Commun. Fac. Sci. Univ. Ank. Series A1
V. 46. pp. 225-247 (1997)

THE IMAGE OF THE MOORE COMPLEX OF SIMPLICIAL GROUPS

Zekeriya ARVASI

Department of Mathematics, Faculty of Science, Osmangazi University, Eskişehir, TURKEY

(Received Dec. 16, 1996; Accepted June 25, 1997)

INTRODUCTION

Simplicial groups model all connected homotyp types. In particular certain simplical groups, namely those with vanishing Moore complex in dimensions greater than n, provide algebraic models for n-types of simplical groups. This result has been crucial in development of homological algebra in the last thirty years.

R. Brown and J.L. Loday [5] examined that if the second dimension G_2 of a simplicial group G is generated by the degenerate elements, that is elements coming from lower dimensions, then the image of the second term NG_2 of the Moore complex (NG, ∂) of G by the differential, ∂, is

$$(Kerd_1, Kerd_0]$$

where the square brackets denote the commutator subgroup. An easy argument then shows that this subgroup of NG_1 is generated by elements of the form $(s_0d_1(x)ys_0d_1(x)^{-1}(xy^{-1}x^{-1})$ and that it is thus exacly the Peiffer subgroup of NG_1, the vanishing of which is equivalent to $\partial_1: NG_1 \to NG_0$ being a crossed module. For implicial algebras, this was carried out by the author ([2]).

In this paper we give a generalisation of the Peiffer elements for the group cases to dimensions 2, 3 and obtain partial results in higher dimensions. In order to present this argumert, we will need to examine part of the hypercrossed complex structure of the Moore complex (cf. Carrasco and Cegarra [7]). More precisely, we have:

Let G be a simplicial group with Moore complex **NG** and for $n > 1$, let D_n be the normal subgroup generated by the degenerate elements in dimension n. If $G_n = D_n$, then

$$\partial_n(NG_n) = \partial_n(N_n) \text{ for all } n > 1$$

where N_n is a normal subgroup in G_n generated by an explicitly given fairly small set of elements.

If $n = 2, 3$, then the image of the Moore complex of the simplicial group **G** can be given in the form

$$\partial_n(NG_n) = \prod_{\{I,J\}} [K_I, K_J]$$

for $\emptyset \neq I, J \subset [n-1] = \{0, 1, ..., n-1\}$ with $I \cup J = [n-1]$, where

$$K_I = \bigcap_{i \in I} \mathrm{Ker} d_i \text{ and } K_J = \bigcap_{j \in J} \mathrm{Ker} d_j$$

In general for $n > 3$, we can only prove

$$\prod_{\{I,J\}} [K_I, K_J] \subseteq \partial_n(NG_n)$$

but suspect the opposite inclusion holds as well.

Finally Curtis [9] stated that if **G** is simplicial group and if $x \in \pi_p(\mathbf{G})$ and $y \in \pi_q(\mathbf{G})$ with $\bar{x} \in G_p$, $\bar{y} \in G_q$, then

$$[x, y] = \prod_{\{a;b\}} [s_b\bar{x}, s_a\bar{y}]$$

where (a;b) varies over all shuffles. The normal subgroup N_n is generated by the component within NG_n of these $[s_b\bar{x}, s_a\bar{y}]$.

1. DEFINITIONS AND NOTATION

A simplicial group **G** is a sequence of groups, $\mathbf{G} = \{G_0, G_1, ..., G_n, ...\}$, together with face and degeneracy maps

$$d_i = d^n_i : G_n \to G_{n-1},\ 0 \leq i \leq n\ (n \neq 0)$$

$$s_i = s^n_i : G_n \to G_{n+1},\ 0 \leq i \leq n.$$

These maps are reqired to satisfy the simplicial identities

$$d_i d_j = d_{j-1} d_i \qquad \text{for} \quad i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \text{identity} & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \qquad \text{for} \quad i \leq j.$$

G can be completely described as a functor **G**: $\Delta^{op} \to$ Grp where Δ is the category of finite ordinals $[n] = \{0 < 1 < ... < n\}$ and increasing maps.

We recall the following notation and terminology referring the reader to the work of Carrasco and Cegarra [7] for more motivation and some related results.

For the ordered set $[n] = \{0 < 1 < ... < n\}$, let $\alpha^n_i: [n+1] \to [n]$ be the increasing surjective map given by

$$\alpha^n_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

Let S(n, n-r) be the set of all monotone increasing surjective maps from [n] to [n-r]. This can be generated from the various α^n_i by composition. The composition of these generating maps is subject to the following rule $\alpha_j\alpha_i = \alpha_{i-1}\alpha_j$, with $j < i$. This implies that every element $\alpha \in S(n, n\text{-}r)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ ... \circ \alpha_{i_r}$ with $0 \leq i_1 < i_2 < ... < i_r \leq n\text{-}1$, where the indices i_k are the elements of [n] at which $\{i_1, ..., i_r\} = \{i: \alpha(i) = \alpha(i+1)\}$. We thus can identify S(n, n-r) with the set $\{(i_r, ..i_1) : 0 \leq i_1 < i_2 < ... < i_r \leq n\text{-}1\}$. In particular, the single element of S(n, n), defined by the identity map on [n], corresponds to the empty 0-tuple ( ) denoted by $\varnothing_n$. Similarly the only element of S(n, 0) is (n-1, n-2, ..., 0). For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq r \leq n} S(n, n\text{-}r).$$

We say that $\alpha = (i_r, ..., i_1) < \beta = (j_s, ..., j_1)$ in S(n)

if $i_1 = j_1, ..., i_k = j_k$ but $i_{k+1} > j_{k+1}$ $(k \geq 0)$ or

if $i_1 = j_1, ..., i_r = j_r$ and $r < s$.

This makes S(n) an ordered set. For instance, the order in S(2) and in S(3) are respectively:

$$S(2) = \{\emptyset_2 < (1) < (0) < (1, 0)\};$$

$$S(3) = \{\emptyset_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0)\}.$$

We define $\alpha \cap \beta$ as a set of indices which belong to both of them and will take the Moore complex (**NG**, ∂) of a simplicial group **G** to be defined by

$$(NG)_n = \bigcap_{i=0}^{n-1} \operatorname{Ker} d_i$$

with ∂_n: NG_n - NG_{n-1} induced from d_n by restriction. Its homology gives the homotopy groups of the simplicial algebra.

The Moore complex, **NG**, carries a hypercrossed complex structure (see Carrasco and Cegarra [7]) which allows the reconstruction of the original G. We recall briefly some of the aspects of this recontruction which we will need later.

The Semidirect Decomposition of Simplicial Group. The fundamental idea behind this can be found in Conduché [8]. A detailed investigation of this for the case of a simplicial group is given in Carassco and Cegarra [7].

Lemma 1.1. Let **G** be a simplicial group. Then G_n can be decomposed as a semidirect product:

$$G_n \cong \operatorname{Ker} d_n^n \rtimes s_{n-1}^{n-1}(G_{n-1}).$$

Proof: The isomorphism can be defined as follows:

$$\theta : G_n \to \operatorname{Ker} d_n^n \rtimes s_{n-1}^{n-1}(G_{n-1})$$

$$g \mapsto \left(g^{-1} s_{n-1} d_n g,\ s_{n-1} d_n g\right).$$

Since we have the isomorphism $G_n \cong \operatorname{Ker} d_n \rtimes s_{n-1} G_{n-1}$, we can repeat this process as often as necessary to get each of the G_n as a multiple semidirect product of degeneracies of terms in the Moore complex.

