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ON GENERAL HELICES AND PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT

In a Riemannian manifold, a regular curve is called a general helix if $\frac{\alpha}{\beta}$ is constant and its firs and second curvatures are not constant [4]. If its first and second curvatures are constant the third curvature is zero then the regular curve is called helix. For helices in a Lorentzian manifold, there is a research of T. Ikawa, who investigated and obtained the differential equation;

$$D_X D_X D_X X = K D_X X \quad , \quad (K = \alpha^2 - \beta^2)$$

for the circular helix which corresponds to the case that the curvatures α and β of the timelike curve c(t) on the Lorentzian manifold M are constant [3]. Later, N. Ekmekçi and H.H. Hacısalihoğlu obtained the differential equation

$$D_X D_X D_X X = K D_X X + 3\alpha' D_X Y \quad , \quad \left(K = \frac{\alpha''}{\beta} + \alpha^2 - \beta^2\right)$$

for the case of general helix [2]. Recently, T. Nakanishi [5] prove the following lemma about a helix in Pseudo-Riemannian manifold which is stated as,

"A unit speed curve c in M_α is a helix if and only if there exist a constant λ such that $D_X D_X D_X X = \lambda D_X X$"

This paper generalizes the lemma stated above to the case of a general helix.

1. PRELIMINARIES

R^n with the metric tensor

$$\langle V_p, W_p \rangle = -\sum_{j=1}^{v} V_J W_J + \sum_{k=v+1}^{n} V_k W_k \quad , \quad V_p, W_p \in R^n$$

is called semi-Euclidean space and is denoted by R^n_i where v is called the index of the metrics [6].

Let M be an n-dimensional smooth manifold equipped with a metric $\langle\ ,\ \rangle$. If the index of the metric $\langle\ ,\ \rangle$ is v, then we call M a pseudo-Riemannian manifold of index ν and denote by M_ν. If $\langle \ , \ \rangle$ is positive definite, then M is a Riemannian manifold. Especially if $\nu = 1$, then M is called a Lorentzian manifold. A tangent vector x of M_ν is said to be spacelike if $\langle x, x \rangle > 0$, timelike if $\langle 0, x \rangle < 0$ and null if $\langle x, x \rangle = 0$ and $x \neq 0$. In particular, on the Lorentzian manifold, null vectors are also said to be lightlike.

Let $x_1, x_2, ..., x_n$ be tangent vectors of M_ν. Assume that they satisfy $\langle x_A, x_B \rangle = \varepsilon_A \delta_{AB}$ where $\varepsilon_A = \langle x_A, x_B \rangle = 1$ (resp. -1) if x_A is spacelike (resp. Timelike) then $\{x_A : A \in [1, n]\}$ is called an orthonormal basis of M_ν [6].

2. CURVES

A curve in a pseudo-Riemannian manifold M_ν is a smooth mapping

$$c : I \to M_\nu$$

where I is an open interval in the real line R. The interval I has a coordinate system consisting of the identity map u of I. The velocity vector of c at $t \in I$ is

$$c'(t) := \frac{dc(t)}{du}$$

A curve c(t) is sait to be regular if c′(t) does not vanish for any t. A curve c(t) in a pseudo-Riemannian manifold M_ν is said to be spacelike if its velocity vectors c′(t) are spacelike for all $t \in I$; similarly for timelike and null.

We define here a circle and circular helix in a pseudo-Riemannian manifold M_ν (cf[5], [1]). Let $c = c(t)$ be a curve in M_ν. By $k_j(t)$, we denote the j-th curvature of c(t). If $k_j(t) \equiv 0$ for $j > 2$ and if the principal vector field Y and binormal vector field Z, then we have the following Frenet formulas along c(t):

$$
\begin{aligned}
c'(t) &:= X \\
D_X X &= \varepsilon_1 \alpha(t) Y \\
D_X Y &= -\varepsilon_0 \alpha(t) X + \varepsilon_2 \beta(t) Z \\
D_X Z &= -\varepsilon_1 \beta(t) Y
\end{aligned}
\tag{2.1}
$$

where D denotes the covariant differentiation in M_v. A curve c is called a circle if $\beta \equiv 0$ and α is β positive constant along c. If both α and β are positive constants along c(t), then c(t) is caled a circular helix [1].

3. HELICES IN A PSEUDO-RIEMANNIAN MANIFOLD

Let c = c(t) be a regular curve in a pseudo-Riemannian manifold M_v. We denote the tangent vector field c′ by the letter X. When ⟨X, X⟩ = + 1 or -1, c is called a unit speed curve. In this paper, a unit speed curve c in M_v is said to be a general helix if only if there exists constants $\frac{\alpha}{\beta}$, where α and β respectively is first and second curvature and vector fields Y, Z of constant length along c such that X, Y, Z are orthogonal and the following equations hold.

$$
\begin{aligned}
D_X X &= \varepsilon_1 \alpha Y \\
D_X Y &= -\varepsilon_0 \alpha X + \varepsilon_2 \beta Z \\
D_X Z &= -\varepsilon_1 \beta Y
\end{aligned}
\tag{3.1}
$$

where, ⟨X, X⟩ = ε_0 (=1 or -1), ⟨X, X⟩ = ε_1, ⟨X, X⟩ = ε_2. If one of the X, Y and Z is timelike then others are spacelike. Especially, if Z = 0 in this equation the curve is called a circle [3].

