

## ON THE (k+1)-DIMENSIONAL SPACE-LIKE RULED SURFACES IN THE MINKOWSKI SPACE $R_1^n$

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### ABSTRACT

In this paper, space-like ruled surfaces in the Minkowski n-space are defined. Moreover, some results and theorems related with the Riemannian curvature K and mean curvature vector H of the (k+1)-dimensional space-like ruled surface are given.

### 1. INTRODUCTION

We shall assume throughout this paper all manifolds, maps, vector fields, etc... are differentiable of class  $C^\infty$ . Consider a general Semi-Riemannian submanifold M of dimension (k+1) of the Minkowski space  $R_1^n$  ( $n \geq 3$ ). If  $\bar{D}$  (resp. D) is the Levi-Civita connection of  $R_1^n$  (resp. M) and if X and Y are tangent vector fields of M, then we find by decomposing  $\bar{D}_X Y$  into a tangent and normal component

$$\bar{D}_X Y = D_X Y + V(X, Y) \quad (1.1)$$

$V(X, Y)$  is a normal vector field on M and is symmetric in X and Y. A vector field Z of M is called an asymptotic vector field if  $V(Z, Z) = 0$ . A curve on M is an asymptotic curve if its tangent vector field T is an asymptotic vector field along the curve [1].

Let  $\xi$  be a normal vector field on M, then, by decomposing  $\bar{D}_X \xi$  in a tangent and a normal component, we find that

$$\bar{D}_X \xi = -A_\xi(X) + D_X^\perp \xi \quad (1.2)$$

which determines, at each point, a self-adjoint linear map, where  $D^\perp$  is a metric connection in the normal bundle  $\chi^\perp(M)$ . We use the same notation  $A_\xi$  to show

the linear map and the matrix of the linear map. A normal vector field  $\xi$  on  $\chi(M)$  is called parallel on the normal bundle  $\chi^\perp(M)$  if  $D_X^\perp \xi = 0$  for each vector field  $X$ . A subbundle  $F$  of  $\chi^\perp(M)$  is said to be parallel in  $\chi^\perp(M)$  if for each vector field  $\eta$  of  $F$  and each vector field  $X$  of  $\chi^\perp(M)$ ,  $D_X^\perp \eta$  is again a vector field of  $F$ , [2].

Suppose that  $X$  and  $Y$  are vector fields on  $\chi(M)$  while  $\xi$  is a normal vector field, then, if the standard metric tensor of  $R_1^n$  is denoted by  $\langle \cdot, \cdot \rangle$ ,

$$\langle \bar{D}_X Y, \xi \rangle = \langle V(X, Y), \xi \rangle = \langle A_\xi(X), Y \rangle. \quad (1.3)$$

If  $\xi_1, \xi_2, \dots, \xi_{n-k-1}$  constitute an orthonormal base field of the normal bundle  $\chi^\perp(M)$ , then we put

$$\langle V(X, Y), \xi_j \rangle = V_j(X, Y) \quad (1.4)$$

or

$$V(X, Y) = \sum_{j=1}^{n-m} V(X, Y) \xi_j.$$

The mean curvature vector  $H$  of  $M$  at the point  $P$  is given by

$$H = \sum_{j=1}^{n-k-1} \frac{\text{tr } A_{\xi_j}}{\text{boy } M} \xi_j. \quad (1.5)$$

$\|H\|$  shows the mean curvature. If  $H = 0$  at each point  $P$  of  $M$ , then  $M$  is said to be minimal, [1]. Let  $R_1^n$  be a Minkowski space in the Levi-Civita connection  $D$ . The function,

$$\bar{R}: \chi(R_1^n) \times \chi(R_1^n) \times \chi(R_1^n) \rightarrow \chi(R_1^n)$$

given by

$$\bar{R}(X, Y)Z = \bar{D}_{[X, Y]}Z - \bar{D}_X \bar{D}_Y Z + \bar{D}_Y \bar{D}_X Z \quad (1.6)$$

is a (1,3) tensor field on  $\chi(R_1^n)$  called the curvature tensor field of  $R_1^n$ . If  $X, Y \in T_{R_1^n}(p)$  the linear operator

$$R_{XY}: T_{R_1^n}(p) \rightarrow T_{R_1^n}(p)$$

sending each  $Z$  to  $R_{XY}Z$  is called a curvature operator, [3]. The function

$$R: T_M(\rho) \times T_M(\rho) \times T_M(\rho) \times T_M(\rho) \rightarrow R$$

given by

$$R(X_1, X_2, X_3, X_4) = \langle X_1, R(X_3, X_4)X_2 \rangle \quad (1.7)$$

is a covariant tensor field of order 4 on  $\chi(M)$  called the Riemannian curvature tensor field of M.