We can thus decompose G_n as follows:

Proposition 1.2. If G is a simplicial group, then for any $n \geq 0$

$$G_n \cong (\ldots (NG_n \rtimes s_{n-1} NG_{n-1}) \rtimes \ldots \rtimes s_{n-2} \ldots s_0 NG_1) \rtimes$$
$$(\ldots (s_{n-2} NG_{n-1} \rtimes s_{n-1}s_{n-2}NG_{n-2}) \rtimes \ldots \rtimes s_{n-1}s_{n-2} \ldots s_0 NG_0).$$

The bracketting and the order of terms in this multiple semidirect product are generated by the sequence:

$$G_1 \cong NG_1 \rtimes s_0 NG_0$$
$$G_2 \cong (NG_2 \rtimes s_1 NG_1) \rtimes (s_0 NG_1 \rtimes s_1 s_0 NG_0)$$
$$G_3 \cong ((NG_3 \rtimes s_2 NG_2) \rtimes (s_1 NG_2 \rtimes s_2 s_1 NG_1)) \rtimes$$
$$((s_0 NG_2 \rtimes s_2 s_0 NG_1) \rtimes (s_1 s_0 NG_1 \rtimes s_2 s_1 s_0 NG_0)).$$

and

$$G_4 \cong \big(((NG_4 \rtimes s_3 NG_3) \rtimes (s_2 NG_3 \rtimes s_3 s_2 NG_2)) \rtimes$$
$$((s_1 NG_3 \rtimes s_3 s_1 NG_2) \rtimes (s_2 s_1 NG_2 \rtimes s_3 s_2 s_1 NG_1))\big) \rtimes$$
$$s_0 \text{ (decomposition of } G_3).$$

Note that the term corresponding to $\alpha = (i_r, \ldots, i_1) \in S(n)$ is

$$s_\alpha(NG_{n-\#\alpha}) = s_{i_r \ldots i_1}(NG_{n-\#\alpha}) = s_{i_r} \ldots s_{i_1}(NG_{n-\#\alpha}),$$

where $\#\alpha = r$. Hence any element $x \in G_n$ can be written in the form

$$x = y \prod_{\alpha \in S(n)} s_\alpha(x_\alpha) \qquad \text{with } y \in NG_n \text{ and } x_\alpha \in NG_{n-\#\alpha}.$$

Crossed modules of groups. A crossed module, (M, P, ∂), is a group homomorphism $\partial\colon M \to P$, together with an action $(m, p) \mapsto m^p$ of P on M satisfying the two rules:

(CM1) $\qquad \partial(m^p) = p^{-1}\,\partial(m)p$

(CM2) $\qquad m^{-1}nm = n^{\partial m}$

for all $m, n \in M$, $p \in P$. The last condition CM2 is called the Peiffer identity. Examples of crosed modules are: an ordinary P-module, when $\partial = 0$;

a normal subgroup, when ∂ is an inclusion. There are lots of good examples of crossed modules. The notion is due to J.H.C. Whitehead [17].

The proposition above can be considered a 'lifting' of a result of Conduché stated in the next section.

2. HIGHER ORDER PEIFFER IDENTITIES

The following lemma is noted by Conduché [8]. A proof is included for completeness.

Lemma 2.1. For a simplicial group **G**, there is a bijection between

$$NG_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i \text{ and } \overline{NG}_n^{(r)} = \bigcap_{i \neq r} \mathrm{Ker} d_i$$

in G_n.

Proof. The bijection is given as follows;

$$\varphi : NG_n \to \overline{NG}_n^{(r)}$$

$$g \mapsto \varphi(g) = g^{-1} \prod_{k=0}^{n-r} s_{n-k} d_n g^{(-1)^{k-1}}.$$

Note that φ is not a homomorphism. The following is an elemantary consequence of 2.1 (cf. Carrasco and Cegarra[7]).

Lemma 2.2. Given a simplicial group **G** then we have the following

$$d_n(NG_n) = d_r\left(\overline{NG}_n^{(r)}\right).$$

Proposition 2.3. Let **G** be a simplicial group, then for $n \geq 2$ and, $I, J \subseteq [n-1]$ with $I \cup J = [n-1]$

$$\left[\bigcap_{i \in I} \mathrm{Ker} d_i , \bigcap_{j \in J} \mathrm{Ker} d_j\right] \subseteq \partial_n NG_n.$$

Proof: For any $J \subset [n-1]$, $J \neq \emptyset$, let r be the smallest element of J. If $r = 0$, then replace J by I and restart, and if $0 \in I \cap J$, then redefine r to be tha smallest nonzero element of J. Otherwise continue. Let $g_0 \in \bigcap_{j \in J} \mathrm{Ker} d_j$ and $g_1 \in \bigcap_{i \in I} \mathrm{Ker} d_i$, one obtains

$$d_i[s_{r-1}\, g_0,\, s_r g_1] = 1 \text{ for } i \neq r$$

and hence $[s_{r-1}g_0, s_r g_1] \in \overline{NG}_n^{(r)}$. It follows that

$$[g_0,g_1] = d_n[s_{r-1}g_0,\, s_r g_1] \in d_r\left(\overline{NG}_n^{(r)}\right) = d_n NG_n \text{ by the provious lemma,}$$

and this implies

$$\left[\bigcap_{i\in I} \operatorname{Ker} d_i\ ,\ \bigcap_{j\in J} \operatorname{Ker} d_j\right] \subseteq \partial_n NG_n.$$

Writting the abbreviations

$$K_I = \bigcap_{i\in I} \operatorname{Ker} d_i \text{ and } K_J = \bigcap_{j\in J} \operatorname{Ker} d_j,$$

then 2.3., becomes

$$\prod_{\{I,J\}} [K_I, K_J] \subseteq \partial_n NG_n$$

for $\varnothing \neq I,\ J \subset [n-1]$ and $I \cup J = [n-1]$.

Corollary 2.4. Let **G** be a simplicial group and let **G′** be the corresponding truncated simplicial group of order n - 1, so we have the canonical morphism **G** → **G′**. Then **G′** verifies the following property:

For all nonempty sets of indexes $(I \neq J)$ $I,\ J \subset [n-1]$ with $I \cup J = [n-1]$,

$$\left[\bigcap_{j\in J} \operatorname{Ker} d_j^{n-1}\ ,\ \bigcap_{j\in I} \operatorname{Ker} d_i^{n-1}\right] = 1.$$

Proof: Since $\partial_n NG'_n = 1$, this follows from proposition 2.3.

