ABOUT REEB FOLIATION

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ABSTRACT

The aim of this paper is to compute the volume of codimension-one foliation of three-sphere S^3, defined in [5].

1. INTRODUCTION

The study of foliations on manifolds has a long history, but it was not known very well by the time of Ehresmann and Reeb's work in 1940's. In 1944, G. Reeb [8] gave the first codimension-one foliation of the round three-sphere S^3. In 1985, H. Gluck and W. Ziller asked the question about foliations;

"Among them which one has the minimum volume?".

In one-dimensional case, the same authors proved that the only volume minimizing foliations of S^3 are the Hopf foliations, $F_H: S^3 \rightarrow S^2$, defined in section 3.1[2]. They also showed that

$$\text{Vol}(F_H) = 2\text{Vol}(S^3).$$

In [6], we showed that the Reeb foliation F_R, defined in section 4.2.1, is locally volume minimizing two-dimensional (codimension-one) foliation of S^3. In this paper, we will prove that

$$\text{Vol}(F_R) = 4\text{Vol}(S^3).$$

2. BASIC DEFINITIONS

2.1. Definition

By a p-dimensional, C^r class foliation of an n-dimensional manifold M, we mean a decomposition of M into a union of disjoint connected

subsets $\{L_\alpha\}_{\alpha \in A}$, called the leaves of the foliation, with the following property: For every $a \in M$ there exists a neighborhood U of a and a system of local C^τ coordinates $x = (x^1, x^2, ..., x^n)$: $U \to \mathbb{R}^n$ such that for every leaf L_α, the components of $U \cap L_\alpha$ are defined by the equations $x^{p+1} = $ constant, ... $x^n = $ constant (Figure 1). We will denote such a foliation by $F = \{L_\alpha\}_{\alpha \in A}$. It is common to say that F is a codimension-q foliation, $(q = n - p)$ rather than dimension-p [7].
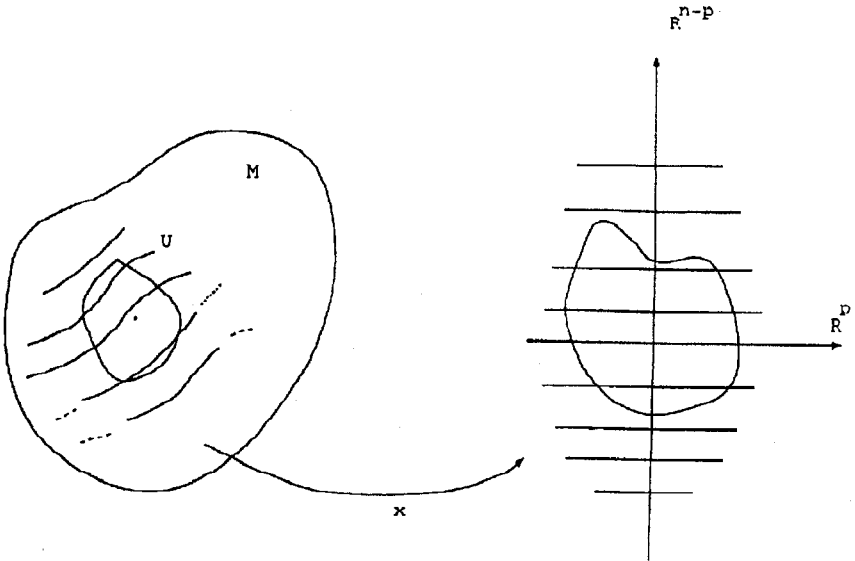

Figure 1.

2.2. Definition

Any transversely-oriented codimension-q foliation F on a Riemannian manifold M defines a section of the Grassmann bundle of n - q planes tangent to M, $\varphi : M \to G(n - q, T_*(M))$, by mapping $x \in M$ to $\varphi(x) :=$ $F_x T_*(M)$. In this setting the image of φ is thought of as the graph of the foliation F. Thus, the volume of foliation is in fact the n-dimensional measure of its graph. i.e. $Vol(F) := H^n(M)$ [4].