The function given by (1.7), at each point P, is called the Riemannian curvature and we denote

$$K(P) = \langle X, R(X, Y)Y \rangle. \quad (1.8)$$

If V is the second fundamental form of Semi-Riemannian manifold M, then we obtain

$$\langle X, R(X, Y)Y \rangle = \langle V(X, Y), V(X, Y) \rangle - \langle V(X, X), V(Y, Y) \rangle. \quad (1.9)$$

A two-dimensional subspace  $\pi$  of the tangent space  $T_M(\rho)$  is called a tangent plane to M at P. For tangent vectors  $X_p, Y_p \in T_M(\rho)$  defined by

$$K(X_p, Y_p) = \frac{\langle R(X_p, Y_p)X_p, Y_p \rangle}{\langle X_p, X_p \rangle \langle Y_p, Y_p \rangle - \langle X_p, Y_p \rangle^2} \quad (1.10)$$

is called the sectional curvature of M at P, [3].

## 2. (k+1)-DIMENSIONAL RULED SURFACE IN $R_1^n$

Let  $\{e_1(s), e_2(s), \dots, e_k(s)\}$  be a system of orthonormal vector fields, which is defined for each point of a space-like curve  $\alpha$  in the n-dimensional Minkowski space  $R_1^n$ . This system spans a k-dimensional subspace of the tangent space  $T_{R_1^n}(\alpha(s))$  at each point. This subspace is denoted by  $E_k(s)$ , that is

$$E_k(s) = \text{Sp}\{e_1(s), e_2(s), \dots, e_k(s)\}.$$

We get a (k+1)-dimensional surface in  $R_1^n$  if the subspace  $E_k(s)$  moves along the curve  $\alpha$ . We call this space a (k+1)-dimensional generalized space-like ruled surface in the n-dimensional Minkowski Space  $R_1^n$ . A parametrization of this ruled surface is

$$\phi(s, u_1, \dots, u_k) = \alpha(s) + \sum_{i=1}^k u_i e_i(s) \quad (2.1)$$

Throughout this paper  $E_k(s) = \text{Sp}\{e_1(s), e_2(s), \dots, e_k(s)\}$  denotes a subspace which is a space-like subspace,  $\alpha$  is a space-like curve which is an orthogonal trajectory of the  $k$ -dimensional generating space  $E_k(s)$  ( $k \geq 1$ ). We denote this ruled surface by  $M$ . If we take the partial derivate of  $\phi$  we get

$$\phi_s = \alpha(s) + \sum_{i=1}^k u_i e_i(s) ,$$

$$\phi_{u_i} = e_i(s) , \quad 1 \leq i \leq k .$$

Throughout our paper we assume that the system

$$\left\{ \alpha(s) + \sum_{i=1}^k u_i e_i(s) , e_1, \dots, e_k \right\}$$

is linear independent.

Let  $\{e_0, e_1, \dots, e_k\}$  be an orthonormal base of  $\chi(M)$  (i.e.  $e_0$  is the unit tangent vector of the orthonogal trajectories of the generating spaces). Suppose that timelike subspace  $\{\xi_1, \xi_2, \dots, \xi_{n-k-1}\}$  is an orthonormal base field of  $T_M^\perp(p)$ . Then  $\{e_0, e_1, \dots, e_k, \xi_1, \xi_2, \dots, \xi_{n-k-1}\}$  is a base field of  $T_{R_1^n}(p)$  at the point  $P \in R_1^n$ . Then we have

$$\langle e_0, e_0 \rangle = 1, \langle e_i, e_0 \rangle = 0, \langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 0 & , i \neq j \\ 1 & , i = j \end{cases} , \langle \xi_j, \xi_j \rangle = \varepsilon_j = \pm 1 . \quad (2.2)$$

Then  $M$  is said to be  $m$ -developable if

$$\text{rank} [e_0, e_1, \dots, e_k, \bar{D}_{e_0} e_1, \dots, \bar{D}_{e_0} e_k] = 2k - m \quad (2.3)$$

at each point  $P \in M$ . If  $m = -1$ , then the space-like ruled surface  $M$  is called non-developable; if  $m = k-1$ , then  $M$  is said to be total developable, [4].