Hypercrossed complex pairings: In the following we will define a normal subgroup N_n. First of all we recall from Carrasco [6] the construction of a useful family of pairings. We define a set P(n) consisting of pairs of elements (α, β) from S(n) with $\alpha \cap \beta = \varnothing$ and $\beta < \alpha$ where $\alpha = (i_r, ..., i_1)$, $\beta = (j_s, ..., j_1) \in S(n)$. The pairings that we will need,

$$\{F_{\alpha,\beta} : NG_{n-\#\alpha} \times NG_{n-\#\beta} \to NG_n : (\alpha, \beta) \in P(n),\ n \geq 0\}$$

are given as composites by the diagrams

$$\begin{array}{ccc} NG_{n-\#\alpha} \times NG_{n-\#\beta} & \xrightarrow{F_{\alpha,\beta}} & NG_n \\ \downarrow{\scriptstyle s_\alpha \times s_\beta} & & \uparrow{\scriptstyle p} \\ G_n \times G_n & \xrightarrow{\mu} & G_n \end{array}$$

where

$$s_\alpha = s_{i_r} \dots s_{i_1} : NG_{n-\#\alpha} \to G_n,\ s_\beta = s_{j_s} \,..\, s_{j_1} : NG_{n-\#\beta} \to G_n,$$

$p : G_n \to NG_n$ is defined by composite projections $p = p_{n-1} \cdots p_0$ where

$$p_j(z) = zs_jd_j(z)^{-1} \qquad \text{with } j = 0, 1, \dots, n\text{-}1$$

and μ: $G_n \times G_n \to G_n$ is given by commuttor. Thus

$$\begin{aligned} F_{\alpha,\beta}(x_\alpha,y_\beta) &= p\mu(s_\alpha \times s_\beta)\ (x_\alpha, y_\beta) \\ &= p[s_\alpha(x_\alpha), s_\beta(y_\beta)]. \end{aligned}$$

We now define the normal subgroup N_n to be that generated by elements of the form

$$F_{\alpha,\beta}(x_\alpha,y_\beta)$$

where $x_\alpha \in NG_{n-\#\alpha}$ and $y_\beta \in NG_{n-\#\beta}$.

We illustrate this normal subgroup for n = 2 and n = 3 to show what it looks like.

Example. For n = 2, suppose $\beta = (0)$, $\alpha = (1)$ and x, y $\in NG_1 = \mathrm{Ker}d_0$. It follows that

$$\begin{aligned} F_{(1)(0)}(x, y) &= p_1p_0([s_0(x), s_1(y)]) \\ &= p_1[s_0(x), s_1(y)] \\ &= [s_0(x), s_1(y)]\ [s_1(y), s_1(x)] \end{aligned}$$

which is a generator element of the normal subgroup N_2.

For n = 3, the linear morphisms are the following

$F_{(1,0)(2)}$, $F_{(2,0)(1)}$, $F_{(2,1)(0)}$,

$F_{(2)(0)}$, $F_{(2)(1)}$, $F_{(1)(0)}$.

For all $x \in NG_1$, $y \in NG_2$, the corresponding generators of N_3 are:

$$F_{(1,0)(2)}(x, y) = [s_1s_0(x), s_2(y)][s_2(y), s_2s_0(x)],$$

$$F_{(2,0)(1)}(x, y) = [s_2s_0(x), s_1(y)][s_1(y), s_2s_1(x)][s_2s_1(x), s_2(y)][s_2(y), s_2s_0(x)]$$

and $x \in NG_2$, $y \in NG_1$,

$$F_{(2,1)(0)}(x, y) = [s_2s_1(x), s_0(y)][s_1(y), s_2s_1(x)][s_2s_1(x), s_2(y)];$$

whilst for all $x,y \in NG_2$,

$$F_{(1)(0)}(x, y) = [s_1(x), s_0(y)][s_1(y), s_1(x)][s_2(x), s_2(y)],$$

$$F_{(2)(0)}(x, y) = [s_2(x), s_0(y)],$$

$$F_{(2)(1)}(x, y) = [s_2(x), s_1(y)][s_2(y), s_2(x)].$$

In the following we analyse various types of elements in N_n and show that products of them give elements that we want in giving an alternative description of $\partial_n NG_n$ in certain cases.

Lemma 2.5. Given $x_\alpha \in NG_{n-\#\alpha}$, $y_\beta \in NG_{n-\#\beta}$ with $\alpha = (i_r, ..., i_1)$, $\beta = (j_s, ..., j_1) \in S(n)$. If $\alpha \cap \beta = \emptyset$ with $\beta < \alpha$ and $u = [s_\alpha(x_\alpha), s_\beta(y_\beta)]$, then

(i) if $k \leq i_1$, then $p_k(u) = u$,

(ii) if $k > i_r + 1$ or $k > j_s + 1$, then $p_k(u) = u$,

(iii) if $k \in \{j_1, ..., j_s, j_s+1\}$ and $k = i_\ell + 1$ for some ℓ, then

$$p_k(u) = [s_\alpha(x_\alpha), s_\beta(y_\beta)]s_k(z_k)^{-1},$$

(iv) if $k \in \{i_1, ..., i_r, j_r + 1\}$ and $k = j_m + 1$ for some m, then

$$p_k(u) = [s_\alpha(x_\alpha), s_\beta(y_\beta)]s_k(z_k)^{-1},$$

where $z_k \in G_{n-1}$ and $0 \leq k \leq n-1$.

Proof: Assuming $\beta < \alpha$ and $\alpha \cap \beta = \varnothing$ which implies $i_1 < j_1$. In the range $0 \leq k \leq i_1$,

$$p_k(u) = [s_\alpha(x_\alpha), s_\beta(y_\beta)] \text{ since } d_k(y_\beta) = 1.$$

Similarly if $k > i_r + 1$, then $p_k(u) = [s_\alpha(x_\alpha), s_\beta(y_\beta)]$ since $d_{k-r}(x_\alpha) = 1$. Clearly the same sort of argument works if $k > j_s + 1$.

If $k \in \{j_1, ..., j_r, j_r + 1\}$ and $k = i_\ell + 1$ for some ℓ, then $p_k(u) = [s_\alpha(x_\alpha), s_\beta(y_\beta)]s_k(z_k)^{-1}$ where $z_k = [s_{\alpha'}(x_{\alpha'}), s_{\beta'}(y_{\beta'})] \in G_{n-1}$ for new strings α', β' as it is clear. The proof of (iv) is same so we will leave it out.

Lemma 2.6. If $\alpha \cap \beta = \varnothing$ then,

$$p_{n-1} \cdots p_0[s_\alpha(x_\alpha), s_\beta(y_\beta)] = [s_\alpha(x_\alpha), s_\beta(y_\beta)] \prod_{k=1}^{n-1} s_k(z_k)^{-1}$$

where $z_k \in G_{n-1}$.

Proof: We prove this by using the induction hypothesis on n. Write $u = [s_\alpha(x_\alpha), s_\beta(y_\beta)]$. For $n = 1$, it is clear to see that the equality is verified. We suppose that it is true for $n - 2$. It then follows that

$$\begin{aligned} p_{n-1} \cdots p_0(u) &= p_{n-1}\left(u\prod_{k=1}^{n-2} s_k(z_k)^{-1}\right) \\ &= p_{n-1}(u)\, p_{n-1}\left(\prod_{k=1}^{n-2} s_k(z_k)^{-1}\right). \end{aligned}$$

Next look at $p_{n-1}(u) = us_{n-1}\underbrace{\left(d_{n-1}u^{-1}\right)}_{z'} = us_{n-1}(z')^{-1}$ and

$$\begin{aligned} p_{n-1}\left(\prod_{k=1}^{n-2} s_k(z_k)^{-1}\right) &= \prod_{k=1}^{n-2} s_k(z_k)^{-1} s_{n-1}\underbrace{\left(\prod_{k=1}^{n-2} s_k(z_k)^{-1}\right)^{-1}}_{z''} \\ &= \prod_{k=1}^{n-2} s_k(z_k)^{-1} s_{n-1}(z'')^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} p_{n-1} \cdots p_0(u) &= u\prod_{k=1}^{n-2} s_k(z_k)^{-1} s_{n-1}\underbrace{(z'z'')}_{z_{n-1}}{}^{-1}. \\ &= u\prod_{k=1}^{n-2} s_k(z_k)^{-1} s_{n-1}(z_{n-1})^{-1} \\ &= u\prod_{k=1}^{n-2} s_k(z_k)^{-1}. \end{aligned}$$

as required.