2.3. Definition

If F is an oriented flow, one-dimensional foliation, on a manifold M, with unit vector field ξ, the volume of the flow is given by

$$Vol(F) = \int_M \sqrt{1 + \left(\nabla\xi\right)^2 + ... + \underbrace{\left(\nabla\xi\wedge ... \wedge\nabla\xi\right)^2}_{n-1}} \, dV_M$$

where the vector wedge is interpreted by $\nabla\alpha\wedge\nabla\beta(X,Y) := \frac{1}{2}\left(\nabla_X\alpha\wedge\nabla_Y\beta - \nabla_Y\alpha\wedge\nabla_X\beta\right)$
etc., so that $(\nabla\xi)^{\wedge k}(X_1, X_k) = \nabla_{X1}\xi\wedge ... \wedge\nabla_{Xk}\xi$.

The sum is taken over wedges of order up to n - 1 because the top n^{th} wedge will vanish since ξ has values in the tangent unit sphere bundle. The metric on the tangent bundle used here is defined by Sasaki, and is the natural metric on T_*M induced from the Riemannian metric on M [4, 9].

If F is a condimension-one foliation, then $F_x^\perp = L_x = \{\lambda\xi_x\}$, where ξ is a line field. If F is a transversely-oriented, ξ can be taken to be a unit vector field, $\xi \in \chi(M)$, $|\xi| = 1$. Hence, Vol(F) := Vol(L), where L is a flow.

3. EXISTENCE OF FOLIATIONS OF S^3

3.1. One-dimensional case (Flows)

The existence problem has been studied back in the 1930's. In 1931, Heinz Hopf [3] introduced the first known foliations of sphere as follows:

$$S^1 \hookrightarrow S^{2n-1} \to CP^{n-1} = \text{complex projective n - 1 space}$$

$$S^3 \hookrightarrow S^{4n-1} \to HP^{n-1} = \text{quaternionic projective n - 1 space}$$

$$S^7 \hookrightarrow S^{15} \to S^8.$$

The simplest case occurs when n = 2 which is the three-sphere S^3,

$$S^1 \hookrightarrow S^3 \to CP^1 = S^2.$$

We call such foliation as above Hopf Foliations and denote by F_H.

3.2. Two-dimensional case

As mentioned in introduction, in 1944, George Reeb introduced the first known codimension-one foliation of S^3 as follows:

First, consider the C^∞- foliation of the (x, y)-plane given by the lines x = c for |c| ≥ 1 together with the graphs of the functions y = f(x) + c', - 1 < x < 1 and c' ∈ R, where f has the property that $\lim\limits_{|x|\to 1} f^{(k)}(x) = \infty$ for all k.

Consider now the foliation of the solid cylinder obtained by rotating the strip {(x, y) ∈ R²: -1 ≤ x ≤ 1} about the y-axis in 3-space. This foliation is invariant by vertical translations, and so we can obtain a foliation of the solid torus where each non-compact leaf has the form of a snake eternally eating its tail (Figure 2). We call such foliation of a solid torus by Reeb Component. In general, a codimension-one foliation of a manifold M may have several Reeb components as part of the total foliation.
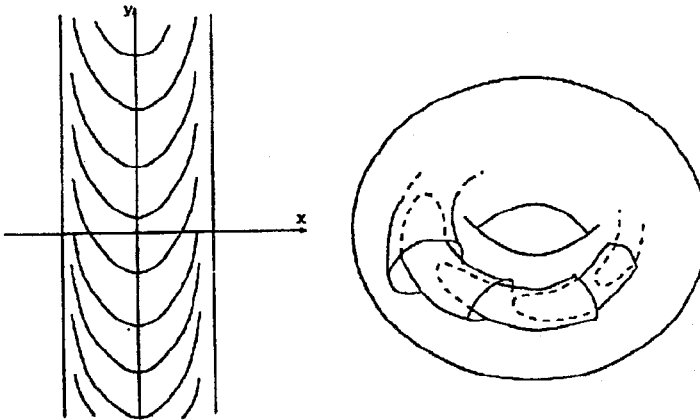

Figure 2.

