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FIXED COEFFICIENTS FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT

In this paper we consider the class $T_p(n,\alpha)$ consisting of analytic and univalent functions with negative coefficients and with fixed second coefficient. The object of the present paper is to show coefficient estimates, convex linear combinations, some distortion theorems and radius of convexity for f(z) in the class $T_p(n,\alpha)$. The results are generalized to families with finetely many fixed coefficients.

1. INTRODUCTION

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the unit disc $U=\{z:|z|<1\}$. For a function f(z) in S, we define

$$D^0 f(z)=f(z), \tag{1.2}$$

$$D^1 f(z)=Df(z)=zf(z), \tag{1.3}$$

and

$$D^n f(z)=D(D^{n-1}f(z)) \quad (n\in N=\{1,2,...\}). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [6]. With the help of the differential operator D^n, we say that a function f(z) belonging to S in the class $S_n(\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \alpha \quad (n\in N_0=N\cup\{0\}) \tag{1.5}$$

for some $\alpha(0 \le \alpha < 1)$, and for all $z \in U$. The class $S_n(\alpha)$ was defined by Salagean [6].

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \ge 0). \tag{1.6}$$

Further, we define the class $T(n,\alpha)$ by

$$T(n,\alpha) = S(\alpha) \cap T. \tag{1.7}$$

The class T(n,a) was studied by Hur and Oh [3]. We note that $T(0,\alpha) = T^*(\alpha)$ and $T(1,\alpha) = C(\alpha)$ were studied by Silverman [7].

For the class $T(n,\alpha)$, Hur and Oh [3] showed the following lemma.

Lemma 1. A function f(z) defined by (1.6) is in the class $T(n,\alpha)$ if and only if

$$\sum_{k=2}^{\infty} k^n(k-\alpha)\ a_k \le 1-\alpha\ . \tag{1.8}$$

The result is sharp.

In view of Lemma 1, we can see that the functions f(z) defined by (1.6) in the class $T(n,\alpha)$ satisfy

$$a_2 \le \frac{1-\alpha}{2^n(2-\alpha)}\ . \tag{1.9}$$

Let $T_p(n,\alpha)$ denote the class of functions f(z) in T(n,a) of the form

$$f(z) = z - \frac{p(1-\alpha)}{2^n(2-\alpha)} z^2 - \sum_{k=3}^{\infty} a_k z^k \quad (a_k \ge 0), \tag{1.10}$$

where $0 \le p \le 1$.

We note that $T_p(0,\alpha)=T_p{}^*(\alpha)$ and $T_p(1,\alpha)=C_p(\alpha)$, were studied by Silverman and Silvia [8]. Owa [4,5], Ganigi [2] and Ahuja and Silverman [1] showed many interesting results for certain subclasses of univalent functions with a fixed second coefficient.

2. COEFFICIENT ESTIMATES

Theorem 1. Let the function f(z) be defined by (1.10). Then $f(z) \in T_p(0,\alpha)$ if and only if.

$$\sum_{k=3}^{\infty} k^n(k-\alpha)\, a_k \le (1-p)(1-\alpha) \ . \tag{2.1}$$

The result is sharp.

Proof. Putting

$$a_2 = \frac{p(1-\alpha)}{2^n(2-\alpha)} \ , \ 0 \le p \le 1, \tag{2.2}$$

in (1.8) and simplifying we get the result. The result is sharp for the function

$$f(z) = z - \frac{p(1-\alpha)}{2^n(2-\alpha)} z^2 - \frac{(1-p)(1-\alpha)}{k^n(k-\alpha)} z^k \quad (k \ge 3). \tag{2.3}$$

Corollary 1. Let the function f(z) defined by (1.10) be in the class $T_p(n,\alpha)$. Then

$$a_k \le \frac{(1-p)(1-\alpha)}{k^n(k-\alpha)} \ (k \ge 3). \tag{2.4}$$

The result is sharp for the function f(z) given by (2.3).

3. EXTREME POINTS

Employing the techniquies used earlier by Silverman and Silvia [8], Owa [4,5], Ganigi [2] and Ahuja and Silverman [1], with the aid of Lemma 1, we can prove the following:

Theorem 2. Let

$$f_2(z) = z - \frac{p(1-\alpha)}{2^n(2-\alpha)} z^2 \tag{3.1}$$

and

$$f_k(z) = z - \frac{p(1-\alpha)}{2^n(2-\alpha)} z^2 - \frac{(1-p)(1-\alpha)}{k^n(k-\alpha)} z^k \tag{3.2}$$

for k = 3,4,.... The f(z) is in the class $T_p(n,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z), \tag{3.3}$$

where $\lambda_k \ge 0$ and $\sum_{k=2}^{\infty} \lambda_k = 1$.

Corollary 2. The extreme points of the class $T_p(n,\alpha)$ are the functions $f_k(z)$ $(k \geq 2)$ given by Theorem 2.

4. DISTORTION THEOREMS

In the first place of this section, we need the following lemmas.

