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ON THE APPROXIMATION OF FUNCTIONS TOGETHER WITH DERIVATIVES BY CERTAIN LINEAR POSITIVE OPERATORS

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ABSTRACT

A generalization of Baskakov operator is considered and the order of approximation of derivative of functions are etablished.

1. INTRODUCTION

By $C^k[a,\infty]$ $(0\leq k\leq\infty)$ we denote the set of all real-valued functions k-times continuously differentiable on the interval $[a,\infty)$ $a\geq 0$ and by $C^{k,m}[a,\infty)$ $(0\leq k,m<\infty)$ the set of all functions $f\in C^k[a,\infty)$, with $f^{(k)}(x)=O(x^m)$ $(x\to\infty)$.

Let also $\{\varphi_n\}$ $(n=1,2,...)$ φ_n: $C\to C$ is a sequence of functions, having the following properties:

(i) φ_n $(n=1,2,...)$ is analytic on a domain D containing the disc $B=\{z\in C: |z-b|\leq b\}$;

(ii) $\varphi_n(0)=1 (n=1,2,...)$;

(iii) φ_n $(n=1,2,...)$ is completely monotone on $[0,b]$, i.e., $(-1)^k\varphi_n^{(k)}(x)\geq 0$ for any $k=0,1,2,...$;

(iv) there exists a positive integer m(n), such that $\varphi_n^{(k)}(x)=-n\varphi_{m(n)}^{(k-1)}(x)(1+\alpha_{k,n}(x))$, $x\in[0,b]$ $(n,k=1,2,...)$ where $\alpha_{k,n}(x)$ converges to zero for $n\to\infty$ uniformly in k and $x\in[0,b]$;

(v) $\lim\limits_{n\to\infty}\dfrac{n}{m(n)}=1.$

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Under these conditions, Baskakov [1] defined the sequence of linear positive operators:

$$\left(L_n f\right)(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x)\ f\left(\frac{k}{n}\right) \tag{1}$$

and proved its uniformly convergence to continuous function f on [0,b].

Later these operators and its different generalizations were investigated by several authors [[2], [3], [4], [5]]. In [5], additionaly, the convergence of the derivatives of the operators defined by $(L_n f)^{(r)} = L_n^{(r)} f$ to the r^{th} -derivative of a function f is investigated. As a special case (1) includes the classical Bernstein polynomials. Note that Stancu [6] gives the following generalization of Bernstein polynomials:

$$B_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right) C_n^k x^k (1-x)^{n-k} \quad (\alpha,\ \beta \geq 0)$$

which contrary to Bernstein polynomials, are not interpolation polynomials, and proved the theorem about its convergence to function f.

In this work, we define and investigate a Stancu type generalization of Baskakov operator in the form

$$\left(L_n f\right)(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x)\ f\left(\frac{k+\alpha}{n+\beta}\right), \ (\alpha,\ \beta \geq 0)\ (n = 1,2,\ldots) \tag{2}$$

We shall derive conditions under which the r^{th} derivative of a function $f \in C^r[0,\infty)$ can be approximated by the elements of the corresponding sequence of r^{th} derivatives $\{L_n^{(r)} f\}$, where

$$L_n^{(r)} f = (L_n f)^{(r)} \quad (n = 1,2,\ldots;\ r = 0,1,\ldots) \tag{3}$$

2. EXISTENCE AND FORMULA FOR $L_n^{(r)} f$

In order to establish the existence of the derivative $L_n^{(r)} f$ we need the following result.

Lemma 1. Let φ_n: C → C, satisfy (1-i), (1-iii) and $f \in C^{0,m}[0,\infty)$. Then the function $L_n f$ defined by (2) is infinitely differentiable on [0,b] and

$$\left(L_n f\right)(x) = \sum_{k=0}^{\infty} (-1)^r \frac{(-x)^k}{k!} \varphi_n^{(k+r)}(x)\ \Delta^r_{(n+\beta)^{-1}} f\left(\frac{k+\alpha}{n+\beta}\right) \tag{4}$$

where $\Delta^r_{h^{-1}} f(x_0)$ represents the difference of order r of the function f with the step $\frac{1}{n}$ starting from the value x_0. This difference of order r is defined by

$$\Delta^{1}_{(n+\beta)^{-1}} f\left(\frac{k+\alpha}{n+\beta}\right) = \Delta_{(n+\beta)^{-1}} f\left(\frac{k+\alpha}{n+\beta}\right) = f\left(\frac{k+\alpha+1}{n+\beta}\right) - f\left(\frac{k+\alpha}{n+\beta}\right)$$

$$\Delta^{r+1}_{(n+\beta)^{-1}} f\left(\frac{k+\alpha}{n+\beta}\right) = \Delta_{(n+\beta)^{-1}}\left(\Delta^{r}_{(n+\beta)^{-1}} f\left(\frac{k+\alpha}{n+\beta}\right)\right), \ (r = 1,2,\ldots)$$