Lemma 2.7. Let $x_\alpha \in NG_{n-\#\alpha}$, $y_\beta \in NG_{n-\#\beta}$ with α, $\beta \in S(n)$. If $\alpha \cap \beta \neq 0$, then

$$[s_\alpha(x_\alpha), s_\beta(y_\beta)] = s_{\alpha\cap\beta}(z_{\alpha\cap\beta})$$

where $z_{\alpha\cap\beta}$ has the form $[s_{\alpha'}(x_\alpha), s_{\beta'}(y_\beta)]$ and $\alpha' \cap \beta' \neq \emptyset$.

Proof: If $\alpha' \cap \beta' \neq \emptyset$, then this is trivially true. Assume $\#(\alpha \cap \beta) = t$, with $t \in \mathbb{N}$. Take $\alpha = (i_r, \ldots, i_1)$ and $\beta = (j_s, \ldots, j_1)$ with $\alpha \cap \beta = (k_t, \ldots, k_1)$,

$$s_\alpha(x_\alpha) = s_{i_r} \ldots s_{k_t} \ldots s_{i_1}(x_\alpha) \text{ and } s_\beta(y_\beta) = s_{j_s} \ldots s_{k_t} \ldots s_{j_1}(y_\beta).$$

Using repeatedly the simplicial axiom $s_e s_d = s_d s_{e-1}$ for $d < e$ until obtaining that $s_{k_t} \ldots s_{k_1}$ is at beginning of the string, one gets the following

$$s_\alpha(x_\alpha) = s_{k_t} \ldots {}_{k_1}(s_{\alpha'} x_\alpha) \text{ and } s_\beta(y_\beta) = s_{k_t} \ldots s_{k_1}(s_{\beta'} y_\beta).$$

and take the commutator

$$[s_\alpha(x_\alpha), s_\beta(y_\beta)] = [s_{k_t} \ldots s_{k_1}(s_{\alpha'} x_\alpha), s_{k_t} \ldots s_{k_1}(s_{\beta'} y_\beta)]$$

$$= s_{k_t} \ldots s_{k_1}[s_{\alpha'}(x_\alpha), s_{\beta'}(y_\beta)]$$

$$= s_{\alpha\cap\beta}(z_{\alpha\cap\beta}),$$

where $z_{\alpha\cap\beta} = [s_{\alpha'}(x_\alpha), s_{\beta'}(y_\beta)] \in G_{n-\#(\alpha\cap\beta)}$ and where $\alpha \backslash \alpha \cap \beta = \alpha'$, $\beta \backslash \alpha \cap \beta = \beta'$. Hence $\alpha' \cap \beta' = \emptyset$. Moreover $\alpha' < \alpha$ and $\beta' < \beta$ as $\#\alpha' < \#\alpha$ and $\#\beta' < \#\beta$.

Proposition 2.8. Let $\mathbf{G}$ be a simplicial group and $n > 0$, and D_n the normal subgroup in G_n generated by degenerate elements. We suppose $G_n = D_n$, and let N_n be the normal subgroup generated by elements of the form

$$F_{\alpha,\beta}(x_\alpha, y_\beta) \quad \text{with } (\alpha,\beta) \in P(n)$$

where $x_\alpha \in NG_{n-\#\alpha}$, $y_\beta \in NG_{n-\#\beta}$ with $1 \leq r,s \leq n$. Then

$$\partial_n(NG_n) = \partial_n(N_n).$$

Proof: From proposition 1.2, G_n is isomorphic to

$$NG_n \rtimes s_{n-1} NG_{n-1} \rtimes s_{n-2} NG_{n-1} \rtimes \ldots \rtimes s_{n-1}s_{n-2} \ldots s_0 NG_0,$$

here $NG_n = \bigcap_{i=0}^{n-1} \operatorname{Ker} d_i$ and $NG_0 = G_0$. Hence any element x in G can be written in the following from

$$x = g_n s_{n-1}(x_{n-1})\, s_{n-2}(x'_{n-1}) s_{n-1}s_{n-2}(x_{n-2}) \ldots s_{n-1}s_{n-2} \ldots s_0(x_0),$$

with $g_n \in NG_n$, x_{n-1}, $x'_{n-1} \in NG_{n-1}$, $x_{n-2} \in NG_{n-2}$, $x_0 \in NG_0$ etc.

We start by comparing N_n with NG_n. We show $NG_n = N_n$. It is enough to prove that, equivalently, any element in G_n/N_n can be written

$$s_{n-1}(x_{n-1})\, s_{n-2}(x'_{n-1})\, s_{n-1}s_{n-2}(x_{n-2}) \ldots s_{n-1}s_{n-2} \ldots s_0(x_0)N_n$$

which implies, for any $b \in G_n$,

$$bN_n = s_{n-1}(x_{n-1})\, s_{n-2}(x'_{n-1}) \ldots s_{n-1}s_{n-2} \ldots s_0(x_0)N_n.$$

for some $x_{n-1} \in NG_{n-1}$ etc.

If $b \in G_n$, it is a product of degeneracies so first of all assume it to be a product of degeneracies and that will suffice for the general case.

If b is itself a degenerate element, it is obvious that it is in some semidirect factor $s_\alpha(G_{n-\#\alpha})$. Assume therefore that provided an element b can be written as a commutator of k-1 degeneracies it has the desired form mod N_n, now for an element b which needs k degenerate elements

$$b = [s_\beta(y_\beta), b'] \quad \text{with } y_\beta \in NG_{n-\#\beta}$$

where b' needs fewer than k and so

$$\begin{aligned} bN_n &= \left[s_\beta(y_\beta), b'\right] N_n \\ &= \left[s_\beta(y_\beta),\, s_{n-1}(x_{n-1})\, s_{n-2}(x'_{n-1}) \ldots s_{n-1}s_{n-2} \ldots s_0(x_0)\right]N_n \\ &= \prod_{\alpha \in S(n)} \left[s_\alpha(x_\alpha),\, s_\beta(y_\beta)\right]N_n. \end{aligned}$$

Next we ignore this product for a moment and just look at

$$[s_\alpha(x_\alpha),\, s_\beta(y_\beta)] \qquad (*).$$

We check this commutator case by case as follows:

If $\alpha \cap \beta = \emptyset$, then there exists by lemma 2.5 and 2.6, an element

$$\left[s_\alpha(x_\alpha), s_\beta(y_\beta)\right] \prod_{k=1}^{n-1} s_k(z_k)^{-1}$$

in N_n with $z_k \in G_{n-1}$ and $k \in \alpha$ so that

$$\left[s_\alpha(x_\alpha), s_\beta(y_\beta)\right] \equiv \prod_{k=1}^{n-1} s_k(z_k) \quad \text{mod } N_n.$$

