Commun. Fac. Sci. Univ. Ank. Series A1
V. 46. pp. 77-81 (1997)

# RESTRICTIVE SEMIGROUPS OF HOLOMORPHIC ENDOMORPHISMS ON RIEMANN SURFACES

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(Received April 15, 1997; Accepted June 9, 1997)

**ABSTRACT**

Let R be a Riemann surface and X be any nonempty subset of R. E(R,X) denotes the semigroup, under composition, of all holomorphic selfmaps of R which carry X into X and is referred to as a restrictive semigroup of holomorphic endomorphisms. Let $R_1$, $R_2$ be Riemann surfaces which have bounded nonconstant holomorphic functions and X, Y be any subsets of $R_1$, $R_2$, respectively. If $\phi$ : E($R_1$ , X) - E($R_2$ , Y) is an isomorphism of semigroups, then there exist a conformal (or anticonformal) isomorphism $\psi : X \rightarrow Y$ such that $\phi(f) = \psi \circ f \circ \psi^{-1}$ for every $f \in E(R_1, X)$.

## 1. INTRODUCTION

Let R be a Riemann surface and X be a nonempty subset of R. The semigroup, under composition, of all holomorphic selfmaps of R which carry X into X is denoted by E(R,X) and is referred to as a restrictive semigroup. E(R,X) is clearly a subsemigroup of E(R) and it coincides with E(R) precisely when X is all of R.

A well-known theorem by L. Bers states that two plane domains are conformally (or anticonformally) equivalent if and only if their rings of holomorphic functions are isomorphic [1]. This result has been generalized to Riemann surfaces and its nonempty subsets [3, 4, 5]. A. Eremenko has shown that if $R_1$ and $R_2$ are two Riemann surfaces which have bounded nonconstant holomorphic functions and E($R_1$) and E($R_2$) are the semigroups of all holomorphic endomorphisms of $R_1$ and $R_2$, respectively, then any isomorphism of E($R_1$) with E($R_2$) induces a conformal (or anticonformal) isomorphism $R_1$ with $R_2$ [2].

## 2. ISOMORPHISIMS BETWEEN RESTRICTIVE SEMIGROUPS

**Definition 1.** $R_1$, $R_2$ are two Riemann surfaces and X,Y be any nonempty subsets of $R_1$, $R_2$, respectively. A function $\psi$: X $\rightarrow$ $R_2$ is said to be holomorphic if for each point P $\in$ X there exists an open neighborhood $U_P$ of P and a holomorphic function $\varphi_P$: $U_p \rightarrow R_2$ such that $\varphi_P$ and $\psi$ coincide on $U_P \cap X$. This is equivalent to assuming that there is a single open set U $\supset$ X and a holomorphic functions $\varphi$: U $\rightarrow$ $R_2$ such that $\varphi|X = \psi$ . $\psi$: X $\rightarrow$ Y is said to be a conformal (anticonformal) mapping if $\psi$ is holomorphic (or antiholomorphic, i.e., $\overline{\psi}$ is holomorphic), one-to-one and onto [3].

Let $R_1$, $R_2$ be Riemann surfaces which have bounded nonconstant holomorphic functions and X,Y be any nonempty subsets of $R_1$, $R_2$, respectively. It is immediate that each conformal (or anticonformal) mapping $\psi$: $R_1 \rightarrow R_2$ which carries X onto Y induces an isomorphism $\phi$: $E(R_1,X) \rightarrow E(R_2,Y)$ such that $\phi(f) = \psi \circ f \circ \psi^{-1}$, $f \in E(R_1,X)$.

The purpose of this paper is to prove the following theorem. So we generalize the Eremenko's result to nonempty subsets of Riemann surfaces.

**Theorem:** Let $R_1$, $R_2$ be Riemann surfaces which have bounded nonconstant holomorphic functions and X,Y be any nonempty subsets of $R_1$, $R_2$, respectively. Suppose that $\phi$: $E(R_1,X) \rightarrow E(R_2,Y)$ is an isomorphism of semigroups of holomorphic endomorphisms, then there exists a conformal (or anticonformal) isomorphism $\psi$: X $\rightarrow$ Y such that $\phi(f) = \psi \circ f \circ \psi^{-1}$, for each $f \in E(R_1,X)$.

**Proof.** We denote the constant mapping which maps $R_1$ to P $\in$ X by $c_P$ and denote the set of all constant endomorphisms by $C(R_1,X)$ the subsemigroup of $E(R_1,X)$. Then $c_P(P') = P$ for all $c_P \in C(R_1,X)$ and $P' \in R_1$;

$$f \circ c_P = c_{f(P)} \text{ and } c_P \circ f = c_P \text{ for all } f \in E(R_1,X).$$

We first prove that $\phi$: $C(R_1,X) \rightarrow C(R_2,Y)$, i.e., $\phi$ maps constants to constants. Let $c_P \in C(R_1,X)$, P $\in$ X. For any Q $\in$ Y there exists an $f \in E(R_1,X)$ such that $\phi(f) = c_Q$ since $\phi$ is onto. Hence, for all $Q' \in Y$

$$\phi(c_P)(Q) = \phi(c_P) \circ \phi(f)(Q') = \phi(c_P \circ f)(Q') = \phi(c_P)(Q'),$$

which shows that $\phi(c_P) \in C(R_2,Y)$. Thus, we can define $\psi: X \to Y$ by

$$\phi(c_P) = c_{\psi(P)} \text{ for all } P \in X \text{ (i.e., } \psi(P) = Q).$$

Then $\psi$ is one-to-one, because $\psi(P) = \psi(P')$ implies that $\phi(c_P) = \phi(c_{P'})$, which leads to $P = P'$. Further, $\psi$ is onto, beacuse for any $Q \in Y$, if we take $f \in E(R_1,X)$ such that $\phi(f) = c_Q$, then we can show in the same way as above that $f \in C(R_1,X)$, or $f = c_P$ for some $P \in X$, and hence, $Q = \psi(P)$.