The 3-sphere can be decomposed as two solid tori joined along their common 2-torus boundary. Indeed, if one removes the solid torus of rotation from $R^3 = S^3 \sim \infty$ what remains is homeomorphic to a solid torus minus an interior point (cosider the vertical coordinate axis as the core circle). Gluing together two copies of our foliated solid torus gives a Reeb foliation of the 3-sphere, S^3 (Figure 3) [7].

4. MINIMIZING PROBLEM

4.1. One-dimensional case

In [2], Gluck and Ziller proved the following theorem:

Figure 3.

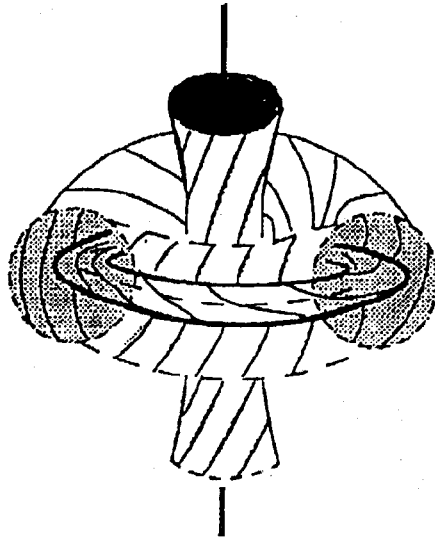### 4.1.1. Theorem

The one-dimensional foliations of minimum volume on S^3 are Hopf foliations, and no others.

They also showed that $\text{Vol}(F_H) = 2\text{Vol}(S^3)$.

We refer the reader to [2] for detail.

4.2. Two-dimensional case

In [6], we proved the following theorem:

4.2.1. Theorem

The two-dimensional foliation of (locally) minimum volume on S^3 is Reeb foliation, F_R.

In general, a foliation can be defined locally by level sets $F(x,y,z) = c$, where each value of c determines a leaf of foliation. In [6], we first found the level sets of Reeb foliation as

$$F_R(x,y,z) = \frac{1}{2} \arctan\left(\frac{2r - 1}{\sqrt{1 - 2r^2}}\right) - \frac{1}{2} \arctan\left(\frac{2r + 1}{\sqrt{1 - 2r^2}}\right) - z = c$$

where $r = \sqrt{x^2 + y^2}$, c = constant. Then, we showed that it is a local minimum.

5. MAIN RESULT

Now, we are going to prove the main goal, that is, $Vol(F_R) = 4V(S^3)$.

For n = 3, the equation of volume is obviously

$$V(F) = \int_M \sqrt{1 + \sum_{i=1}^{3} \|\nabla_{e_i}\xi\|^2 + \sum_{i,j=1}^{3} \|\nabla_{e_i}\xi \wedge \nabla_{e_j}\xi\|^2} \ dV \qquad (5.1)$$

where ξ is a unit normal vector to the foliation F and $\{e_1,e_2,e_3\}$ is orthonormal basis. Recall the properties of covariant derivative ∇,

$$\nabla_{fX}Y = f\nabla_X Y$$

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y$$

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_i} = \sum \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

where Γ_{ij}^k are the Christoffel symbols and recall the theorem of calculus of variations for variational problems [1];

5.1. Theorem

Let J(y) be a functional of the form

$$\int_a^b F(x,y,y') \ dx$$

defined on the set of functions y(x) which have continuous first derivatives on [a,b] and satisfy the boundary condition y(a) = A, y(b) = B. Then, a necessary condition for J(y) to have an extremum for a given function y(x) is that y(x) satisfy the Euler's equation

$$F_y - \frac{d}{dx} F_{y'} = 0 \ .$$

Using the computer program MapleV Release 3, we first computed the integrand term of the equation (5.1), denote by INT, as

INT $= [1 + 2H(r)^2 - 4H(r)^4r^6 + 6H(r)^4r^4 - 4H(r)^4r^2 - 6H(r)^2r^4 + H(r)^4 +$

$4r^7\left(\dfrac{d}{dr}H(r)\right)H(r) - 12r^5\left(\dfrac{d}{dr}H(r)\right)H(r) + 12r^3\left(\dfrac{d}{dr}H(r)\right)H(r) -$