If $\alpha \cap \beta = \emptyset$, then one gets from lemma 2.7, the following

$$[s_\alpha(x_\alpha), s_\beta(y_\beta)] = s_{\alpha\cap\beta}(z_{\alpha\cap\beta})$$

where $z_{\alpha\cap\beta} = [s_{\alpha'}(x_{\alpha'}), s_{\beta'}(y_{\beta'})] \in G_{n-\#(\alpha\cap\beta)}$ with $\#(\alpha \cap \beta) = t \in \mathbb{N}$. Since $\alpha' \cap \beta' = \emptyset$, we can use lemma 2.6 to from an equailty

$$\left[s_\alpha(x_\alpha), s_\beta(y_\beta)\right] \equiv \prod_{k=1}^{n-1} s_{k'}(z_{k'}) \quad \text{mod } N_n$$

where $z_{k'} \in E_{n-1}$. It then follows that

$$s_{\alpha\cap\beta}(z_{\alpha\cap\beta}) = s_{\alpha\cap\beta}\left[s_{\alpha'}(x_\alpha), s_{\beta'}(y_\beta)\right]$$

$$\equiv \prod_{k=1}^{n-1} s_{\alpha\cap\beta}s_{k'}(z_{k'}) \quad \text{mod } N_n.$$

Thus we have shown that every commutator which can be formed in the required form are in N_n. Therefore $\partial v(N_n) = \partial_n(NG_n)$.

3. THE CASES n = 2 AND n = 3

3.1. Case n = 2

We know that any element g_2 of G_2 can be expressed in the form

$$g_2 = bs_1ys_0xs_0u$$

with $b \in NG_2$, $x, y \in NG_1$ and $u \in s_0G_0$. We suppose $D_2 = G_2$. For $n = 1$, we take $\beta = (1)$, $\alpha = (0)$ and $x, y \in NG_1 = \text{Ker}d_0$. The normal subgroup N_2 is generated by elements of the form

$$F_{(1)(0)}(x, y) = [s_0(x), s_1(y)][s_1(y), s_1(x)].$$

The image of N_2 by ∂_2 is known to be $[\text{Ker}d_1, \text{Ker}d_0]$ by direct calculation. Indeed,

$$d_2[F_{(1)(0)}(x, y)] = d_2([s_0(x), s_1(y)][s_1(y), s_1(x)])$$
$$= [s_0d_1(x), y][y, x]$$

where $y \in Kerd_0$ and $x^{-1}s_0d_1(x) \in Kerd_1$ and all elements of $Kerd_1$ have this form since lemma 2.1.

The usefulness of the above for us is that it gives us a way of constructing a crossed module directly from a simplicial group.

We consider the truncated simlicial group of order 2.

$$\mathbf{G} \ : \ G_1/\partial_2NG_2 \overset{d_0,d_1}{\underset{s_0}{\rightleftarrows}} G_0$$

To get a crossed module we merely have to divide NG_1 by ∂_2NG_2 (which is the same as $[Kerd_1, Kerd_0]$). The crossed module is

$$\delta : NG_1/\partial_2NG_2 \to NG_0$$

where δ is induced by d_1. NG_0 acts on NG_1/∂_2NG_2 by multiplication via s_0 i.e.

$$NG_1/\partial_2NG_2 \times NG_0 \to NG_1/\partial_2NG_2$$
$$(\bar{x}, y) \mapsto {}^{\bar{x}}y = \overline{s_0(y)xs_0(y)^{-1}}$$

where $\bar{x}$ denotes the corresponding element of NG_1/∂_2NG_2 whilst $x \in NG_1$.$(NG_1/\partial_2NG_2, NG_0, \delta)$ is the crossed module. We note:

For all $x\partial_2NG_2$, $y\partial_2NG_2$ with $x, y \in NG_1$,

$$\begin{aligned} {}^{\delta(x\partial_2NG_2)}(y\partial_2NG_2) &= {}^{\delta(x)\partial_2NG_2}(y\partial_2NG_2) \\ &= {}^{d_1(x)}y\partial_2NG_2 \\ &= s_0d_1(x)ys_0d_1(x)^{-1}\partial_2NG_2 \quad \text{by the action} \\ &= xyx^{-1}\partial_2NG_2 \quad \text{mod } \partial_2NG_2 \\ &= (x\partial_2NG_2)(y\partial_2NG_2)(x^{-1}\partial_2NG_2) \end{aligned}$$

as required.

3.2. Case n = 3

This subsection provides analogues in dimension 3 of the Peiffer elements.

Proposition 3.1.

$$\partial_3(NG_3) = \prod_{\{I,J\}} [K_I, K_J] \left([K_{\{0,2\}}, K_{\{0,1\}}]\, [K_{\{1,2\}}, K_{\{0,2\}}]\, [K_{\{1,2\}}, K_{\{0,1\}}]\right)$$

where $I \cup J = [2]$, $I \cap J = \emptyset$.

Proof: By proposition 2.8, we know the generator elements of the normal subgroup N_3 and $\partial_3(N_3) = \partial_3(NG_3)$. For each pair $\alpha, \beta \in S(3)$ with $\emptyset_3 < \alpha < \beta$ and $\alpha \cap \beta = \emptyset$, we take $x \in NG_{3-\#\alpha}$, $y \in NG_{3-\#\beta}$ and set $F_{\alpha,\beta}(x, y) = p_3p_2p_1[s_\alpha(x), s_\beta(y)]$ where $p_i(g) = gs_{i-1}d_ig^{-1}$. This element is thus in NG_3. The valid pairs together with their corresponding pairing functions is given in the following table:

	α	β	$F_{\alpha,\beta}(x, y)$
1	(1, 0)	(2)	$[s_1s_0(x), s_2(y)][s_2(y), s_2s_0(x)]$
2	(2, 0)	(1)	$[s_2s_0(x),s_1(y)][s_1(y),s_2s_1(x)][s_2s_1(x),s_2(y)][s_2(y),s_2s_0(x)]$
3	(2, 1)	(0)	$[s_2s_1(x), s_0(y)][s_1(y), s_2s_1(x)][s_2s_1(x), s_2(y)]$
4	(2)	(1)	$[s_2(x), s_1(y)][s_2(y), s_2(x)]$
5	(2)	(0)	$[s_2(x), s_0(x)]$
6	(1)	(0)	$[s_1(x), s_0(y)][s_1(y), s_1(x)][s_2(x), s_2(y)]$

The explanation of this table is the following:

Row 1. Firstly we look at the case of $\alpha = (1, 0)$ and $\beta = (2)$. For $x \in NG_1$ and $y \in NG_2$,

$$d_3(F_{(1,0)(2)}(x, y)) = d_3[s_1s_0(x), s_2(y)][s_2(y), s_2s_0(x)]$$
$$= [s_1s_0d_1(x), y][y, s_0(x)]$$

and so

$$d_3(F_{(1,0)(2)}(x, y)) = [s_1s_0d_1(x), y][y, s_0(x)] \in [\mathrm{Ker}d_2, \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1].$$

We have denoted $[\mathrm{Ker}d_2, \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1]$ by $[K_{\{2\}}, K_{\{0,1\}}]$ where $I = \{2\}$ and $J = \{0, 1\}$.

