we have

$$\varepsilon_j = he_j,\ 1 \leq j \leq k. \tag{4.4}$$

From (4.4.) by differentiation we observe that

$$\dot{\varepsilon}_i = \dot{h}e_i + h\dot{e}_i,\ 1 \leq i \leq m.$$

Using (4.3) and (4.4) we get

$$\dot{\varepsilon}_i = (\dot{h}h^{-})^1\ \varepsilon_i + h\left(\sum_{j=1}^{k} \alpha_{ij}e_j + K_i a_{k+i}\right)$$

$$\varepsilon_i = (\dot{h}h^{-1})\ \varepsilon_i + \sum_{j=1}^{k} \alpha_{ij}\ (he_j) + K_i\ (ha_{k+i})$$

$$\dot{\varepsilon}_i = (\dot{h}h^{-1} + \alpha_{ii})\,\varepsilon_i + \sum_{ji=1}^{k} \alpha_{ij}\varepsilon_j + (h_{Ki})\ a_{k+i},\ 1 \leq i \leq m. \tag{4.5}$$

Moreover, if we set the equation (4.2) in the equation (3.9), we get

$$\dot{\varepsilon}_i = -S\left(\sum_{j=1}^{k} \bar{\alpha}_{ij}\bar{e}_j + K\bar{a}_{k+i}\right),\ 1 \leq i \leq m. \tag{4.6}$$

Since S is a linear transformation, we get

$$\dot{\varepsilon}_i = \sum_{j=1}^{k} \bar{\alpha}_{ij}\ (-S\bar{e}_j) + \bar{K}_i\ (-S\bar{a}_{k+i})\ .$$

From this last equation, (3.2) and (3.21), we find

$$\dot{\varepsilon}_i = \sum_{j=1}^{k} \bar{\alpha}_{ij}e_j + (h\bar{K}_i)\ a_{k+i},\ 1 \leq i \leq m. \tag{4.7}$$

If we consider the equations (4.5) and (4.7), then the theorem is proved.

Theorem 4.2. Let $\bar{\varnothing}$ and $\varnothing$ be the moving and fixed ruled surfaces with the leading curves $\bar{\alpha}$ and α resp. under the symmetric homothetic motion of order k. Then we have following results.

$$\zeta_j = -h\zeta_j,\ 1 \leq j \leq k, \tag{4.8}$$

$$\eta_{m+1} = h\bar{\eta}_{m+1},$$

where

$$\dot{\bar{\alpha}} = \sum_{j=1}^{k} \bar{\zeta}_j \bar{e} + \bar{\eta}_{m+1} \bar{a}_{k+m+1} \tag{4.9}$$

and

$$\dot{\alpha} = \sum_{j=1}^{k} \zeta_j e_j + \eta_{m+1} a_{k+m+1}. \tag{4.10}$$

Proof: From (3.4) we get

$$S\dot{\bar{\alpha}} = S \left(\sum_{j=1}^{k} \bar{\zeta}_j \bar{e}_j + \bar{\eta}_{m+1} \bar{a}_{k+m+1} \right).$$

Since S is a linear transformation,we have

$$S\dot{\bar{\alpha}} = \sum_{j=1}^{k} \bar{\zeta}_j (S\bar{e}_j) + \bar{\eta}_{m+1} (S\bar{a}_{k+m+1}). \tag{4.11}$$

From this last equation, (3.4) and (3.27) we find

$$\dot{\alpha} = \sum_{j=1}^{k} (-h\bar{\zeta}_j)\, e_j + (h\bar{\eta}_{m+1})\, a_{k+m+1}. \tag{4.12}$$

If we consider the equations (4.10) and (4.12), then the theorem is proved.

From the Theorem 4.1, Theorem 4.2 and the equations (2.3), (2.4), (2.5), (2.6), (2,7), (2.8) we get the following corolaries.