Denote of  $\bar{D}$  the Levi-Civita connection of the Minkowski space  $R_1^n$ . For the orthonormal base  $\{e_1, \dots, e_k\}$  of the generating space  $E_k(s)$ , we observe that

$$\bar{D}_{e_i} e_j = 0 , \quad 1 \leq i , j \leq k .$$

Hence, if  $V$  denotes the second fundamental form of  $R_1^n$ , we must have

$$V(e_i, e_j) = 0 , \quad 1 \leq i , j \leq k . \quad (2.4)$$

Let  $X = \sum_{i=1}^k a_i e_i + a e_0$  and  $Y = \sum_{i=1}^k b_i e_i + b e_0$  be two vector fields of  $\chi(M)$ . So we find that

$$V(X,Y) = \sum_{i=1}^k (a_i b + b_i a)V(e_0, e_i) + abV(e_0, e_0) . \tag{2.5}$$

The normal subbundle of  $\chi^\perp(M)$  spanned by the normal fields  $V(e_0, e_i), 1 \leq i \leq k$  is denoted by  $F$ .

**Theorem 2.1.**  $M$  is  $m$ -developable iff the normal subbundle  $F$  is  $(k-m-1)$ -dimensional.

**Proof.** Suppose that we have (2.3). Because of (1.1) we can write

$$\bar{D}_{e_0} e_i = D_{e_0} e_i + V(e_0, e_i) , 1 \leq i \leq k .$$

But  $D_{e_0} e_i$  is a linear combination of the vector fields  $\{e_0, e_1, \dots, e_k\}$  and so we may replace the fields  $\bar{D}_{e_0} e_i$  by  $V(e_0, e_i)$  in (2.3). Now, the tangent space spanned by  $e_0, e_1, \dots, e_k$  is at each point normal to  $F$  and thus we find  $k+1+\dim F = 2k-m$  or  $\dim F = k-m-1$ , which completes the proof of the theorem.

From (2.2) we observe that  $\bar{D}_{e_i} e_0 \perp e_0$  and  $\bar{D}_{e_i} e_0 \perp e_j$ . This means that  $\bar{D}_{e_i} e_0$  is a normal vector field or

$$\bar{D}_{e_i} e_0 = V(e_i, e_0) , 1 \leq i \leq k . \tag{2.6}$$

Suppose that  $\{\xi_1, \xi_2, \dots, \xi_{n-k-1}\}$  is an orthonormal base field of the normal bundle  $\chi^\perp(M)$ , then we have the following Weingarten equations

$$\begin{aligned} \bar{D}_{e_0} \xi_j &= a_{00}^j e_0 + \sum_{r=1}^k a_{or}^j e_r + \sum_{s=1}^{n-k-1} b_{os}^j \xi_s , 1 \leq j \leq n-k-1 , \\ \bar{D}_{e_1} \xi_j &= a_{10}^j e_0 + \sum_{r=1}^k a_{1r}^j e_r + \sum_{s=1}^{n-k-1} b_{1s}^j \xi_s \\ &\dots\dots\dots \\ \bar{D}_{e_k} \xi_j &= a_{k0}^j e_0 + \sum_{r=1}^k a_{kr}^j e_r + \sum_{s=1}^{n-k-1} b_{ks}^j \xi_s . \end{aligned} \tag{2.7}$$

These equation together with (2.4) and (1.3) yield

$$\begin{aligned} a_{or}^j &= a_{ro}^j \\ a_{ir}^j &= 0 \quad 1 \leq j \leq n-k-1 , 1 \leq i,r \leq k . \end{aligned} \tag{2.8}$$

So the matrix of  $A_{e_j}$  has the form

$$A_{\xi_j} = - \begin{bmatrix} a_{00}^j & a_{01}^j & \dots & a_{0k}^j \\ a_{01}^j & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{0k}^j & 0 & \dots & 0 \end{bmatrix} \quad (2.9)$$

and this means  $\det A_{\xi_j} = 0$  if  $k \geq 2$ , from which we have:

**Corollary 2.2.** If  $k \geq 2$ , then the Lipschitz-Killing curvature of  $M$  is zero at each point in each normal direction.