Lemma 2. Let the function $f_3(z)$ be defined by

$$f_3(z) = z - \frac{p(1-\alpha)}{2^n(2-\alpha)} z^2 - \frac{(1-p)(1-\alpha)}{3^n(3-\alpha)} z^3. \tag{4.1}$$

Then, for $0 \leq r < 1$ and $0 \leq p \leq 1$,

$$\left|f_3(re^{i\theta})\right| \geq r - \frac{p(1-\alpha)}{2^n(2-\alpha)} r^2 - \frac{(1-p)(1-\alpha)}{3^n(3-\alpha)} r^3. \tag{4.2}$$

with equality for $\theta=0$. For either $0 \leq p < p_0$ and $0 \leq r \leq r_0$ or $p_0 \leq p \leq 1$.

$$\left|f_3(re^{i\theta})\right| \leq r + \frac{p(1-\alpha)}{2^n(2-\alpha)} r^2 - \frac{(1-p)(1-\alpha)}{3^n(3-\alpha)} r^3 \tag{4.3}$$

with equality for $\theta=\pi$. Further, for $0 \leq p \leq p_0$ and $r_0 \leq r < 1$.

$$\left|f_3(re^{i\theta})\right| \leq r \left\{\left(1 + \frac{3^n p^2(1-\alpha)(3-\alpha)}{4.4^n(1-p)(2-\alpha)^2}\right) + \left(\frac{p^2(1-\alpha)^2}{2.4^n(2-\alpha)^2} + \frac{2(1-p)(1-\alpha)}{3^n(3-\alpha)}\right)r^2 + \left(\frac{(1-p)^2(1-\alpha)^2}{9^n(3-\alpha)^2} + \frac{p^2(1-p)(1-\alpha)^3}{4.12^n(2-\alpha)^2(3-\alpha)}\right)r^4\right\}^{\frac{1}{2}} \tag{4.4}$$

with equality for

$$\theta = \cos^{-1}\left(\frac{p(1-p)(1-\alpha)r^2-3^n p(3-\alpha)}{4.2^n(1-p)(2-\alpha)r}\right) \tag{4.5}$$

where

$$p_0 = \frac{\left[(1-\alpha)-4.2^n(2-\alpha)-3^n(3-\alpha)\right]+p_0^*}{2(1-\alpha)} \tag{4.6}$$

and

$$r_0 = \frac{-2.2^n(1-p)(2-\alpha)+r_0^*}{p(1-p)(1-\alpha)} \tag{4.7}$$

where

$$p_0^* = \sqrt{\left[(1-\alpha)-4.2^n(2-\alpha)-3^n(3-\alpha)\right]^2+16.2^n(1-\alpha)(2-\alpha)},$$

and

$$r_0^* = \sqrt{4.4^n(1-p)^2(2-\alpha)^2+3^n p^2(1-p)(1-\alpha)(3-\alpha)}.$$

Lemma 3. Let the function $f_k(z)$ be defined by (3.2) and $k \geq 4$. Then

$$|f_k(re^{i\theta})| \leq |f_4(-r)| \tag{4.8}$$

Theorem 3. Let the function f(z) defined by (1.10) belong to the class $T_p(n,\alpha)$. Then for $0 \leq r < 1$,

$$\left|f(re^{i\theta})\right| \geq r - \frac{p(1-\alpha)}{2^n(2-\alpha)} r^2 - \frac{(1-p)(1-\alpha)}{3^n(3-\alpha)} r^3 \tag{4.9}$$

with equality for $f_3(z)$ at $z = r$, and

$$\left|f(re^{i\theta})\right| \leq \text{Max}\left\{\underset{\theta}{\text{Max}} \left|f_3(re^{i\theta})\right| , -f_4(-r)\right\}, \tag{4.10}$$

where $\underset{\theta}{\text{Max}} \left|f_3(re^{i\theta})\right|$ is given by Lemma 2.

The proof of Theorem 3 is obtained by comparing the bounds of Lemma 2 and Lemma 3.

Remark. Putting $p = 1$ in Theorem 3 we obtain the following result obtained by Hur and Oh [3].

Corollary 3. Let the function f(z) defined by (1.6) be in the class $T(n,\alpha)$. Then for $|z| = r < 1$, we have

$$r - \frac{(1-\alpha)}{2^n(2-\alpha)} r^2 \leq |f(z)| \leq r + \frac{(1-\alpha)}{2^n(2-\alpha)} r^2. \tag{4.11}$$

The result is sharp.