Row 2. For $\alpha = (2, 0)$ and $\beta = (1)$ with $x \in NG_1$, $y \in NG_2$,

$$d_3(F_{(2,0)(1)}(x,y)) = d_3[s_2s_0(x),s_1(y)][s_1(y),s_2s_1(x)][s_2s_1(x),s_2(y)][s_2(y),s_2s_0(x)]$$
$$= [s_0(x),\ s_1d_2(y)][s_1d_2(y),\ s_1(x)][s_1(x),\ y][y,\ s_0(x)]$$

and so

$$d_3(F_{(2,0)(1)}(x,\ y)) \in [Kerd_1,\ Kerd_0 \cap Kerd_2] = [K_{\{1\}},\ K_{\{0,2\}}].$$

Row 3. For $\alpha = (2, 1)$ and $\beta = (0)$ with $x \in NG_1$, $y \in NG_2$,

$$d_3(F_{(2,1)(0)}(x,\ y)) = d_3\big([s_2s_1(x),s_0(y)][s_1(y),s_2s_1(x)][s_2s_1(x),s_2(y)]\big)$$
$$= [s_1(x),\ s_0d_2(y)][s_1d_2(y),\ s_1(x)][s_1(x),\ y]$$

and hence

$$d_3(F_{(2,1)(0)}(x,\ y)) \in [Kerd_1 \cap Kerd_2,\ Kerd_0] = [K_{\{1,2\}},\ K_{\{0\}}].$$

Row 4. For $\beta = (1)$ and $\alpha = (2)$ with $x, y \in NG_2 = Kerd_0 \cap Kerd_1$,

$$d_3(F_{(2)(1)}(x,\ y)) = d_3\big([s_2(x),s_1(y)][s_2(y),s_2(x)]\big)$$
$$= [x,\ s_1d_2(y)][y,\ x].$$

It follows that

$$d_3(F_{(2)(1)}(x,\ y)) \in [Kerd_0 \cap Kerd_2,\ Kerd_0 \cap Kerd_1]$$
$$= [K_{\{0,2\}},\ K_{\{0,1\}}].$$

Row 5. For $\beta = (0)$ and $\alpha = (2)$ with $x, y \in NG_2 = Kerd_0 \cap Kerd_1$,

$$d_3(F_{(2)(0)}(x,\ y)) = d_3[s_2(x),s_0(y)]$$
$$= [x,\ s_0d_2(y)]$$

We can assume, for $x, y \in NG_2$,

$$x \in Kerd_0 \cap Kerd_1 \text{ and } ys_0d_2(y)s_1d_2(y)^{-1} \in Kerd_1 \cap Kerd_2$$

and

$$
\begin{aligned}
[x, ys_0d_2(y)s_1d_2(y)^{-1}] &= [x, y][x, xs_0d_2(y)][s_1d_2y, x] \\
&= [x, s_1d_2y]\ [y, x][x, s_0d_2y] \\
&= d_3(F_{(2)(1)}(x, y))d_3(F_{(2)(0)}(x, y))
\end{aligned}
$$

and so

$$
\begin{aligned}
d_3(F_{(2)(1)}(x, y)) &\in [K_{\{1,2\}}, K_{\{0,1\}}]\ d_3(F_{(1)(2)}(x, y)) \\
&\subseteq [K_{\{1,2\}}, K_{\{0,1\}}][K_{\{0,2\}}, K_{\{0,1\}}].
\end{aligned}
$$

Row 6. For $\beta = (0)$ and $\alpha = (1)$ and $x, y \in NG_2 = Kerd_0 \cap Kerd_1$,

$$
\begin{aligned}
d_3(F_{(1)(0)}(x, y)) &= d_3\big([s_1(x),s_0(y)][s_1(y),s_1(x)][s_2(x),s_2(y)]\big) \\
&= [s_1d_2(x), s_0d_2(y)][s_1d_2(y), s_1d_2(x)][x, y]
\end{aligned}
$$

We can take the following elements

$$xs_1d_2(x)^{-1}s_0d_2(x) \in Kerd_1 \cap Kerd_2 \text{ and } s_1d_2(y)y^{-1} \in Kerd_0 \cap Kerd_2$$

When taking the commutator of these elements, one get

$$
\begin{aligned}
&[xs_1d_2(y)^{-1}, s_1d_2(y)y^{-1}] = \\
&{}^{xs_1d_2(x)^{-1}y^{-1}}\big([y, xs_0d_2(x)][s_0d_2(x), s_0d_2(y)][s_1d_2(x), s_1d_2(y)][y, x]\big) \\
&{}^{xs_1d_2(x)^{-1}y^{-1}}\big\{[y, x][s_1d_2(x), y]\big\}^{y^{-1}}\big\{[y, x][x, s_1d_2(y)]\big\} \\
&[s_0d_2(x),y]^{xs_1d_2(x)x^{-1}}\big\{[xs_1d_2(x)^{-1},s_0d_2(x),s_1d_2(y)y^{-1}]^{y^{-1}}\big\{[s_1d_2(x),y][x,y]\big\}\big\} \\
&\qquad [s_1d_2(x),y][y,x]
\end{aligned}
$$

and hence

$$d_3(F_{(1)(0)}(x, y)) \in [K_{\{1,2\}},K_{\{0,1\}}][K_{\{0,2\}},K_{\{0,1\}}][K_{\{1,2\}},K_{\{0,2\}}][K_{\{0,2\}},K_{\{0,1\}}]$$

So we have shown

$$\partial_3(NG_3) \subseteq \prod_{\{I,J\}} [K_I, K_J]([K_{\{0,2\}}, K_{\{0,1\}}][K_{\{1,2\}},K_{\{0,2\}}][K_{\{1,2\}},K_{\{0,1\}}]).$$

The opposite inclusion can be verified by using proposition 2.3. Therefore

$$\begin{aligned}\partial_3(NG_3) = {} & [Kerd_2, Kerd_0 \cap Kerd_1][Kerd_1, Kerd_0 \cap Kerd_2] \\ & [Kerd_1 \cap Kerd_2, Kerd_0][Kerd_0 \cap Kerd_2, Kerd_0 \cap Kerd_1] \\ & [Kerd_0 \cap Kerd_2, Kerd_1 \cap Kerd_2][Kerd_0 \cap Kerd_1, Kerd_1 \cap Kerd_2].\end{aligned}$$

This completes the proof of the proposition.