Proof. Applying the Martini's [5] calculations with the simple modifications, we can show that the series in right hand side of (2) is uniformly convergence on $[0,b]$.

By formal differentiation of the function (2) we get

$$\left(L_n^{(1)} f\right)(x) = \sum_{k=0}^{\infty} -\frac{(-x)^k}{k!}\varphi_n^{(k+1)}(x)\left\{f\left(\frac{k+\alpha+1}{n+\beta}\right) - f\left(\frac{k+\alpha}{n+\beta}\right)\right\}$$

$$= \sum_{k=0}^{\infty} -\frac{(-x)^k}{k!}\varphi_n^{(k+1)}(x)\,\Delta^{1}_{(n+\beta)^{-1}} f\left(\frac{k+\alpha}{n+\beta}\right).$$

Repeating the process of differentiation, which can be justified in the same way, we arrive at formula (4).

3. A CONVERGENCE THEOREM AND AN ESTIMATION OF $f^{(r)} - L_n^{(r)} f$

Theorem 1. Let $\{\varphi_n\}$ $(n = 1,2,\ldots)$ be a sequence of functions $\varphi_n : C \to C$ satisfying (1-i), (1-iii), with

$$\varphi_n^{(k)}(0) = (-n)^k + o(n^k) \quad (k = 0,1,2,\ldots). \tag{5}$$

Moreover, let $f \in C^{r,2}[0,\infty)$ and $L_n^{(r)}$ be the operators defined in (3). Then $\lim_{n\to\infty}\left(L_n^{(r)} f\right)(x) = f^{(r)}(x)$ uniformly on $[0,b]$.

Proof. Consider the difference

$|f^{(r)}(x) - (L_n^{(r)} f)(x)|$, where $x \in [0,b]$.

Denoting

$$i(x) = \frac{(-1)^r}{n^r}\varphi_n^{(r)}(0)\, f^{(r)}(x) - \left(L_n^{(r)} f\right)(x) \tag{6}$$

we obtain

$$f^{(r)}(x) - \left(L_n^{(r)} f\right)(x) = \left(1 - \frac{(-1)^r}{n^r}\varphi_n^{(r)}(0)\right) f^{(r)}(x) + \frac{(-1)^r}{n^r}\varphi_n^{(r)}(0) f^{(r)}(x) - \left(L_n^{(r)} f\right)(x) =$$

$$= \left(1 - \frac{(-1)^r}{n^r}\varphi_n^{(r)}(0)\right) f^{(r)}(x) + i(x) \tag{7}$$

By using (4), (6) and the equality

$$\varphi_n^{(r)}(0) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\, \varphi_n^{(k+r)}(x) \tag{8}$$

we observe that the second term in right hand side of (7) may be written in the form

$$\begin{aligned} i(x) &= \sum_{k=0}^{\infty} \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!}\, \varphi_n^{(k+r)}(x)\, f^{(r)}(x) - \sum_{k=0}^{\infty} (-1)^r \frac{(-x)^k}{k!}\, \varphi_n^{(k+r)}(x) \Delta^r_{(n+\beta)^{-1}} f\left(\frac{k+\alpha}{n+\beta}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!}\, \varphi_n^{(k+r)}(x) \left\{ f^{(r)}(x) - n^r \Delta^r_{(n+\beta)^{-1}} f\left(\frac{k+\alpha}{n+\beta}\right)\right\} \end{aligned}$$