Now let $f \in E(R_1,X)$ and $P, P' \in X$ such that $f(P) = P'$ and $\psi(P) = Q \in Y$. Then for all $Q' \in Y$,

$$\begin{aligned} \phi(f)(\psi(P)) &= \phi(f)(Q) &&= [\phi(f) \circ (c_Q)](Q') \\ &= [\phi(f) \circ c_{\psi(P)}](Q') &&= [\phi(f) \circ \phi(c_P)](Q') \\ &= \phi(f \circ c_P)(Q') &&= \phi(c_{P'})(Q') \\ &= c_{\psi(P')}(Q') &&= c_{\psi \circ f(P)}](Q'). \end{aligned}$$

Hence $\phi(f) \circ \psi = \psi \circ f$ so $\phi(f) = \psi \circ f \circ \psi^{-1}$.

Now we show that $\psi$ is continuous. Firstly, we give following definition:

**Definition 2:** Let f be an element of E(R,X). f is called a good element if for any iterate $f^n$ of f, $f^n(R)$ is relatively compact image in R.

If f is a good element, then the existence of a fixed point in R follows from relatively compactness of the image. Every element of E (R,X) which is different from identity has at most one fixed point in R. If R is a hyperbolic Riemann surface, i.e., the universal covering of R, is the unit disk U, then there exists a Riemannian metric on R which is called Poincaré metric. Denote by $\rho$ the distance in the Poincaré metric in R. The invariant form of the Schwarz lemma states that $\rho(f(P), f(Q)) \leq \rho(P,Q)$ for every P and Q. If f(R) is relatively compact, then f cannot be a covering so f strictly decreases the Poincaré distance. It follows that the sequence $f(R) \supset f^2(R) \supset \ldots$ has one point of intersection and this point P is the unique attractive fixed point of f in R. (Attractive means that $|f'(P)| < 1$.

The derivative at a fixed point does not depend on the choice of local coordinate.).

Now let $f \in E(R_1,X)$ be a good element. Then f has a fixed point $P_0 \in X$ and f is univalent in a neighborhood of this fixed point and

$$\bigcap_{n\in N} f^n(R_1) = \{P_0\}.$$

Eremenko showed that $\{f^n(R_1)\}$ forms a fundamental set of neighborhoods of $P_0$ [2]. Now let $Q_0 = \psi(P_0) \in Y$. Since f is good, $\phi(f) = g$ is a good element in $E(R_2,Y)$ which fixes $Q_0$. We also have $\psi(f^n(R_1) \cap X) = g^n(R_2) \cap Y$. So $\psi$ maps a fundamental set of neighborhoods of $P_0$ to a fundamental set of neighborhoods of $Q_0$, in the relative topologies. Thus $\psi$ is continuous.

Next, we show that $\psi$ is conformal (or anticonformal). Let

$$P(f) = \{h \in E(R_1,X) \mid h \circ f = f \circ h,\ f \in E(R_1,X)\}.$$

This is a semigroup of $E(R_1,X)$. Denote by S the group of all linear self-maps of the field C, i.e.,

$$S = \{z \to \lambda z \mid \lambda \in C^* = C \setminus \{0\}\}.$$

The group S is isomorphic to the multiplicative group $C^*$. There exists a neighborhood $O_1 \subset R_1$ of $P_0$ and a local coordinate $F: (O_1,P_0) \to (C,0)$ which conjugates P(f) to some subsemigroup $S_1 \subset S$. In other words $s(h) = F \circ h \circ F^{-1} \in S$ if $h \in P(f)$ and $h \to s(h)$ is an isomorphism of semigroups $P(f) \to S_1$. Similarly consider a local coordinate $G: (O_2,Q_0) \to (C,0)$, $Q_0 \in O_2 \subset R_2$, which conjugates P(g) to a subsemigroup $S_2 \subset S$. If $S_1$ and $S_2$ are considered as subsets of $C^*$, then they contain some punctured neighborhoods of 0.

**Lemma.** Let $S_1$ and $S_2$ be subsemigroups of the multiplicative group $C^*$ both containing some punctured neighborhoods of 0. If V is a continuous injective mapping in a neighborhood of 0 which conjugates $S_1$ to $S_2$, then

$$V(z) = az^A \bar{z}^B, \tag{1}$$

where $a \in C^*$, $A, B \in C$ and $A - B = \pm 1$ [2].

Note that V given by (1) is differentiable (as a function from $R^2$ to $R^2$) and nondegenerate in $C^*$. It is differentiable and nondegenerate at 0 iff A + B = 1. In the latter case V is conformal (or anticonformal) because A + B = 1 and A - B = ±1 imply A = 1 or B = 1.

Now the function $V_{P_0} = G \circ \psi \circ F^{-1}$ maps a neighborhood of 0 to some neighborhood of 0 and conjugates $S_1$ to $S_2$. According to the Lemma, for arbitrary $P \in O_1 \setminus \{P_0\}$ the function $V_P$ is differentiable and nondegenerate. Therefore $\psi \mid X \cap (O_1 \setminus \{P_0\})$ is differentiable and nondegenerate. So $V_P$ is conformal (or anticonformal) this implies that $\psi$ is conformal (or anticonformal).

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