**Corollary 2.3.** The matrix  $A_{\xi_j}$  of the shape operator of  $M$  is of the form (2.9) and is symmetric.

Because of the equations (2.7), we get

$$a_{0i}^j = \langle \bar{D}_{\xi_j} \xi_j, e_0 \rangle = - \langle \xi_j, \bar{D}_{\xi_j} e_0 \rangle \quad (2.10)$$

and from (2.6) together with (2.10) we receive

$$\bar{D}_{\xi_j} e_0 = V(e_j, e_0) + \sum_{j=1}^{n-k-1} \xi_j \langle \xi_j, \bar{D}_{\xi_j} e_0 \rangle \xi_j = - \sum_{j=1}^{n-k-1} \xi_j a_{0i}^j \xi_j. \quad (2.11)$$

**Theorem 2.4.** Let  $M$  be a  $(k+1)$ -dimensional space-like ruled surface of  $\mathbb{R}_1^n$ . Then the Riemannian curvature of  $M$  in the two-dimensional direction spanned by  $e_1$  and  $e_0$  is given by

$$K(e_i, e_0) = \langle \bar{D}_{\xi_i} e_0, \bar{D}_{\xi_i} e_0 \rangle, \quad 1 \leq i \leq k.$$

**Proof:** Let  $R$  be the Riemannian curvature tensor field of  $M$ . From (1.10) and (2.2), we find

$$K(e_i, e_0) = \langle R(e_i, e_0)e_i, e_0 \rangle. \quad (2.12)$$

If we connect (2.12) with (1.9) and (2.4), then we get

$$K(e_i, e_0) = \langle V(e_i, e_0), V(e_i, e_0) \rangle$$

or

$$K(e_i, e_0) = \langle \bar{D}_{\xi_i} e_0, \bar{D}_{\xi_i} e_0 \rangle. \quad (2.13)$$

From (2.11) and (2.13) we receive the following corollary.

**Corollary 2.5.** The Riemannian curvature of  $M$  in the two-dimensional direction spanned by  $e_1$  and  $e_0$  can be written with the entries of the Matrix  $A_{\xi_j}$  as follows

$$K(e_i, e_0) = \sum_{j=1}^{n-k-1} \varepsilon_j (a_{0i}^j)^2, \quad 1 \leq i \leq k, \quad \varepsilon_j = \langle \xi_j, \xi_j \rangle = \pm 1. \quad (2.14)$$

It is easy to see that (1.10) and (2.4) gives

$$K(e_i, e_j) = 0, \quad 1 \leq i, j \leq k. \quad (2.15)$$

**Theorem 2.6.** Let M be a (k+1)-dimensional space-like ruled surface in  $R_1^n$  and  $e_0$  be the tangent vector field of the base curve of M. The mean curvature is

$$H = \varepsilon_j \frac{V(e_0, e_0)}{k+1}, \quad \varepsilon_j = \langle \xi_j, \xi_j \rangle = \pm 1.$$

**Proof:** From (1.5) we known that

$$H = \sum_{j=1}^{n-k-1} \frac{\text{tr} A_{\xi_j}}{k+1} \xi_j. \quad (2.16)$$

Using (1.4), we can write

$$V(e_0, e_0) = \sum_{j=1}^{n-k-1} \xi_j \langle \bar{D}_{e_0} e_0, \xi_j \rangle \xi_j, \quad \varepsilon_j = \langle \xi_j, \xi_j \rangle = \mp 1$$

Because of the last equation and equation (2.7), we get

$$V(e_0, e_0) = - \sum_{j=1}^{n-k-1} \xi_j (a_{00}^j) \xi_j \quad (2.17)$$

For the matrix  $A_{\xi_j}$  given (2.9) we find

$$\text{tr} A_{\xi_j} = - a_{00}^j. \quad (2.18)$$

If we substitute (2.17) and (2.18) in (2.16), we observe that

$$H = \varepsilon_j \frac{V(e_0, e_0)}{k+1}, \quad \varepsilon_j = \langle \xi_j, \xi_j \rangle = \mp 1$$

From Theorem 2.6 we have immediately:

**Corollary 2.7.** The space-like ruled surface M is minimal iff each orthogonal trajectory of the generating spaces is an asymptotic line of M.