Since by the mean-value theorem

$$\Delta^r_{(n+\beta)^{-1}} f\left(\frac{k+\alpha}{n+\beta}\right) = \frac{1}{(n+\beta)^r} f^{(r)}\left(\frac{k+\alpha+\theta_k r}{n+\beta}\right), \quad (0 < \theta_k < 1)$$

we obtain

$$\begin{aligned} i(x) = \sum_{k=0}^{\infty} \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!}\, \varphi_n^{(k+r)}(x) \left(\frac{n}{n+\beta}\right)^r \Bigg\{ \left[f^{(r)}(x) - f^{(r)}\left(\frac{k+\alpha+\theta_k r}{n+\beta}\right)\right] + \\ + \frac{1}{(n+\beta)^r} \sum_{k=1}^{r} \binom{r}{k} \beta^k n^{r-k} f^{(r)}(x) \Bigg\} . \end{aligned}$$

Using this equality in (7) gives

$$\begin{aligned} f^{(r)}(x) - \left(L_n^{(r)} f\right)(x) &= \left(1 - \frac{(-1)^r}{n^r} \varphi_n^{(r)}(0) + \frac{1}{(n+\beta)^r} \sum_{k=1}^{r} \binom{r}{k} \beta^k n^{r-k}\right) f^{(r)}(x) + \\ &+ \sum_{k=0}^{\infty} \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!}\, \varphi_n^{(k+r)}(x) \left(\frac{n}{n+\beta}\right)^r \left(f^{(r)}(x) - f^{(r)}\left(\frac{k+\alpha+\theta_k r}{n+\beta}\right)\right) \end{aligned}$$

and denoting

$$I_{n,r}(x) = \sum_{k=0}^{\infty} \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!}\, \varphi_n^{(k+r)}(x) \left(\frac{n}{n+\beta}\right)^r \left| f^{r}(x) - f^{(r)}\left(\frac{k+\alpha+\theta_k r}{n+\beta}\right)\right| \tag{9}$$

we obtain

$$\left|f^{(r)}(x) - \left(L_n^{(r)} f\right)(x)\right| \le \left|1 - \frac{(-1)^r}{n^r} \varphi_n^{(r)}(0) + \frac{1}{(n+\beta)^r} \sum_{k=1}^{r} \binom{r}{k} \beta^k n^{r-k}\right| \left|f^{(r)}(x)\right| + I_{n,r}(x) \tag{10}$$

From the condition $f \in C^{r,2}[0,\infty)$ to any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$, such that for each pair $x', x'' \in [0,b]$, with $|x' - x''| \le \delta$

$$|f^{(r)}(x') - f^{(r)}(x'')| \le \varepsilon$$

holds. Also there exists a positive constant A such that for each pair $x' \in [0,b]$, $x'' \in [0,\infty)$, with $|x' - x''| > \delta$

$$|f^{(r)}(x') - f^{(r)}(x'')| \leq A\,(x' - x'')^2$$

holds.

The above two inequalities imply

$$\left|f^{(r)}(x') - f^{(r)}(x'')\right| \leq \varepsilon + \frac{A}{\delta^2}\left(x' - x''\right)^2$$

for each pair $x' \in [0,b]$, $x'' \in [0,\infty)$. Hence it is possible to continue estimate (9) by the following way

$$I_{n,r}(x) \leq \sum_{k=0}^{\infty} \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!} \varphi_n^{(k+r)}(x) \left\{\varepsilon + \frac{A}{\delta^2}\left(x - \frac{(k+\alpha)+\theta_k r}{n+\beta}\right)^2 \leq\right.$$

$$\leq \sum_{k=0}^{\infty} \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!} \varphi_n^{(k+r)}(x) \left\{\varepsilon + \frac{A}{\delta^2}\left[\frac{(k+\alpha)\,(k+\alpha-1)}{(n+\beta)^2} +\right.\right.$$

$$\left.\left. + \left(\frac{k+\alpha}{n+\beta}\right)\left[-2x + \frac{2r+1}{n+\beta}\right] + \left(x^2 + \frac{r^2}{(n+\beta)^2}\right)\right]\right\}$$

since

$$\left(x - \frac{(k+\alpha)+\theta_k r}{n+\beta}\right)^2 \leq \left(x^2 + \frac{r^2}{(n+\beta)^2}\right) + \frac{(k+\alpha)\,(k+\alpha-1)}{(n+\beta)^2} + \left(\frac{k+\alpha}{n+\beta}\right)\left[-2x + \frac{2r+1}{n+\beta}\right].$$