$4\left(\dfrac{d}{dr}H(r)\right)r\,H(r) + H(r)^4r^8 + 4H(r)^2r^6 + \dfrac{d}{dr}H(r)^2 + r^8\dfrac{d}{dr}H(r)^2 - 4r^6\dfrac{d}{dr}H(r)^2 +$

$6r^4\dfrac{d}{dr}H(r)^2 - 4\dfrac{d}{dr}H(r)^2r^2]^{1/2}\quad \dfrac{\sqrt{-r^2 - H(r)^2 + 2H(r)^2r^2 - H(r)^2r^4}}{r(H(r)^2 - 2H(r)^2r^2 + H(r)^2r^4 + 1)^{3/2}\ \sqrt{r-1}\ \sqrt{r+1}}$

Then using the theorem of calculus of variations, above, we get the second order ordinary differential equation. Before giving the differential equation, let's note that this particular foliation was found by reducing the variational question of a volume-minimizing Reeb component on the sphere to an ordinary differential equation, by making the additional assumptions that the foliation be invariant under translations along the component (invariance under translations in the x_3-direction) as well as rotational invariance around the core circle. Under these assumptions the Euler-Lagrange equations of the first variation of the volume, as an ordinary differential equation reduce to the following: The graphs of $x_3 = f(r)$ are critical for the volume of the foliation generated by revolving and translating this graph in the interior of the Clifford torus, $S^1 \times S^1$, in S^3, subject to the boundary being tangent to the distribution $(f'(1/\sqrt{2}) = \infty)$ and the foliation being at least C^1 along the core circle $(f'(0) = 0)$ if and only if the derivative $f'(r) := H(r)$ satisfies:

$[-2(H(r))^4 - (H(r))^6 + 2r^4 - r^2 - (H(r))^2 - r^6 - (H(r))^6r^{16} + 4(H(r))^4r^{12} - 24(H(r))^4r^4 -$

$3(H(r))^4r^{10} + 25(H(r))^4r^6 - 10(H(r))^4r^8 + 7(H(r))^2r^8 - 2(H(r))^2r^{10} + 2(H(r))^2r^2 +$

$2(H(r))^2r^4 - 28(H(r))^6r^{12} + 8(H(r))^6r^{14} - 70(H(r))^6r^8 + 56(H(r))^6r^6 + 56(H(r))^6r^{10} +$

$8(H(r))^6r^2 - 8(H(r))^2r^6 - 28(H(r))^6r^4 + 11(H(r))^4r^2 - (H(r))^4r^{14}]\left(\dfrac{d^2}{dr^2}H(r)\right) +$

$[7r^3 - 21r^{11}(H(r))^2 + 11r^{11} + 30r^7 - 25r^9 + 7(H(r))^2r^{13} - r + r(H(r))^2 + 21r^5(H(r))^2 -$

$2r^{13} - r^{15}(H(r))^2 - 35r^7(H(r))^2 - 20r^5 + 35r^9(H(r))^2 - 7r^3(H(r))^2]\left(\dfrac{d}{dr}H(r)\right)^3 +$

$[-16r^2(H(r))^5 - 16(H(r))^5r^{14} + 2(H(r))^5 - 93(H(r))^3r^{10} + 33(H(r))^3r^{12} + 55r^{10}H(r) -$

$23(H(r))^3r^2 + 3r^2H(r) + 86r^6H(r) + H(r) + 3(H(r))^3 - 99r^8H(r) - 12r^{12}H(r) -$

$112(H(r))^3r^2 + 75(H(r))^3r^4 - 34r^4H(r) + 145(H(r))^3r^8 - 112r^{10}(H(r))^5 + 56(H(r))^6r^{12} +$

$2(H(r))^5r^{16} - 135(H(r))^3r^6 + 56r^4(H(r))^5 + 140r^8(H(r))^5 - 5(H(r))^3r^{14}]\left(\dfrac{d}{dr}H(r)\right)^2 +$

$[9r^3 + (H(r))^4r - 8r^5 + r(H(r))^2 + 27r^5(H(r))^2 + 21(H(r))^6r^{11} - (H(r))^6r - 70(H(r))^4r^5 -$

