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THREE LORENTZIAN PLANES MOVING WITH RESPECT TO ONE ANOTHER AND POLE POINTS

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ABSTRACT

In this paper we have considered a Lorentzian plane A moving with respect to the planes L and L'. We have discussed here on the three Lorentzian planes A, L and L' instead of the two planes in [1]. Also, Lorentzian pole points and pole line of the mentioned planes are evaluated at a time "t".

I. INTRODUCTION

Let A and L be two moving Lorentzian planes with coordinate systems $\{B; \vec{a}_1, \vec{a}_2\}$ and $\{O; \vec{l}_1, \vec{l}_2\}$ respectively and L′ be a fixed Lorentzian plane with coordinate system $\{O'; \vec{l}_1{}' \vec{l}_2\}$.

At first let's evaluate some resembling equations given in [1]. By using Figure. 1 we find

$$\left.\begin{aligned} \vec{a}_1 &= \vec{l}_1 \operatorname{ch}\varnothing + \vec{l}_2 \operatorname{sh}\varnothing \\ \vec{a}_2 &= \vec{l}_1 \operatorname{sh}\varnothing + \vec{l}_2 \operatorname{ch}\varnothing \\ \vec{OB} &= \vec{b} = \vec{a}_1 b_1 + \vec{a}_2 b_2 \end{aligned}\right\} \quad (2)$$

And we obtain equations of LM motion A / L as

$$\begin{aligned} \vec{da}_1 &= \vec{a}_2 \, d\varnothing \\ \vec{da}_2 &= \vec{a}_1 \, d\varnothing \\ \vec{db} &= \vec{a}_1 (db_1 + b_2 \, d\varnothing) + \vec{a}_2 (db_2 + b_1 \, d\varnothing) \end{aligned} \quad (2)$$

Likewise, equations of LM motion A/L′ are obtained as follows

$$\vec{d'a_1} = \vec{a}_2 \, d\varnothing'$$

$$\vec{d'a_2} = \vec{a}_1 \, d\varnothing' \qquad (3)$$

$$\vec{d'b'} = \vec{a}_1 \, (db_1' + b_2' d\varnothing') + \vec{a}_2 \, (db_2' + b_1' d\varnothing')$$

For the sake of shortness let's use

$d\varnothing = \tau$, $d\varnothing' = \tau'$, $db_1 + b_2 \, d\varnothing = \sigma_1$,

$db_2 + b_1 \, d\varnothing = \sigma_2$, $db_1' + b_2' \, d\varnothing' = \sigma_1'$, $db_2' + b_1' \, d\varnothing' = \sigma_2'$. And σ_j, σ_j', τ and τ' will be called the Lorentzian Pfaffian forms of the 1–parameter Lorentzian motion, where $1 \leq j \leq 2$. So we obtain (2) and (3) again as follows, respectively,

$$\left.\begin{aligned} \vec{da_1} &= \vec{a}_2 \, \tau \\ \vec{da_2} &= \vec{a}_1 \, \tau \\ \vec{db} &= \vec{a}_1 \, \sigma_1 + \vec{a}_2 \, \sigma_2 \end{aligned}\right\} \qquad (4)$$

and

$$\left.\begin{aligned} \vec{d'a_1} &= \vec{a}_2 \, \tau' \\ \vec{d'a_2} &= \vec{a}_1 \, \tau' \\ \vec{d'b} &= \vec{a}_1 \, \sigma_1' + a_2 \, \sigma_2' \end{aligned}\right\} \qquad (5)$$

Now let us consider a point $X = (x_1, x_2)$ on the moving plane A. Then the following vectors are obtained.