4. APPLICATIONS TO 2-CROSSED MODULES AND CROSSED SQUARES

Generating elements of $\partial_3 NG_3$ allow us to examine the identities to be satisfied in truncated simplicial groups of order 3, i.e.

$$\mathbf{G}^2 \ : \ G_2/\partial_3 NG_3 \underset{s_0,s_1}{\overset{d_0,d_1,d_2}{\rightleftarrows}} G_1 \underset{s_0}{\overset{d_0,d_1}{\rightleftarrows}} G_0$$

Dividing NG_2 by $\partial_3 NG_3$ gives a 2-crossed module of commutative groups. Before verifying this we recall from [8] the definition of 2-crossed module:

Definition 4.1. A 2-crossed module of groups consists of a complex of C_0-groups

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

and ∂_2, ∂_1 morphisms of C_0-groups, where the group C_0 acts on itself by multiplication such that $\partial_2 : C_2 \to C_1$ is a crossed module. Thus C_1 act on C_2 via C_0 and we require that for all $x \in C_2$, $y \in C_1$ and $z \in C_0$, there is a C_0-bilinear function, i.e. the Peiffer lifting, giving

$$\{\ ,\ \} : C_1 \times C_1 \to C_2$$

which verifies the following axioms:

$$\begin{array}{lrl}
\text{PL1:} & \partial_2\{y_0, y_1\} & = {}^{\partial_1 y_0}y_1{}^{-1} y_0 y_1 y_0{}^{-1}, \\
\text{PL2:} & \{\partial_2(x_1), \partial_2(x_2)\} & = [x_2, x_1], \\
\text{PL3:} & \{y, \partial_2(x_2)\}\{\partial_2(x), y\} & = {}^{\partial_1 y}x x^{-1}, \\
\text{PL4:} & \{y_0, y_1 y_2\} & = \{y_1, y_2\}^{(y_0 y_1 y_0^{-1})}\{y_0, y_2\} \\
\text{PL5:} & \{y_0 y_1, y_2\} & = {}^{\partial_1 y_0}\{y_1, y_2\}\{y_0, y_1 y_2 y_1{}^{-1}\}, \\
\text{PL6:} & {}^{z}\{y_0, y_1\} & = \{{}^{z}y_0, {}^{z}y_1\},
\end{array}$$

for all x_1, $x_2 \in C_2$, y_0, y_1, $y_2 \in C_1$ and $z \in C_0$.

Proposition 4.2. Let **G** be a simplicial group with the Moore complex **NG**. Then the complex of groups

$$NG_2/\partial_3(NG_3 \cap D_3) \xrightarrow{\bar{\partial}_2} NG_1 \xrightarrow{\partial_1} NG_0$$

is a 2-crossed module of groups, where the Peiffer map is defined as follows:

$$\{\ ,\ \} : NG_1 \times NG_1 \to NG_2/\partial_3(NG_3 \cap D_3)$$

given by $\{x, y\} \mapsto \overline{\left(s_0xs_1ys_0x^{-1}\right)\left(s_1xs_1y^{-1}s_1x^{-1}\right)}$

Proof: We will show that all axioms of a 2-crossed module are verified. It is readily checked that morphism

$$\bar{\partial}_2 : NG_2/\partial_3(NG_3 \cap D_3) \to NE_1$$

is a crossed module (see proposition 3.4). In the following calculations we display the elements omitting the overlines as:

PL1:

$$\begin{aligned}\bar{\partial}_2\{y_0,y_1\} &= \partial_2\left((s_0(y_0)s_1(y_1)s_0(y_0)^{-1})(s_1(y_0)s_1(y_1)^{-1}s_1(y_0)^{-1})\right) \\ &= (s_0d_1(y_0)y_1s_0d_1(y_0)^{-1})(y_0y_1^{-1}y_0^{-1}) \\ &= {}^{\partial_1 y_0}y_1y_0y_1^{-1}y_0^{-1}.\end{aligned}$$

PL2: From

$$d_3(F_{(1)(0)}(x_1, x_2)) = [s_0d_2(x_1), s_1d_2(x_2)][s_1d_2(x_2), s_1d_2(x_1)][x_1, x_2]$$

$\in \partial_3(NG_3 \cap D_3)$, one obtains

$$\begin{aligned}\{\bar{\partial}_2(x_1), \bar{\partial}_2(x_2)\} &= (s_0d_2(x)s_1d_2(y)s_0d_2x^{-1})(s_1d_2(x)s_1d_2y^{-1}s_1d_2x^{-1}) \\ &\equiv [x_2, x_1] \bmod \partial_3(NG_3 \cap D_3)\end{aligned}$$

PL3: a) From $d_3(F_{(0)(2,1)}(y,x))=[s_0d_2(y)s_1(x)][s_1(x)s_1d_2(y)][y,s_1(x)]$ $\in \partial_3(NG_3 \cap D_3)$,

$$\begin{aligned}\{\bar{\partial}_2(x), y\} &\equiv [s_1(y), x] \bmod \partial_3(NG_3 \cap D_3) \\ &= s_1(y)xs_1(y)^{-1}x^{-1} \\ &= ({}^{y}x)x^{-1} \quad \text{by the definition of the action}\end{aligned}$$

and b) since $d_3(F_{(1,0)(2)}(x, y))$ and $d_3(F_{(2,0)(1)}(x, y))$ are in $\partial_3(NG_3 \cap D_3)$

$$\begin{aligned}\{y, \bar{\partial}_2(x)\} &= s_0y(s_1d_2xs_0y^{-1}s_1ys_1d_2x^{-1})s_1y^{-1}\\ &\equiv (s_1(y)xs_1(y)^{-1})(s_0(y)x^{-1}s_0(y)^{-1}) \bmod \partial_3(NG_3 \cap D_3)\\ &\equiv (s_1s_0d_1(y)xs_1s_0d_1(y)^{-1})(s_0(y)x^{-1}s_0(y)^{-1}) \bmod \partial_3(NG_3 \cap D_3)\\ &= ({}^{\partial_1(y)}x)^y x^{-1} \quad \text{by the definition of the action}\end{aligned}$$

and thus

$$\{y, \bar{\partial}_2(x)\}\{\bar{\partial}_2(x), y\} = {}^{\partial_1(y)}xx^{-1}$$

PL4: The following equalities are easily verified:

$$\begin{aligned}\{y_0, y_1y_2\} &= s_0(y_0)s_1(y_1)s_2(y_2)s_0(y_0)^{-1}\\ &\quad s_1(y_0)s_1(y_2)^{-1}s_1(y_1)^{-1}s_1(y_0)^{-1}\\ &= s_0(y_0)s_1(y_1)s_0(y_0)^{-1}s_1(y_0)s_1(y_1)^{-1}s_1(y_0)^{-1}\\ &\quad s_1(y_0)s_1(y_1)s_1(y_0)^{-1}s_0(y_0)s_1(y_2)s_0(y_0)^{-1}\\ &\quad s_1(y_0)s_1(y_2)^{-1}s_1(y_0)^{-1}s_1(y_0)s_1(y_1)^{-1}s_1(y_0)^{-1}\\ &\quad s_1(y_2)^{-1}s_1(y_0)^{-1}s_1(y_0)s_1(y_1)^{-1}s_1(y_0)^{-1}\\ &= \{y_0, y_1\}^{(y_0y_1y_0^{-1})}\{y_0, y_2\}.\end{aligned}$$

PL5:

$$\begin{aligned}\{y_0y_1, y_2\} &= s_0(y_0)s_0(y_1)s_1(y_2)s_0(y_1)^{-1}s_0(y_0)^{-1}\\ &\quad s_1(y_0)s_1(y_1)s_1(y_2)^{-1}s_1(y_1)^{-1}s_1(y_0)^{-1}\\ &= s_0(y_0)s_0(y_1)s_1(y_2)s_0(y_1)^{-1}s_1(y_1)s_1(y_2)^{-1}\\ &\quad s_1(y_1)s_0(y_1)^{-1}s_0(y_0)s_1(y_1)s_1(y_2)s_1(y_1)^{-1}\\ &\quad s_0(y_0)s_1(y_0)s_1(y_1)s_1(y_2)s_1(y_1)^{-1}s_1(y_0)^{-1}\\ &= {}^{\partial_1 y_0}\{y_1, y_2\}\{y_0, y_1y_2y_1^{-1}\}.\end{aligned}$$

PL6:

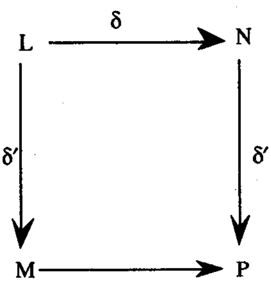
$$ {}^{z}\{y_0, y_1\} = s_1s_0(z)s_0y_0s_1y_1s_0y_0^{-1}s_1s_0(z)^{-1}s_1s_0(z)s_1y_0s_1y_1^{-1}s_1y_0^{-1}s_1s_0(z)^{-1} $$

$$ = \{{}^{z}y_0, {}^{z}y_1\}. $$

where $x, x_1, x_2 \in NG_2/\partial_3(NG_3 \cap D_3)$, $y, y_0, y_1, y_2 \in NG_1$ and $z \in NG_0$. This completes the proof of the proposition.