Corollary 4.3.

$$P_i = h\bar{P}_i,\ 1 \leq i \leq m,$$
$$P = h\bar{P}, \tag{4.13}$$
$$D = h^{2m}\bar{D}.$$

Corollary 4.4. Let $\bar{L}_i$ and L_i be i-pitches of $\bar{\varnothing}$ and $\varnothing$ resp. under the closed symm etric homothetic motion of orderk, Then we have

$$dL_i = -hd\bar{L}_i,\ 1 \leq i \leq m = k. \tag{4.14}$$

Theorem 4.5. Let $\{\bar{e}_1, \ldots, \bar{e}_k, \bar{a}_{k+1}, \ldots, \bar{a}_{k+m}\}$ and $\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}\}$ be ONFs of asymptotic bundles $\bar{A}(t)$ and $A(t)$ resp. which correspond to each other under the symmetric homothetic motion of order k. Then we have the following results.

$$\overline{w}_i = -w_i \tag{4.15}$$

$$\overline{\gamma}_{i\lambda} = -\gamma_{i\lambda},\ 1 \leq i \leq m,\ 2 \leq \lambda \leq n-k-m, \tag{4.16}$$

Proof: From the equation (2.7) and since S is a linear transformation, we observe that

$$S(\dot{\bar{a}}_{k+i}) = -K_i S(\bar{e}_i) + \sum_{j=1}^{m} \overline{\tau}_{ij}\, S(\bar{a}_{k+j}) + \overline{w}_i S(\bar{a}_{k+m+1}) + \sum_{\lambda=1}^{n-k-m} \overline{\gamma}_{i\lambda} S(\bar{a}_{k+m+\lambda}).$$

If we set (3.27) and (3.33) in this last equation we get

$$S(\dot{\bar{a}}_{k+i}) = (h\overline{K}_i)\, e_i + \sum_{j=1}^{m} (-h\overline{\tau}_{ij})\, a_{k+j} + (h\overline{w}_i)\, a_{k+m+1} + \sum_{\lambda=1}^{n-k-m} (h\overline{\gamma}_{i\lambda})\, a_{k+m+\lambda}. \tag{4.18}$$

Moreover, since

$$S\bar{a}_{k+i} = -h a_{k+i},$$

by differentiation, we observe that

$$S\dot{\bar{a}}_{k+i} = -h\dot{a}_{k+i} + (-\dot{h} + h\dot{S}S^{-1})\, a_{k+i},\ 1 \leq i \leq m. \tag{4.19}$$

On the other hand, from the (2.7) we can write

$$\dot{a}_{k+i} = -K_i e_i + \sum_{j=1}^{m} \tau_{ij} a_{k+j} + w_i a_{k+m+1} + \sum_{\lambda=1}^{n-k-m} \gamma_{i\lambda} a_{k+m+\lambda}. \tag{4.20}$$

If we set (4.20) in (4.19), we get

$$S\dot{\bar{a}}_{k+i} = (hK_i)\, e_i + \sum_{j=1}^{m} [h(\dot{S}S^{-1} - \tau_{ij}) - \dot{h}]\, a_{k+j} + (-hw_i)\, a_{k+m+1} + \sum_{\lambda=1}^{n-k-m} (-h\gamma_{i\lambda})\, a_{k+m+\lambda}.$$

From (4.18) and (4.21) we find

$$\bar{w}_i = -w_i$$

$$\bar{\gamma}_{i\lambda} = -\gamma_{i\lambda},\ 1 \leq i \leq m,\ 2 \leq \lambda \leq n-k-m.$$

Now, we can give the following corollary.

Corollary 4.6. Let $\bar{\lambda}_i$ and λ_i i-apex angles of $\bar{\varnothing}$ and $\varnothing$ resp. under the closed symmetric homothetic motion of order k. Then we have the following results.

$$\lambda_i = -\bar{\lambda}_i,\ 1 \leq i \leq m \tag{4.22}$$

and

$$\lambda = \bar{\lambda}, \tag{4.23}$$

where $\bar{\lambda}$ and λ the apex of $\bar{\varnothing}$ and $\varnothing$ resp.

Proof: From the definition of i-apex angle of $\varnothing$ and $\varnothing$, we have

$$\bar{\lambda}_i = \int_0^p \bar{w}_i(t)\, dt$$

and

$$\lambda_i = \int_0^p w_i(t)\, dt,\ 1 \leq i \leq m.$$

Using (4.15) we get

$$\bar{\lambda}_i = -\int_0^p w_i(t)\, dt = -\lambda_i.$$

Moreover, from the definition of apex angle of a closed generalized ruled surface we have

$$\lambda = \sqrt[m]{|\lambda_1 \lambda_2 \ldots \lambda_m|}\ .$$

If we set $\lambda_i = -\bar{\lambda}_i$ in this last equation we find

$$\lambda = \sqrt[m]{|\lambda_1 \lambda_2 \ldots \lambda_m|} = \bar{\lambda}.$$

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