Commun. Fac. Sci. Univ. Ank. Series A
V. 41. pp. 111-117 (1992)

ON THE DEGREE OF APPROXIMATION OF A PERIODIC FUNCTION F BY ALMOST RIESZ - MEANS OF ITS CONJUGATE SERIES

By

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(Received Dec. 28, 1990; Accepted July 9, 1992)

ABSTRACT

The present paper is concerned with the degree of approximation of certain functions belonging to the class Lip ($\rho(t)$, p) by almost Riesz means.

1. Let f be a 2π–periodic function integrable L^p $(p > 1)$ and let

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \operatorname{Cos} nx + b_n \operatorname{Sin} nx) \tag{1.1}$$

be its Fourier series.

The conjugate series of the Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty} (b_n \operatorname{Cos} nx - a_n \operatorname{Sin} nx) \tag{1.2}$$

A function $f \in \operatorname{Lip}(\rho(t), p)$ $(p > 1)$ if

$$\left\{ \int_0^{2\pi} | f(x+t) - f(x) |^p \, dt \right\}^{1/p} = O(\rho(t)) \tag{1.3}$$

when $\rho(t)$ is a positive increasing function.

1. Definition (Lorentz [2]). A sequence $\{S_n\}$ is said to be almost convergent to a limit S,

$$\text{if} \quad \lim_{n \to \infty} \frac{1}{(n+1)} \sum_{k=p}^{n+p} S_k = S \tag{1.4}$$

with respect to p.

An almost convergence is a generalization of ordinary convergence.

2. Definition (Sharma and Qureshi [4]). A series $\sum_{n=0}^{\infty} U_n$ with the sequence of partial sums $\{S_n\}$ is said to be almost Riesz summable to S, provided

$$T_{n,p} = \frac{1}{P_n} \sum_{k=0}^{n} p_k \, S_{k,p} \to S \text{ as } n \to \infty$$

uniformly with respect to p, where

$$S_{k,p} = \frac{1}{k+1} \sum_{\mu=1}^{k+p} S_\mu$$

and $\{p_n\}$ be a sequence of non-negative constants, such that $p_0 > 0$, $P_n = p_0 + p_1 + \cdots\cdots + p_n$.

The Riesz means is regular if and only if $P_n \to \infty$ with n.

(see Theorem 1.4.4 of Peterson [3]).

Qureshi [1] proved the following theorem:

Theorem: The degree of approximation of a periodic function f(x), conjugate to a 2π-periodic function f(x) and belonging to the class Lip α, by almost Riesz means of its conjugate series, is given by

$$\max_{0 \le x \le 2\pi} | f(x) - \bar{T}_{n,p}(x) | = \begin{cases} O\left\{\left(\frac{p_n}{P_n}\right)\right\}^{\alpha} ; & 0 < \alpha < 1 \\ O\left\{\frac{p_n}{P_n} \log \frac{P_n}{p_n}\right\} ; & \alpha = 1 \end{cases}$$

where, $\bar{T}_{n,p}(x)$ is the almost Riesz means of series (1.2) and Riesz means are regular such that $0 < p_n \uparrow$ with $n \geq n_0$. The object of this paper is to prove the following theorem.

Theorem: The degree of approximation of a periodic function $\bar{f}(x)$, conjugate to a 2π-periodic function f(x) and belonging to the class Lip $(\rho(t), p)$, $(p > 1)$, by almost Riesz means of its conjugate series is given by

$$\max_{0 \le x \le 2\pi} |\bar{f}(x) - \bar{T}_{n,p}(x)| = O\left\{\rho\left(\frac{p_n}{P_n}\right)\left(\frac{p_n}{P_n}\right)^{-1/p}\right\}.$$

where $\bar{T}_{n,p}(x)$ is the almost Riesz means of the series (1.2) and Riesz means are regular such that $0 < p_n \uparrow$ with $n > n_0$ where $\rho(t)$ is a positive increasing function and satisfies the following conditions–

$$\text{(i)} \quad \left\{ \int_0^{\frac{p_n}{P_n}} \left(\frac{\rho(t)}{t^{1/p}} \right)^p dt \right\}^{1/p} = O\left(\rho\left(\frac{p_n}{P_n}\right)\right)$$

$$\text{(ii)} \quad \left\{ \int_{\frac{p_n}{P_n}}^{\pi} \left(\frac{\rho(t)}{t^{1/p+1}} \right)^p dt \right\}^{1/p} = O\left(\rho\left(\frac{p_n}{P_n}\right)\left(\frac{p_n}{P_n}\right)^{-1}\right)$$