Crossed squares of groups are another type of 2-dimensional crossed module defined by D. Guin-Walery and J.L. Loday [12]. Thus we show that $NG_2/\partial_3(NG_3 \cap D_3)$ occurs in the crossed square.

A crossed square of groups is a commutative diagram of groups.



together wtih actions of P on L, M and N. There are thus actions of M on L and N via ∂, and N act on L and N via ∂' and a function $h: M \times N \to L$ such that, for all $m, m' \in M$, $n, n' \in N$, $p \in P$, $\ell \in L$,;

1. each of the maps δ, δ', ∂, ∂' and the composite $\partial'\delta = \partial\delta'$ are crossed modules
2. the maps δ, δ' preserve the action of P
3. $h(mm',n) = {}^{m}h(m', n)\, h(m, n)$
4. $h(m, nn') = h(m, n)^{n}\, h(m, n')$
5. ${}^{p}h(m,n) = h({}^{p}m, {}^{p}n)$
6. $\delta h(m, n) = {}^{m}nn^{-1}$
7. $\delta' h(m, n) = m^{n}m^{-1}$
8. $h(m, \delta l) = {}^{m}\ell\ell^{-1}$
9. $h(\delta' 1, n) = \ell^{n}\ell^{-1}$.

Proposition 4.3. The following diagram

$$\begin{array}{ccc} NG_2/\partial_3(NG_3 \cap D_3) & \xrightarrow{\delta} & NG_1 \\ \Big\downarrow{\scriptstyle \delta'} & & \Big\downarrow{\scriptstyle \partial'} \\ \overline{NG}_1 & \xrightarrow[\partial]{} & G_1 \end{array}$$

is a crossed square. Here $NG_1 = \mathrm{Ker}d^1_0$ and $\overline{NG}_1 = \mathrm{Ker}d^1_1$.

Proof: Since G_1 acts on $NG_2/\partial_3(NG_3 \cap D_3)$, $\overline{NG}_1$ and NG_1, there are actions of $\overline{NG}_1$, on $NG_2/\partial_3(NG_3 \cap D_3)$ and NG_1 via, ∂, and NG_1 act on $NG_2/\partial_3(NG_3 \cap D_3)$ and $\overline{NG}_1$ via ∂', As ∂ and ∂' are inclusions, all actions can be given by conjugation. The h-map is

$$NG_1 \times \overline{NG}_1 \to NG_2/\partial_3(NG_3 \cap D_3)$$
$$(x, \bar{y}) \mapsto h(x, \bar{y}) = \left(s_0(x)s_1(\bar{y})s_0(x)^{-1}s_1(x)s_1(\bar{y})^{-1}s_1(x)^{-1}\right)\partial_3(NG_3 \cap D_3),$$

where x and y are in NG_1 as there exists a bijection between NG_1 and $\overline{NG}_1$ (by lemma 2.1). It is routine to check that the axioms of crossed sqare holds.

To summarise we have:

Theorem 4.4. Let n = 2, 3 and let **G** be a simplicial group with Moore complex **NG** in which $G_n = D_n$, Then

$$\partial_n(NG_n) = \prod_{\{I,J\}} [K_I, K_J]$$

for any $I, J \subseteq [n - 1]$ with $I \cup J = [n - 1]$, $I = [n - 1] - \{\alpha\}$ and $J = [n - 1] - \{\beta\}$, where $(\alpha, \beta) \in P(n)$.

Theorem 4.5. If $G_n \neq D_n$, then

$$\partial_n(NG_n \cap D_n) = \prod_{\{I,J\}} [K_I, K_J] \qquad \text{with } n = 2, 3.$$

REFERENCES

[1] M. ANDRÉ, Homotologie des algèbres commutatives, *Springer-Verlag*, Die Grundlehren der mathematicshen Wissenschaften in Einzeldarstellungen Band **206** (1974).

[2] Z. ARVASI. Applications in commutative algebra of the Moore complex of a simplicial algebra. *Ph. D. Thesis*, University of Wales, (1994).

[3] Z. ARVASI and T. PORTER. Simplicial and crossed resolutions of commutative algebras. *J. Algebra*, **181**, (1996) 426-448.

[4] H.J. BAUES. Combinatorial homotopy and 4-dimensional complexes. *Walter de Gruyter*, (1991).

[5] R. BROWN and J.L. LODAY. Van Kampen theorems for diagram of spaces. *Topology*, **26** (1987) 311-335.

[6] P. CARRASCO. Complejos hipercruzados, cohomologia y extensiones. *Ph.D. Thesis*, Univ. de Granada, (1987).

[7] P. CARRASCO and A.M. CEGARRA. Group-theoretic algebraic models for homotopy types. *Journal Pure Appl. Algebra*, **75** (1991) 195-235.

[8] D. CONDUCHÉ. Modules croise generalises de longueur 2. *Journal Pure Appl. Algebra*, **34** (1984) 155-178.

[9] E.B. CURTIS. Simplicial homotopy theory. *Adv. in Math.* **6** (1971), 107-209.

[10] G.J. ELLIS. Higher dimensional crossed modules of algebras. *Journal Pure Appl. Algebra*, **52** (1988) 277-282.

[11] A.R. GRANDJEÁN and M.J. VALE. 2-Modulus cruzados en la cohomologia de André-Quillen. *Memorias de la Real Academia de Ciencias*, **22** (1986) 1-28.

[12] D. GUIN-WALERY and J.L. LODAY. Obstructions à l'excision en K-théorie algèbrique. Springer Lecture Notes in Math. **854** (1981) 179-216.

[13] L. ILLUSIE. Complex cotangent et deformations I, II. *Springer Lecture Notes in Math.*, **239** (1971) II **283** (1972).

[14] S. LICHTENBAUM and M. SCHLESSINGER, The cotangent complex of a morphism, *Trans. Amer. Math. Soc.*, **128** (1967), 41-70.

[15] T. PORTER, Homology of commutative algebras and an invariant of Simis and Vasconcelos, *J. Algebra*, **99** (1986) 458-465.

[16] D. QUILLEN, On the homology of commutative rings, *Proc. Sympos. Pure Math*, **17** (1970) 65-87.

[17] J.H.C. WHITEHEAD. Combinatorial homotopy II. *Bull. Amer. Math. Soc.*, **55** (1949) 213-245.