**Theorem 2.8.** If the (k+1)-dimensional m-developable space-like ruled surface M is minimal, then M is necessarily a submanifold of an  $R_1^{2k-m}$ .

**Proof:** Because of Theorem 2.1, we already know that the codimension of M is at least k-m-1 we have two cases:

- 1) First, suppose that the normal subbundle F is zero-dimensional.

Thus

$$V(e_0, e_i) = 0, \quad 1 \leq i \leq k.$$

Because of the second fundamental form  $V$  is symmetric, we find

$$V(e_i, e_0) = 0, \quad 1 \leq i \leq k.$$

If we substitute  $V(e_0, e_0) = 0$  and  $V(e_i, e_0) = 0, 1 \leq i \leq k$  in (2.5), we get

$$V(X, Y) = 0.$$

This says that the space-like ruled surface  $M$  must be totally geodesic, i.e.  $M$  is part of a  $(k+1)$ -dimensional linear space.

2) Next assume that the normal subbundle  $F$  is not zero. Consider an orthonormal base field  $\xi_1, \xi_2, \dots, \xi_{n-k-1}$  of  $\chi^\perp(M)$  such that  $\xi_1, \xi_2, \dots, \xi_{k-m-1}$  is a base field of the normal subbundle  $F$ . Consider the equations (2.7) in this case. Since  $\langle V(e_i, e_0), \xi_j \rangle = -a_{0i}^j, 1 \leq i \leq k, 1 \leq j \leq n-k-1$  we have immediately

$$a_{0i}^j = 0, \quad 1 \leq i \leq k, \quad k-m \leq j \leq n-k-1. \quad (2.19)$$

But  $H = 0$  and hence  $\text{tr } A_{\xi_j} = 0, 1 \leq j \leq n-k-1$  and so we get

$$A_{\xi_{k-m}} = \dots = A_{\xi_{n-k-1}} = 0. \quad (2.20)$$

If we set  $V(X, Y) = \sum_{j=1}^{n-k-1} V_j(X, Y) \xi_j$  for each two vector fields  $X$  and  $Y$  of  $M$ , then we find

$$V_{k-m}(X, Y) = \dots = V_{n-k-1}(X, Y) = 0. \quad (2.21)$$

If  $\bar{R}$  is the curvature tensor of  $R_1^n$  and if  $X, Y, Z$  are vector fields of  $\chi(M)$ , then the Codazzi equation says

$$\begin{aligned} (\bar{R}(X, Y)Z)^\perp &= \sum_{j=1}^{n-k-1} \left\{ (D_Y V_j)(XZ) - (D_X V_j)(YZ) \right\} \xi_j \\ &+ \sum_{j=1}^{n-k-1} V_j(XZ) D_Y^\perp \xi_j - \sum_{j=1}^{n-k-1} V_j(YZ) D_X^\perp \xi_j. \end{aligned} \quad (2.22)$$

Put

$$D_{e_i}^\perp \xi_\ell = \sum_{h=1}^{n-k-1} C_{i\ell}^h \xi_h + \sum_{r=k-m}^{n-k-1} C_{i\ell}^r \xi_r, \quad 1 \leq \ell \leq k-m-1, 1 \leq i \leq k \quad (2.23)$$

Then, from (2.21) and (2.22), we have



$$\begin{aligned} (\bar{R}(e_i, e_0)e_s)^\perp &= \sum_{\ell=1}^{k-m-1} \left\{ (D_{e_0} V_\ell)(e_i, e_s) - (D_{e_i} V_\ell)(e_0, e_s) \right\} \xi_\ell \\ &\quad - \sum_{\ell=1}^{k-m-1} V_\ell(e_0, e_s) D_{e_i}^\perp \xi_\ell + \sum_{\ell=1}^{k-m-1} V_\ell(e_0, e_s) D_{e_0}^\perp \xi_\ell = 0, \quad 1 \leq i, s \leq k. \end{aligned} \quad (2.24)$$