Proof of the Theorem: Let $\bar{S}_k$ be the k–th partial sum of the conjugate series (1.2). It is easy to show that–

$$\bar{S}_k - \bar{f}(x) = \frac{1}{\pi} \int_0^{\pi} \psi(t) \frac{\operatorname{Cos}(k+\frac{1}{2})t}{2 \operatorname{Sin} \frac{t}{2}} dt$$

where $\psi(t) = f(x+t) - f(x-t)$

$$\text{And } \bar{S}_{k,p}(x) - \bar{f}(x) = \frac{1}{k+1} \sum_{\mu=p}^{k+p} \{\bar{S}_\mu(x) - \bar{f}(x)\}.$$

$$= \frac{1}{\pi(k+1)} \int_0^{\pi} \psi(t) \sum_{k=0}^{n} \frac{\operatorname{Cos}(k+\frac{1}{2})t}{2 \text{ x } \operatorname{Sin} \frac{t}{2}} dt$$

$$= \frac{1}{2\pi(k+1)} \int_0^{\pi} \psi(t) \frac{(\operatorname{Sin}(pt) - \operatorname{Sin}(k+p+1)t)}{2 \operatorname{Sin}^2 \frac{t}{2}} dt$$

We have

$$\bar{t}_{n,p}(t) - \bar{f}(t) = \frac{1}{P_n} \sum_{k=0}^{n} p_k \{\bar{S}_{k,p} - \bar{f}(t)\}$$

$$= \frac{1}{2\pi P_n} \int_0^{\pi} \psi(t) \sum_{k=0}^{n} \frac{p_k [\operatorname{Sin}(pt) - \operatorname{Sin}(k+p+1)t]}{(k+1)\, 2 \operatorname{Sin}^2 \frac{t}{2}} dt$$

Therefore

$$| \bar{t}_{n,p}(t) - \bar{f}(t) | \leq \frac{1}{2\pi \; P_n} \int_0^{\pi} | \psi(t) | \sum_{k=0}^{n} \frac{p_k}{k+1} \frac{\operatorname{Cos}(k+2p+1)\frac{t}{2} \operatorname{Sin}(k+1)\,x\,dt}{\operatorname{Sin}^2 \frac{t}{2}}$$

$$= \frac{1}{2\pi \; P_n} \left[\int_0^{\frac{p_n}{P_n}} + \int_{\frac{p_n}{P_n}}^{\pi} \right] | \psi(t) | \, x \, | \sum_{k=0}^{n} \frac{p_k}{k+1} \frac{\operatorname{Cos}(k+2p+1)\frac{t}{2} \operatorname{Sin}(k+1)\frac{t}{2}}{\operatorname{Sin}^2 \frac{t}{2}} | \, dt = I_1 + I_2, \text{ say}$$

Now,

$$I_1 = \frac{1}{2\pi \; P_n} \int_0^{\frac{p_n}{P_n}} | \psi(t) | \; | \sum_{k=0}^{n} \frac{p_k}{k+1} \frac{\operatorname{Cos}(k+2p+1)\frac{t}{2} \operatorname{Sin}(k+1)\frac{t}{2}}{\operatorname{Sin}^2 \frac{t}{2}} | \, dt$$

$$= O \left[\frac{1}{P_n} \int_0^{\frac{p_n}{P_n}} | \psi(t) | \; | \sum_{k=0}^{n} \frac{p_k}{k+1} \frac{\operatorname{Cos}(k+2p+1)\frac{t}{2} \operatorname{Sin}(k+1)\frac{t}{2}}{\operatorname{Sin}^2 \frac{t}{2}} | \, dt \right]$$

$$= O\left[\frac{1}{P_n}\left\{\int_0^{\frac{p_n}{P_n}} |\psi(t)|^p \, dt\right\}^{\frac{1}{p}}\right.$$

$$\left.\times\left\{\int_0^{\frac{p_n}{P_n}} \left|\sum_{k=0}^{n} \frac{p_k}{k+1} \frac{(\mathrm{Cos}\,(k+2p+1)\,\frac{t}{2}\ \mathrm{Sin}\,(k+1)\,\frac{t}{2}}{\mathrm{Sin}^2\,\frac{t}{2}}\right|^q dt\right\}^{\frac{1}{q}}\right]$$