Lemma 4. Let the function $f_3(z)$ be defined by (4.1). Then, for $0 \leq r < 1$ and $0 \leq p \leq 1$,

$$\left|f'_3(re^{i\theta})\right| \geq 1 - \frac{2p(1-\alpha)}{2^n(2-\alpha)} r - \frac{3(1-p)(1-\alpha)}{3^n(3-\alpha)} r^2 \tag{4.12}$$

with equality for $\theta = 0$. For either $0 \leq p < p_1$ and $0 \leq r \leq r_1$ or $p_1 \leq p \leq 1$.

$$\left|f'_3(re^{i\theta})\right| \leq 1 + \frac{2p(1-\alpha)}{2^n(2-\alpha)} r - \frac{3(1-p)(1-\alpha)}{3^n(3-\alpha)} r^2 \tag{4.13}$$

with equality for $\theta = \pi$. For either $0 \leq p < p_1$ and $r_1 \leq r < 1$,

$$\left|f'_3(re^{i\theta})\right| \leq \left\{\left(1 + \frac{3^n p^2(1-\alpha)(3-\alpha)}{3.4^n(1-p)(2-\alpha)^2}\right) + \left(\frac{2p^2(1-\alpha)^2}{4^n(2-\alpha)^2} + \frac{6(1-p)(1-\alpha)}{3^n(3-\alpha)}\right)r^2 + \left(\frac{9(1-p)^2(1-\alpha)^2}{9^n(3-\alpha)^2} + \frac{3p^2(1-p)(1-\alpha)^3}{12^n(3-\alpha)(2-\alpha)^2}\right)r^4\right\}^{\frac{1}{2}} \tag{4.14}$$

with equality for

$$\theta = \cos^{-1}\left(\frac{3p(1-p)(1-\alpha)r^2 - 3^n p(3-\alpha)}{6.2^n(1-p)(2-\alpha)r}\right) \tag{4.15}$$

where

$$p_1 = \frac{\left[3(1-\alpha)-6.2^n(2-\alpha)-3^n(3-\alpha)\right]+p_1^*}{6(1-\alpha)} \tag{4.16}$$

and

$$r_1 = \frac{-6.2^n(1-p)(2-\alpha)+r_1^*}{6p(1-p)(1-\alpha)} \tag{4.17}$$

where

$$p_1^* = \sqrt{\left[3(1-\alpha)-3^n(3-\alpha)-6.2^n(2-\alpha)\right]^2+72.2^n(1-\alpha)(2-\alpha)},$$

and

$$r_1^* = \sqrt{36.4^n(1-p)^2(2-\alpha)^2+12.3^n p^2(1-p)(1-\alpha)(3-\alpha)}.$$

The Proof of Lemma 4 is given in much the same way as Lemma 2.

Theorem 4. Let the function f(z) defined by (1.10) be in the class $T_p(n,\alpha)$. Then, for $0 \le r < 1$,

$$\left|f'_3(re^{i\theta})\right| \ge 1 - \frac{2p(1-\alpha)}{2^n(2-\alpha)}\, r - \frac{3(1-p)(1-\alpha)}{3^n(3-\alpha)}\, r^2 \tag{4.18}$$

with equality for $f'_3(z)$ at $z = r$, and

$$\left|f'_3(re^{i\theta})\right| \le \text{Max}\left\{\underset{\theta}{\text{Max}} \left|f'_3(re^{i\theta})\right| , f'_4(-r)\right\} \tag{4.19}$$

where $\underset{\theta}{\text{Max}} \left|f'_3(re^{i\theta})\right|$ is given by Lemma 4.

5. RADIUS OF CONVEXITY

Theorem 5. Let the function f(z) defined by (1.10) be in the class $T_p(n,\alpha)$. Then f(z) is convex in the disc $|z| < r(n,\alpha,p)$, where $r(n,\alpha,p)$ is the largest value for which

$$\frac{4p(1-\alpha)}{2^n(2-\alpha)}\, r + \frac{k^2(1-p)(1-\alpha)}{k^n(k-\alpha)}\, r^{k-1} \le 1 \quad (k = 3,4,\ldots). \tag{5.1}$$

The result is sharp for the extremal function

$$f_k(z) = z - \frac{p(1-\alpha)}{2^n(2-\alpha)}\, z^2 - \frac{(1-p)(1-\alpha)}{k^n(k-\alpha)}\, z^k \text{ for some } k. \tag{5.2}$$

6. THE CLASS $T_{p_k,n}(n,\alpha)$

Instead of fixing just the second coefficient, we can fix finitely many coefficients. Let $T_{p_k,n}(n,\alpha)$ denote the class of functions in $T(n,\alpha)$ of the form

$$f(z) = z - \sum_{k=2}^{N} \frac{p_k(1-\alpha)}{k^n(k-\alpha)}\, z^k - \sum_{k=N+1}^{\infty} a_k z^k , \tag{6.1}$$

where $0 \le \sum_{k=2}^{N} p_k = p \le 1$. Note that $T_{p_k,2}(n,\alpha)=T_p(n,\alpha)$.

Theorem 6. The extreme poins of $T_{p_k,n}(n,\alpha)$ are

$$z - \sum_{k=2}^{N} \frac{p_k(1-\alpha)}{k^n(k-\alpha)}\, z^k$$

and

$$z - \sum_{k=2}^{N} \frac{p_k(1-\alpha)}{k^n(k-\alpha)} z^k - \frac{(1-p)(1-\alpha)}{k^n(k-\alpha)} z^k \quad \text{for } k = N+1, N+2, \ldots$$

The details of the proof are omitted.

The characterization of the extreme points enables us to solve the standard extremal problems in the same manner as was done for T(n,a). We omit the details.

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