$3r^3(H(r))^2 + r^{15}(H(r))^6 - 73r^7(H(r))^2 - r + 72r^9(H(r))^2 + 11(H(r))^4r^3 - 24r^{11}(H(r))^2 -$

$7(H(r))^6r^{13} + 150(H(r))^4r^7 + 7(H(r))^6r^3 - 16(H(r))^4r^{13} + 79(H(r))^4r^{11} - 155(H(r))^4r^9 -$

$35(H(r))^6r^9 + 35(H(r))^6r^7 - 21(H(r))^6r^5]\left(\dfrac{d}{dr}H(r)\right) +$

$[-(H(r))^7r^{14} - 10r^4(H(r)) + 3r^2H(r) - 9(H(r))^3r^2 - 5(H(r))^3r^4 + 53r^{10}(H(r))^5 -$

$65r^8(H(r))^5 + 10r^6(H(r))^5 + 40r^4(H(r))^5 - 31r^2(H(r))^5 - 33(H(r))^7r^{10} + 9(H(r))^7r^{12} +$

$51(H(r))^7r^4 - 75(H(r))^7r^6 + 65(H(r))^7r^8 - 19(H(r))^7r^2 - 16(H(r))^3r^{10} - 14(H(r))^5r^{12} +$

$7(H(r))^5 + H(r) + 5(H(r))^3 + (H(r))^3r^6 + 3(H(r))^7 + 24(H(r))^3r^8] = 0$

In addition, of course, the function H(r) must satisfy the initial conditions, H(0) = 0 and H'(0) = a, the radius of torus. Although this equation may seem hopelessly complex, it happens that there is a closed-form solution. We found this solution by first determining (using MapleV (rel. 3)) the coefficients of a power series solution of the equation, then, with some luck, being able to recognize the solution series in closed form, which we then verified by direct substitution. We give the detail below:

Let's denote the differential equation, above, by ODE.

H(r) := dsolve(ODE = 0, H(0) = 0, D(H)(0) = a, H(r), series);

We get the following output:

$$H(r) := ar + \left(\frac{a^3}{2} + \frac{3a}{2}\right)r^3 + \left(\frac{3a^5}{8} + \frac{5a^3}{4} + \frac{15a}{8}\right)r^5 + \left(\frac{5a^7}{16} + \frac{21a^5}{16} + \frac{35a^3}{16} + \frac{35a}{16}\right)r^7 +$$

$$\left(\frac{35a^9}{128} + \frac{45a^7}{32} + \frac{189a^5}{64} + \frac{105a^3}{32} + \frac{315a}{128}\right)r^9 +$$

$$\left(\frac{63a^{11}}{256}+\frac{385a^{9}}{256}+\frac{495a^{7}}{128}+\frac{693a^{5}}{128}+\frac{1155a^{3}}{256}+\frac{693a}{256}\right)r^{11} +$$

$$\left(\frac{231a^{13}}{1024}+\frac{819a^{11}}{512}+\frac{5005a^{9}}{1024}+\frac{2145a^{7}}{256}+\frac{9009a^{5}}{1024}+\frac{3003a^{3}}{512}+\frac{3003a}{1024}\right)r^{13} +$$

$$\left(\frac{429a^{15}}{2048}+\frac{3465a^{13}}{2048}+\frac{12285a^{11}}{2048}+\frac{25025a^{9}}{2048}+\frac{32175a^{7}}{2048}+\frac{27027a^{5}}{2048}+\frac{15015a^{3}}{2048}+\frac{6435a}{2048}\right)r^{15} +$$

...