Going on with our estimation by some calculation, we arrive at the form

$$I_{n,r}(x) \leq \frac{A}{\delta^2}\, x^2\left[\frac{(-1)^{r+2}}{(n+\beta)^2} \frac{\varphi_n^{(r+2)}(0)}{n^r} - \frac{2}{(n+\beta)} \frac{(-1)^{r+1}}{n^r} \varphi_n^{(r+1)}(0) + \frac{(-1)^r}{n^r} \varphi_n^{(r)}(0)\right] +$$

$$+ \frac{A}{\delta^2}\, x\left[\frac{(2r+2\alpha+1)}{(n+\beta)^2} \frac{(-1)^{r+1}}{n^r} \varphi_n^{(r+1)}(0) - \frac{2\alpha}{(n+\beta)} \frac{(-1)^r}{n^r} \varphi_n^{(r)}(0)\right] +$$

$$+ \left[\varepsilon + \frac{A}{\delta^2} \frac{1}{(n+\beta)^2} (\alpha+r)^2\right]\left(\frac{(-1)^r}{n^r} \varphi_n^{(r)}(0)\right).$$

From this, the form (10) and the assumption (5) the rest of the proof is trivial. We can write the following inequality:

$$\left|f^{(r)}(x) - \left(L_n^{(r)}f\right)(x)\right| \leq \left|1 - (-1)^r \frac{\varphi_n^{(r)}(0)}{n^r} + \frac{1}{(n+\beta)^r} \sum_{k=1}^{r} \binom{r}{k} \beta^k n^{r-k}\right| \left|f^{(r)}(x)\right| +$$

$$+ \frac{A}{\delta^2}\, x^2\left[\frac{n^2(-1)^{r+2}}{(n+\beta)^2} \frac{\varphi_n^{(r+2)}(0)}{n^{r+2}} - 2\frac{n}{n+\beta}(-1)^{r+1}\frac{\varphi_n^{(r+1)}(0)}{n^{r+1}} + (-1)^r \frac{\varphi_n^{(r)}(0)}{n^r}\right] +$$

$$+ \frac{A}{\delta^2} \frac{x}{(n+\beta)} \left[\frac{(2r+2\alpha+1)n}{(n+\beta)} (-1)^{r+1} \frac{\varphi_n^{(r+1)}(0)}{n^{r+1}} - 2\alpha (-1)^r \frac{\varphi_n^{(r)}(0)}{n^r} \right] +$$

$$+ \left[\varepsilon + \frac{A}{\delta^2} \frac{(\alpha+r)^2}{(n+\beta)^2} \right] \left((-1)^r \frac{\varphi_n^{(r)}(0)}{n^r} \right).$$

Hence, using the condition

$$\varphi_n^{(k)}(0) = (-n)^k + o(n^k), \quad (k = 0,1,2,\ldots),$$

we obtain

$$\lim_{n\to\infty} \left(L_n^{(r)} f\right)(x) = f^{(r)}(x)$$

uniformly on [0,b] and the proof is completed.

Starting from the expression (9) we can also give a direct estimation when $f^{(r)}$ is uniformly continuous on $[0,\infty)$. We obtain

$$I_{n,r}(x) \leq \sum_{k=0}^{\infty} \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!} \varphi_n^{(k+r)}(x) \omega_r \left(\left| x - \frac{(k+\alpha)+\theta_k r}{n+\beta} \right| \right) \tag{11}$$

where ω_r denotes the modulus of continuity of the function $f^{(r)}$ defined as usually by

$$\omega_r(\delta) = \sup \{ f^{(r)}(x') - f^{(r)}(x''): x', x'' \in [0,\infty), |x' - x''| \leq \delta, \delta > 0 \}.$$

Using the well-known formula

$$\omega(\lambda\delta) \leq (1 + [\lambda])\, \omega(\delta), \quad (\lambda \geq 0)$$

and the inequality

$$\left| x - \frac{(k+\alpha)+\theta_k r}{n+\beta} \right| \leq \left| x - \frac{(k+\alpha)}{n+\beta} - \frac{r}{2(n+\beta)} \right| + \frac{r}{2(n+\beta)}, \quad (0 < \theta_k < 1)$$

it is possible to estimate (11) by

$$I_{n,r}(x) \leq \sum_{k=0}^{\infty} \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!} \varphi_n^{(k+r)}(x) \left(1 + \left[\frac{\left| x - \frac{(k+\alpha)}{n+\beta} - \frac{r}{2(n+\beta)} \right| + \frac{r}{2(n+\beta)}}{\delta} \right] \right) \omega_r(\delta) =$$