But  $V(e_i, e_s) = 0$ ,  $1 \leq i, s \leq k$  and so we find from (2.23) and (2.24)

$$\sum_{\ell=1}^{k-m-1} C_{i,\ell}^r V_\ell(e_0, e_s) = 0, \quad 1 \leq i, s \leq k, \quad k-m \leq r \leq n-k-1. \quad (2.25)$$

Now, fix in this expression  $i$  and  $r$  and let  $s$  be variable, then we find a system of  $k$  homogeneous linear equations with  $k-m-1$  unknowns  $C_{i,\ell}^r$ . The matrix of this system is

$$[V_\ell(e_0, e_s)], \quad 1 \leq \ell \leq k-m-1, \quad 1 \leq s \leq k.$$

and its rank is at each point of  $M$  equal to  $k-m-1$  because space-like ruled surface  $M$  is  $m$ -developable. So, it is easy to see that (2.25) gives

$$C_{i,\ell}^r = 0, \quad 1 \leq i \leq k, \quad 1 \leq \ell \leq k-m-1, \quad k-m \leq r \leq n-k-1. \quad (2.26)$$

We also have

$$\begin{aligned} (\bar{R}(e_0, e_i)e_0)^\perp &= \sum_{\ell=1}^{k-m-1} \left\{ (D_{e_i} V_\ell)(e_0, e_0) - (D_{e_0} V_\ell)(e_i, e_0) \right\} \xi_\ell \\ &\quad + \sum_{\ell=1}^{k-m-1} V_\ell(e_0, e_0) D_{e_i}^\perp \xi_\ell - \sum_{\ell=1}^{k-m-1} V_\ell(e_i, e_0) D_{e_0}^\perp \xi_\ell = 0. \end{aligned} \quad (2.27)$$

But  $V(e_0, e_0) = 0$ , and if we put

$$D_{e_0}^\perp \xi_\ell = \sum_{h=1}^{k-m-1} C_{\ell,h}^h \xi_h + \sum_{r=k-m}^{n-k-1} C_{\ell,r}^r \xi_r, \quad 1 \leq \ell \leq k-m-1$$

we find from (2.27)

$$\sum_{\ell=1}^{k-m-1} C_{\ell,i}^r V_\ell(e_i, e_0), \quad 1 \leq i \leq k, \quad k-m \leq r \leq n-k-1.$$

This gives analogously

$$C_{\ell,i}^r = 0, \quad 1 \leq \ell \leq k-m-1, \quad k-m \leq r \leq n-k-1. \quad (2.28)$$

Now, equation (2.26) together with equation (2.28) says that for each unit normal field  $\eta$  in  $F$  and for each vector field  $X$  of  $M$ ,  $D_X^\perp \eta$  has no component in the complementary subbundle  $F^\perp$  i.e. the normal subbundle  $F$  is parallel. If we identify all the tangent spaces of  $R_1^n$  with  $R_1^n$  itself, then, since  $F$  is parallel and because of equation (2.20), we see that the  $(2k-m)$ -dimensional subspaces of  $R_1^n$  spanned at each point of  $M$  by the

tangent space and the normal space  $F$ , are independent of the choice of the point  $P$  of  $M$ , which was to be proved.

**Theorem 2.9.** If the mean curvature vector  $H \neq 0$  of the  $(k+1)$ -dimensional  $m$ -developable space-like ruled surface  $M$  is at each point of  $M$  a vector of the normal subbundle  $F$ , then  $M$  is necessarily a submanifold of an  $R_1^{2k-m}$ .

**Proof:** Take again an orthonormal base field  $\xi_1, \xi_2, \dots, \xi_{n-k-1}$  such that  $\xi_1, \xi_2, \dots, \xi_{k-m-1}$  is a base field of  $F$ . Then, since  $\varepsilon_j(k+1)H = V(e_0, e_0) \in F$  we have again

$$V_{k-m}(X, Y) = \dots = V_{n-k-1}(X, Y) = 0$$

for each two vector fields  $X$  and  $Y$  of  $\chi(M)$ .

Next, if we have (2.23), then we find from (2.24) again (2.26). Moreover, since the vector fields  $D_{\xi_i}^\perp \xi_\ell$ ,  $1 \leq i \leq k$ ,  $1 \leq \ell \leq k-m-1$  have no components in the complementary subbundle  $F^\perp$ , we find because of (2.27) again (2.28) and this completes the proof.

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