By looking at the coefficient of the series, H(r), we get the closed form. Although we checked all of the coefficients, we give only some of them below. For instance, look at the coefficient with respect to "a", that is,

$$r+\frac{3}{2}r^{3}+\frac{15}{8}r^{5}+\frac{35}{16}r^{7}+\frac{315}{128}r^{9}+\frac{693}{256}r^{11}+\frac{3003}{1024}r^{13}+\frac{6435}{2048}r^{15}+...$$
$$=r+\frac{1.3}{2}r^{3}+\frac{1.3.5}{2.4}r^{5}+\frac{1.3.5.7}{2.4.6}r^{7}+\frac{1.3.5.7.9}{2.4.6.8}r^{9}+\frac{1.3.5.7.9.11}{2.4.6.8.10}r^{11}+...$$

The closed form of this is

$$I = \sum_{n=0}^{\infty}\frac{1.3.5...(2n+1)}{2.4.6...(2n)}\frac{a}{1}r^{2n+1}$$

With respect to "a^{3}",

$$\frac{1}{2}r^{3}+\frac{5}{4}r^{5}+\frac{35}{16}r^{7}+\frac{105}{32}r^{9}+\frac{1155}{256}r^{11}+\frac{3003}{512}r^{13}+\frac{15015}{2048}r^{15}+...$$
$$=\frac{1}{2}\frac{1}{3}(\frac{1.3}{1}r^{3}+\frac{1.3.5}{2}r^{5}+\frac{1.3.5.7}{2.4}r^{7}+\frac{1.3.5.7.9}{2.4.6}r^{9}+\frac{1.3.5.7.9.11}{2.4.6.8}r^{11}+...)$$

Hence, the closed form

$$II = \sum_{n=0}^{\infty}\frac{1}{2.3}\frac{1.3.5...(2n+3)}{2.4.6...(2n)}a^{3}r^{2n+3}$$

$$= \sum_{n=0}^{\infty}\frac{1}{2}\frac{1.3.5...(2n+1)}{2.4.6...(2n-2)}\frac{a^{3}}{3}r^{2n+1}$$

Similarly, with respect to "a^{5}", we get

$$III = \sum_{n=0}^{\infty}\frac{1}{2^{3}.5}\frac{1.3.5...(2n+5)}{2.4.6...(2n)}a^{5}r^{2n+5}$$

$$= \sum_{n=0}^{\infty}\frac{1}{2.4}\frac{1.3.5...(2n+1)}{2.4.6...(2n-4)}\frac{a^{5}}{5}r^{2n+1}$$

Continuing the above argument and adding these, we get the closed form of H(r),

$$H(r) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{2 \cdot 4 \ldots (2k)} \frac{1 \cdot 3 \ldots (2n + 1)}{2 \cdot 4 \ldots (2n - 2k)} \frac{a^{2k+1}}{2k + 1} r^{2n+1} \ .$$

Let's denote the interior sum by

$$s = \sum_{k=0}^{n} \frac{1}{2 \cdot 4 \ldots (2k)} \frac{1 \cdot 3 \ldots (2n + 1)}{2 \cdot 4 \ldots (2n - 2k)} \frac{a^{2k+1}}{2k + 1}$$

Since

$$2 \cdot 4 \ldots (2k) = (2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \ldots (2 \cdot k) = 2^k \cdot k!$$

it follows that

$$s = \sum_{k=0}^{n} \frac{1}{2 \cdot 4 \ldots (2k)} \frac{1 \cdot 3 \ldots (2n + 1)}{2 \cdot 4 \ldots (2n - 2k)} \frac{a^{2k+1}}{2k + 1}$$

$$= \sum_{k=0}^{n} \frac{1 \cdot 3 \ldots (2n + 1)}{2^k \cdot k! \cdot 2^{(n-k)!} (n - k)!} \frac{a^{2k+1}}{2k + 1}$$

$$= \frac{1 \cdot 3 \ldots (2n + 1)}{2^n \cdot n!} \sum_{k} \frac{n!}{k! (n - k)} \frac{a^{2k+1}}{2k + 1}$$

$$= \frac{1 \cdot 3 \ldots (2n + 1)}{2^n \cdot n!} \sum_{k} \binom{n}{k} \frac{a^{2k+1}}{2k + 1} \ .$$

Define

$$s_1 = \sum_{k} \binom{n}{k} \frac{a^{2k+1}}{2k + 1} \ .$$

Then, it follows from the Binomial theorem that

$$\frac{\partial s_1}{\partial a} = \sum_{k} \binom{n}{k} a^{2k} = (1 + a^2)^n \ .$$