$$= \left\{ \frac{(-1)^r}{n^r} \varphi_n^{(r)}(0) + {\sum_{k=0}^{\infty}}' \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!} \varphi_n^{(k+r)}(x) \left[\frac{\left| x - \frac{(k+\alpha)}{n+\beta} - \frac{r}{2(n+\beta)} \right| + \frac{r}{2(n+\beta)}}{\delta} \right] \right\} \omega_r(\delta) \tag{12}$$

where the prime denotes the summation to be taken over those terms for which

$$\frac{\left|x - \frac{(k+\alpha)}{n+\beta} - \frac{r}{2(n+\beta)}\right| + \frac{r}{2(n+\beta)}}{\delta} \geq 1.$$

Choosing $\delta > \frac{r}{2(n+\beta)}$, we also obtain

$$I_{n,r}(x) \leq \left\{ \frac{(-1)^r}{n^r} \varphi_n^{(r)}(0) \left(1 + \frac{r}{2(n+\beta)\delta}\right) + \sum_{k=0}^{\infty}{}' \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!} \varphi_n^{(k+r)}(x). \right.$$

$$\left. \cdot \frac{\left|x - \frac{(k+\alpha)}{n+\beta} - \frac{r}{2(n+\beta)}\right|}{\left(\delta - \frac{r}{2(n+\beta)}\right)} \cdot \frac{\left(\delta - \frac{r}{2(n+\beta)}\right)}{\delta} \right\} \omega_r(\delta).$$

We remark that

$$\frac{\left|x - \frac{(k+\alpha)}{n+\beta} - \frac{r}{2(n+\beta)}\right|}{\left(\delta - \frac{r}{2(n+\beta)}\right)} \geq 1$$

and continue our estimation by

$$I_{n,r}(x) \leq \left\{ \frac{(-1)^r}{n^r} \varphi_n^{(r)}(0) + \left(1 + \frac{r}{2(n+\beta)\delta}\right) + \sum_{k=0}^{\infty}{}' \frac{(-1)^r}{n^r} \frac{(-x)^k}{k!} \varphi_n^{(k+r)}(x). \right.$$

$$\left. \cdot \frac{\left(x - \frac{(k+\alpha)}{n+\beta} - \frac{r}{2(n+\beta)}\right)^2}{\delta\left(\delta - \frac{r}{2(n+\beta)}\right)} \right\} .\omega_r(\delta).$$

By the same arguments as made before this equals to

$$\left\{ \left[\frac{(-1)^{r+2}}{(n+\beta)^2} \frac{\varphi_n^{(r+2)}(0)}{n^r} - 2 \frac{(-1)^{r+1}}{(n+\beta)} \frac{\varphi_n^{(r+1)}(0)}{n^r} \frac{(-1)^r}{n^r} \varphi_n^r(0) \right] \frac{x^2}{\delta\left(\delta - \frac{r}{2(n+\beta)}\right)} + \right.$$

$$+ \frac{(-1)^{r+1}}{n^r} \varphi_n^{(r+1)}(0) \left[\frac{\frac{(r+2\alpha+1)x}{(n+\beta)^2}}{\delta\left(\delta - \frac{r}{2(n+\beta)}\right)} \right] +$$

$$\left. + \frac{(-1)^r}{n^r} \varphi_n^{(r)}(0) \left[1 + \frac{r}{2(n+\beta)\delta} + \frac{\frac{r^2+4\alpha^2+4\alpha r}{4(n+\beta)^2} - \frac{(r+2\alpha)x}{(n+\beta)}}{\delta\left(\delta - \frac{r}{2(n+\beta)}\right)} \right] \right\} .\omega_r(\delta).$$

Finally, making the special chose

$$\delta = \frac{r + \sqrt{r^2+16(n+\beta)}}{4(n+\beta)}$$

we have from the above expression

$$\left[\left\{\frac{(-1)^{r+2}}{(n+\beta)^2}\,\frac{\varphi_n^{(r+2)}(0)}{n^r} - 2\,\frac{(-1)^{r+1}}{(n+\beta)}\,\frac{\varphi_n^{(r+1)}(0)}{n^r} + \frac{(-1)^r}{n^r}\,\varphi_n^{(r)}(0)\right\}(n+\beta)x^2 + \right.$$

$$+ \frac{(-1)^{r+1}}{n^r}\,\varphi_n^{(r+1)}(0)\left(\frac{(r+2\alpha+1)x}{(n+\beta)^2}\right) + \frac{(-1)^r}{n^r}\,\varphi_n^{(r)}(0)\left(1 + \frac{r^2+8\alpha^2+8\alpha r+}{8(n+\beta)}\right.$$

$$\left.\left. + r + \sqrt{r^2+16(n+\beta)} - (r+2\alpha)x\right)\right].\omega_r\left(\frac{r + \sqrt{r^2+16(n+\beta)}}{4(n+\beta)}\right). \tag{13}$$