Theorem 3.1.

A unit speed curve in M_v is a general helix if and only if

$$D_X D_X D_X X = \varepsilon_1 \lambda D_X X + 3\varepsilon_1 \alpha' \, D_X Y \tag{3.2}$$

where, $\lambda = \varepsilon_1 \frac{\alpha''}{\alpha} - \varepsilon_0 \alpha^2 - \varepsilon_2 \beta^2$.

Proof. Suppose that c is a general helix. Then, from (2.1), we have,

$$D_XD_XX = D_X(\varepsilon_1\alpha Y)$$
$$= \varepsilon_1\alpha' Y + \varepsilon_1\alpha D_X Y$$
$$= -\varepsilon_0\varepsilon_1\alpha^2 X + \varepsilon_1\alpha' Y + \varepsilon_1\varepsilon_2\alpha\beta Z \qquad (3.3.)$$

and

$$D_XD_XD_XX = -2\varepsilon_0\varepsilon_1\alpha'\alpha X - \varepsilon_0\varepsilon_1\alpha^2 D_X X + \varepsilon_1\alpha'' Y + \varepsilon_1\alpha'(-\varepsilon_1\alpha X + \varepsilon_2\beta Z) + \varepsilon_1\varepsilon_2(\alpha\beta)' Z + \varepsilon_1\varepsilon_2\alpha\beta(-\varepsilon_1\beta Y)$$
$$= -3\varepsilon_0\varepsilon_1\alpha'\alpha X + (\varepsilon_1\alpha'' - \varepsilon_2\alpha\beta^2)Y + (\varepsilon_1\varepsilon_2\alpha'\beta + \varepsilon_1\varepsilon_2(\alpha\beta)')Z - \varepsilon_0\varepsilon_1\alpha^2 D_X X \qquad (3.4)$$

Now, since c is general helix,

$$\frac{\alpha}{\beta} = \text{constant}$$

and this upon the derivation give rise to

$$\alpha'\beta = \alpha\beta'.$$

If we substitue the values

$$Y = \frac{\varepsilon_1}{\alpha} D_X X \qquad (3.5)$$

and

$$(\alpha\beta)' = \alpha'\beta + \alpha\beta' = 2\alpha'\beta$$

in (3.4) we obtain

$$D_XD_XD_XX = \varepsilon_1\left(\varepsilon_1\frac{\alpha''}{\alpha} - \varepsilon_0\alpha^2 - \varepsilon_2\beta^2\right) D_X X + 3\varepsilon_1\alpha' D_X Y$$

Hence we have (3.2).

Conversely let us assume that (3.2) holds. We show that the curve c is a general helix. Differentiating covariantly (3.5) we obtain

$$D_X Y = -\varepsilon_1\left(\frac{\alpha'}{\alpha^2}\right) D_X X + \frac{\varepsilon_1}{\alpha} D_X D_X X$$

and so,

$$D_XD_XY = \left(-\varepsilon_1\frac{\alpha'}{\alpha^2}\right)' D_XX - 2\varepsilon_1\frac{\alpha'}{\alpha^2} D_XD_XX + \frac{\varepsilon_1}{\alpha} D_XD_XD_XX \tag{3.6}$$

If we use (3.2) in (3.6), we get

$$D_XD_XY = \left\{\varepsilon_1\left(-\frac{\alpha'}{\alpha}\right)' + \frac{\lambda}{\alpha}\right\} D_XX - 2\varepsilon_1\frac{\alpha'}{\alpha} D_XD_XX + \left(3\varepsilon_1^2\frac{\alpha'}{\alpha}\right) D_XY.$$

Substituing (3.3) and (2.1) in this last equality we obtain

$$D_XD_XY = \left\{\varepsilon_1\left(-\frac{\alpha'}{\alpha}\right)' + \frac{\lambda}{\alpha}\right\} D_XX - 2\varepsilon_0\alpha' X - 2\left(\frac{\alpha'}{\alpha}\right)Y + \varepsilon_2\left(\frac{\alpha'\beta}{\alpha}\right)Z \tag{3.7}$$

On the other hand substituting the equality

$$D_XD_XY = -\varepsilon_0\alpha' X - (\varepsilon_1\varepsilon_2\beta^2 + \varepsilon_0\varepsilon_1\alpha^2)Y + \varepsilon_2\beta' Z$$

in (3.7) we obtain

$$\beta' = \left(\frac{\alpha'\beta}{\alpha}\right)$$

Integrating this we get

$$\frac{\alpha}{\beta} = \text{constant}$$

Thus c is a general helix. Hence proof is done.

We note that in the special case of c being a circular helix, our theorem coincides with the result of Y. Nakanishi [5].

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