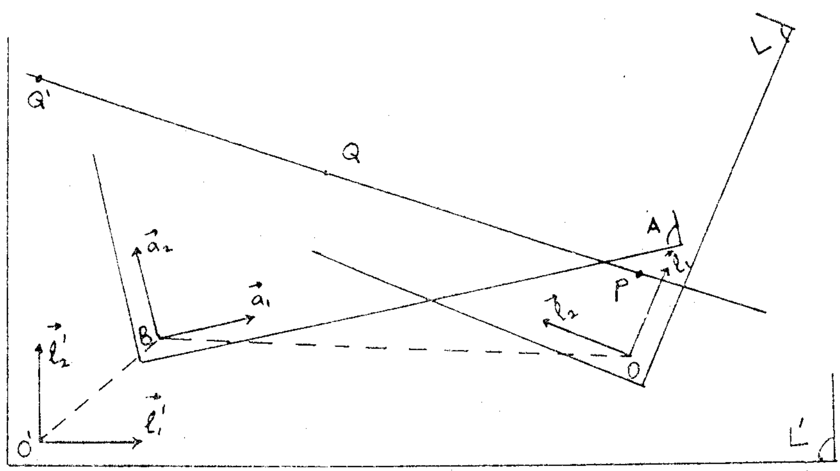

Figure. 1.

$$\vec{BK} = \vec{a}_1 x_1 + \vec{a}_2 x_2$$

$$\vec{x} = \vec{OX} = \vec{OB} + \vec{BK} = \vec{b} + \vec{a}_1 x_1 + \vec{a}_2 x_2$$

$$\vec{x}' = \vec{O'X} = \vec{O'B} + \vec{BX} = \vec{b}' + \vec{a}_1 x_1 + \vec{a}_2 x_2$$

From (4) we find differentiation of X in L as

$$\vec{dx} = \vec{a}_1 (dx_1 + \sigma_1 + x_2 \tau) + \vec{a}_2 (dx_2 + \sigma_2 + x_1 \tau) \qquad (6)$$

And the relative velocity vector is $\vec{V}_r = \frac{\vec{dx}}{dt}$. If $\vec{V}_r = 0$, then X is a fixed point on L. That is, we can find the conditions that X is a fixed point on L as follows:

If $\vec{V}_r = 0$, then $\vec{dx} = 0$ and from (6)

$$\left.\begin{aligned} dx_1 + \sigma_1 + x_2 \tau = 0 \\ dx_2 + \sigma_2 + x_1 \tau = 0 \end{aligned}\right\}$$

and so

$$\left.\begin{aligned} dx_1 = -(\sigma_1 + x_2 \tau) \\ dx_2 = -(\sigma_2 + x_1 \tau) \end{aligned}\right\}$$

Similarly, we find the conditions of X as a constant point on the fixed Lorentzian plane L′ as follows:

$$\vec{d'x'} = \vec{a}_1 (dx_1 + \sigma_1' + x_2 \tau') + \vec{a}_2 (dx_2 + \sigma_2' + x_1 \tau') \quad .. \ (7)$$

and $\vec{V}_a = \frac{\vec{d'x'}}{dt}$ is the absolute velocity vector. If X is a fixed point on L′, then $\vec{V}_a = \vec{o}$. So $\vec{d'x'} = \vec{o}$ and we obtain

$$\left.\begin{aligned} dx_1 = -(\sigma_1 + x_2 \tau') \\ dx_2 = -(\sigma_2 + x_1 \tau') \end{aligned}\right\}$$

Hence we can give the following theorem.

I.1. Theorem: Let A and L be moving and L′ be a fixed Lorentzian planes and X be a point on A. If X is a fixed point on L, then

$$\left.\begin{aligned} dx_1 &= -(\sigma_1 + x_2\,\tau) \\ dx_2 &= -(\sigma_2 + x_1\,\tau) \end{aligned}\right\} \tag{8}$$

and if X is a fixed point on L,, then

$$\left.\begin{aligned} dx_1 &= -(\sigma_1' + x_2\,\tau') \\ dx_2 &= -(\sigma_2' + x_1\,\tau') \end{aligned}\right\} \tag{9}$$

Now let X be a fixed point on L′ then the sliding velocity vector of X according to L′ is $\vec{V}_f = \dfrac{\overrightarrow{d_f x}}{dt}$, which corresponds to the differential $\overrightarrow{d_f x}$ of X with respect to L′. Therefore by using (8), (7) changes into

$$\overrightarrow{d_f x} = \vec{a}_1\,(-\sigma_1 - x_2\,\tau + \sigma'_1 + x_2\,\tau') + \vec{a}_2\,(-\sigma_2 - x_1\,\tau + \sigma_2' + x_1\,\tau')$$

and so

$$\overrightarrow{d_f x} = \vec{a}_1\,\{(\sigma_1' - \sigma_1) - x_2\,(\tau - \tau')\} + \vec{a}_2\,\{(\sigma'_2 - \sigma_2) - x_1\,(\tau - \tau')\} \quad \ldots \tag{10}$$