Hence, it possible to write by (10)

$$\left|f^{(r)}(x) - \left(L_n^{(r)}f\right)(x)\right| \le \left|1 - \frac{(-1)^r}{n^r}\,\varphi_n^{(r)}(0) + \frac{1}{(n+\beta)^r}\sum_{k=1}^{r}\binom{r}{k}\beta^k n^{r-k}\right|\left|f^{(r)}(x)\right| + \left\{\left[\frac{(-1)^{r+2}}{(n+\beta)^2}\,\frac{\varphi_n^{(r+2)}(0)}{n^r}\right.\right.$$

$$\left. - 2\,\frac{(-1)^{r+1}}{(n+\beta)}\,\frac{\varphi_n^{(r+1)}(0)}{n^r} + \frac{(-1)^r}{n^r}\,\varphi_n^{(r)}(0)\right] x^2(n+\beta) + \frac{(-1)^{r+1}}{n^r}\,\varphi_n^{(r+1)}(0)\left(\frac{(r+2\alpha+1)x}{(n+\beta)}\right) +$$

$$\left. + \frac{(-1)^r}{n^r}\,\varphi_n^{(r)}(0)\left(1 + \frac{r^2+8\alpha^2+8\alpha r+r\sqrt{r^2+16(n+\beta)}}{8(n+\beta)} - (r+2\alpha)x\right)\right\}.$$

$$.\omega_r\left(\frac{r + \sqrt{r^2+16(n+\beta)}}{4(n+\beta)}\right).$$

Now we can give the following statement about degree of approximation $f^{(r)}$ by the sequences $L_n^{(r)}f$.

Theorem 2. Let $\omega_r(\delta)$ be the modulus of continuity of r-th derivate of function f. Then in conditions of Theorem 1, the following inequality holds in every point $x \in [0,b]$

$$\left|f^{(r)}(x) - \left(L_n^{(r)}f\right)(x)\right| \le \left|1 - \frac{(-1)^r}{n^r}\,\varphi_n^{(r)}(0) + \frac{1}{(n+\beta)^r}\sum_{k=1}^{r}\binom{r}{k}\beta^k n^{r-k}\right|\left|f^{(r)}(x)\right| +$$

$$+ A\,(n,r,\alpha,\beta).\omega_r\left(\frac{r + \sqrt{r^2+16(n+\beta)}}{4(n+\beta)}\right),$$

where

$$A(n,r,\alpha,\beta) = \left[\frac{(-1)^{r+2}}{(n+\beta)^2}\frac{\varphi_n^{(r+2)}(0)}{n^r} - 2\frac{(-1)^{r+1}}{(n+\beta)}\frac{\varphi_n^{(r+1)}(0)}{n^r} + (-1)^r\frac{\varphi_n^{(r)}(0)}{n^r}\right] x^2(n+\beta) +$$

$$+ \frac{(-1)^{r+1}}{n^r}\varphi_n^{(r+1)}(0)\left(\frac{(r+2\alpha+1)}{(n+\beta)}x\right) +$$

$$+ \frac{(-1)^r}{n^r}\varphi_n^{(r)}(0)\left(1 + \frac{r^2+8\alpha^2+8\alpha r+r\sqrt{r^2+16(n+\beta)}}{8(n+\beta)} - (r+2\alpha)x\right)\Bigg\}.$$

Remark. The simple calculations show that the term $A(n,r,\alpha,\beta)$ tends to zero as $n \to \infty$ for every fixed r,α,β. Also, the term $\left|1 - \frac{(-1)^r}{n^r}\varphi_n^{(r)}(0)\right|$ tends to zero by the condition (5) and the term $\frac{1}{(n+\beta)^r}\sum_{k=1}^{r}\binom{r}{k}\beta^k n^{r-k} = O\left(\frac{1}{n}\right)$. Therefore, our theorem gives the order of convergence of the sequence $L_n^{(r)}f$ to f.

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