$$= O\left[\frac{1}{P_n}\left\{\int_0^{\frac{p_n}{P_n}} |\psi(t)|^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{p_n}{P_n}} \left|\sum_{k=0}^{n} \frac{p_k}{k+1} \times \frac{(k+1)}{t}\right|^q dt\right\}^{\frac{1}{q}}\right]$$

$$= O\left[\left\{\int_0^{\frac{p_n}{P_n}} |\psi(t)|^p dt\right\}^{\frac{1}{p}} \left\{\int_0^{\frac{p_n}{P_n}} \frac{1}{t^q} dt\right\}^{\frac{1}{q}}\right]$$

$$= O\left[\left\{\int_0^{\frac{p_n}{P_n}} \left(\frac{\rho(t)}{t^{1/p}}\right)^p dt\right\}^{\frac{1}{p}} O\left(\frac{p_n}{P_n}\right)^{-1+\frac{1}{q}}\right.$$

$$= O\left(\rho\left(\frac{p_n}{P_n}\right)\right) O\left(\frac{p_n}{P_n}\right)^{-1+\frac{1}{q}}$$

$$= O\left(\rho\left(\frac{p_n}{P_n}\right)\left(\frac{p_n}{P_n}\right)^{-\frac{1}{p}}\right)$$

since $\frac{1}{p} + \frac{1}{q} = 1$, such that $1 \leq q \leq \infty$,

Similarly,

$$I_2 = O\left[\frac{1}{P_n} \int_{\frac{p_n}{P_n}}^{\pi} |\psi(t)| \quad \left|\sum_{k=0}^{n} \frac{p_k}{k+1}\right.\right.$$

$$\frac{\operatorname{Cos}(k+2p+1)\frac{t}{2}\operatorname{Sin}(k+1)\frac{t}{2}}{\operatorname{Sin}^2\frac{t}{2}}\Big|\,dt\Big]$$

$$= O\left[\frac{1}{P_n}\left\{\int_{\frac{p_n}{P_n}}^{\pi}\left|\frac{\rho(t)}{t}\right|^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{p_n}{P_n}}^{\pi}\Big|\,t\sum_{k=0}^{n}\frac{p_k}{k+1}\right.\right.$$

$$\left.\left.\frac{\operatorname{Cos}(k+2p+1)\frac{t}{2}\operatorname{Sin}(k+1)\frac{t}{2}}{\operatorname{Sin}^2\frac{t}{2}}\right|^q dt\right\}^{\frac{1}{q}}\Bigg]$$

$$= O\left[\frac{1}{P_n}\left\{\int_{\frac{p_n}{P_n}}^{\pi}\left(\frac{\rho(t)}{t^{1/p+1}}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{p_n}{P_n}}^{\pi}\Big|\,t\sum_{k=0}^{n}\frac{1}{(k+1)}\right.\right.$$

$$\left.\left.\times\frac{(k+1)\left|\operatorname{Sin}\frac{t}{2}\right|p_k\operatorname{Cos}(k+2p+1)\frac{t}{2}}{\operatorname{Sin}^2\frac{t}{2}}\right|^q dt\right\}^{\frac{1}{q}}\Bigg]$$

$$= O\left[\frac{1}{P_n}\left\{\int_{\frac{p_n}{P_n}}^{\pi}\left(\frac{\rho(t)}{t^{1/p+1}}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{p_n}{P_n}}^{\pi}\left|\,t\frac{\sum_{k=0}^{n}p_k\,(k+2p+1)\frac{t}{2}}{t}\right|^q dt\right\}^{\frac{1}{q}}\right]$$

$$= O\left[\frac{p_n}{P_n}\times\rho\left(\frac{p_n}{P_n}\right)\left(\frac{p_n}{P_n}\right)^{-1}\left\{\int_{\frac{p_n}{P_n}}^{\pi}\frac{1}{t^q}\,dt\right\}^{\frac{1}{q}}\right.$$

$$= O\left[\rho\ \left(\frac{p_n}{P_n}\right)\left(\frac{p_n}{P_n}\right)^{-1+\frac{1}{q}}\right]$$

$$= O\left[\rho\ \left(\frac{p_n}{P_n}\right)\left(\frac{p_n}{P_n}\right)^{-\frac{1}{p}}\right]$$

Since $\{p_n\}$ is monotonic, increasing, we have

$$\sum_{k=0}^{n} p_k \operatorname{Cos}(k+2p+1)\frac{t}{2} \leq p_n \sum_{k=0}^{n} \operatorname{Cos}(k+2p+1)\frac{t}{2}$$

$$= O\left(\frac{p_n}{t}\right)$$

This completes the proof of the theorem.

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