I.2. Theorem: If X is a fixed point on L, then

$$\overrightarrow{d'x'} = \overrightarrow{d_f x} + \overrightarrow{dx}.$$

II. MOVING PLANES WITH RESPECT TO ONE ANOTHER AND ROTATION POLES

In the 1–parameter Lorentzian motion the rotation pole is characterized by vanishing sliding velocity. That is, at the pole point the relative velocity equals to the absolute velocity at a t time. And so, if we take $d_f x = 0$ and use (10), the pole point $P = (p_1, p_2)$ of the 1–parameter Lorentzian motion L / L′ is obtained as

$$\left.\begin{aligned} p_1 = x_1 &= \frac{\sigma_2' - \sigma_2}{\tau - \tau'} \\ p_2 = x_2 &= \frac{\sigma_1' - \sigma_1}{\tau - \tau'} \end{aligned}\right\} \tag{11}$$

where $\overrightarrow{BP} = \vec{a}_1 p_1 + \vec{a}_2 p_2$.

Now let us consider 1–parameter Lorentzian motions A / L and A / L'. Each of them has an exact pole point at a "t" time.

The Lorentzian A / L motion was known by (4), where $\tau = d\varnothing$ was the infinitesimal rotation angle and $\frac{\tau}{dt}$ was the angular velocity.

Differentiating N on A we find the vector

$$\overrightarrow{dBX} = \vec{a}_1 \, dx_1 + \vec{a}_2 \, dx_2$$

which corresponds to the relative velocity vector of X on A. We have given by (6) the differential of X with respect to L. If we write (6) in the following form

$$\overrightarrow{dx} = [\vec{a}_1 \, dx_1 + \vec{a}_2 \, dx_2] + [\vec{a}_1 (\sigma_1 + x_2 \tau) + \vec{a}_2 (\sigma_2 + x_1 \tau)].$$

The first paranthesis corresponds to the differential of the relative velocity vector of X on A, while the second paranthesis corresponds to the sliding velocity vector of X in A / L.

The rotation pole $Q = (q_1, q_2)$, at any t time, of the Lorentzian motion A / L is obtained by the vanishing of the sliding velocity. So

$$\left.\begin{array}{l} \sigma_1 + x_2 \tau = 0 \\ \sigma_2 + x_1 \tau = 0 \end{array}\right\}$$

and from that we find

$$\left.\begin{array}{l} x_1 = q_1 = - \dfrac{\sigma_2}{\tau} \\[2ex] x_2 = q_2 = - \dfrac{\sigma_1}{\tau} \end{array}\right\} \qquad (12)$$

Similarly, the pole point $Q' = (q_1', q_2')$ of the 1–parameter Lorentzian motion A / L, is found as

$$\left.\begin{array}{l} q_1' = - \dfrac{\sigma_2'}{\tau'} \\[2ex] q_2' = - \dfrac{\sigma_1'}{\tau'} \end{array}\right\} \qquad (13)$$

II.1. Theorem: Let P, Q and Q′ be the pole points of the Lorentzian motions L/L', A/L and A/L' respectively, then P, Q and Q′ are collinear.

Proof: The slopes of [PQ], [PQ′] and [QQ′] are all equal to

$$\frac{\sigma_1 \tau' - \sigma_1' \tau}{\sigma_2 \tau' - \sigma_2' \tau}.$$

That completes the proof of the theorem.

II.1. Definition: The straight line, indicated by the points P, Q and Q′ is called the Lorentzian pole line of the Lorentzian motions L/L', A/L and A/L'.

ÖZET

Bu çalışmada L ve L′ Lorentz düzlemlerine göre hareket eden bir A Lorentz düzlemi göz önüne alındı. [1] deki iki Lorentz düzlemine karşılık burada A, L ve L′ gibi üç Lorentz düzlemi üzerinde çalışıldı. Ayrıca bahsedilen düzlemlerin bir "t" anındaki Lorentz anlamında pole noktaları ve pol doğrusu bulundu.

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