Therefore,

$$s = \frac{1 \cdot 3 \ldots (2n + 1)}{2^n \cdot n!} \int_0^a (1 + t^2)^n \, dt \ .$$

It follows that

$$H(r) = \sum_{n=0}^{\infty} \left(\frac{1.3...(2n+1)}{2^n.n!} \int_0^a (1+t^2)^n \, dt\right) r^{2n+1} \; .$$

Notice also that

$$\frac{\partial H}{\partial a} = \sum_{n=0}^{\infty} \frac{1.3...(2n+1)}{2^n.n!} (1+a^2)^n \, r^{2n+1} \; .$$

Let $x = r\sqrt{1+a^2}$. Then,

$$\frac{\partial H}{\partial a} = \frac{1}{\sqrt{1+a^2}} \sum_{n=0}^{\infty} \frac{1.3...(2n+1)}{2^n.n!} x^{2n+1}$$

which is the Taylor series of $x(1-x^2)^{-3/2}$, i.e,

$$\sum_{n=0}^{\infty} \frac{1.3...(2n+1)}{2^n.n!} x^{2n+1} = \frac{x}{(1-x^2)^{3/2}} \; .$$

Thus,

$$\frac{\partial H}{\partial a} = \frac{1}{\sqrt{1+a^2}} \frac{x}{(1-x^2)^{3/2}} = \frac{r}{(1-r^2(1+a^2))^{3/2}} \; .$$

Finally,

$$H(r) = \int_0^a \frac{r}{(1-r^2(1+t^2))^{3/2}} \, dt$$

$$= \frac{-ar}{(r^2-1)\sqrt{1-r^2-a^2r^2}}$$

Recall that $f'(r) := H(r)$. Therefore,

$$f(r) = \int H(r) \, dr$$

$$= \int \frac{-ar}{(r^2-1)\sqrt{1-r^2-a^2r^2}} \, dr$$

$$= \frac{\sqrt{a}}{2} \left(\text{arctanh} \left(\frac{r-1+ar}{\sqrt{a}\sqrt{1-r^2-ar^2}}\right) - \text{arctanh} \left(\frac{r+1+ar}{\sqrt{a}\sqrt{1-r^2-ar^2}}\right) \right)$$

This means that the level sets of the foliation of the three-sphere, S^3, are

$$F(x,y,z) = \frac{\sqrt{-a}}{2}\left(\text{arctanh}\left(\frac{r - 1 + ar}{\sqrt{-a}\,\sqrt{1 - r^2 - ar^2}}\right) - \text{arctanh}\left(\frac{r + 1 + ar}{\sqrt{-a}\,\sqrt{1 - r^2 - ar^2}}\right)\right) - z$$

$$= c \, ,$$

where $r^2 = x^2 + y^2$, $c = $ constant.

We denote the foliation of the Reeb component, $D^2\left(\frac{1}{\sqrt{1 + a^2}}\right) \times S^1\left(\sqrt{1 - \frac{1}{1 + a^2}}\right)$, given by these level sets by $F_0(a)$. Recall the construction of the foliation of the three-sphere from part 3.2 that we 'glue' two copies of the solid tori along their common boundary. So, gluing two copies of the foliation $F_0(a)$, we will get the foliation of S^3 which we denote by $F_R(a)$.

For $a = 1$, we get the level sets of Reeb foliation, we are dealing with, as

$$F_R(x,y,z) = \frac{1}{2}\left(\arctan\left(\frac{2r - 1}{\sqrt{1 - 2r^2}}\right) - \frac{1}{2}\,\text{arctanh}\left(\frac{2r + 1}{\sqrt{1 - 2r^2}}\right)\right) - z = c$$

where $r = \sqrt{x^2 + y^2}$, $c = $ constant.

Now we came to the point of proving the main goal.

Recall that in polar coordinates,

$$\text{Vol} := \iiint \text{INT } r \, dr \, d\theta \, dz$$

where INT is the integrand term of the equation (5.1).

First, we make the following computation by using MapleV.

$f_1 := \text{factor (INT} * r)$; so that $\int \text{INT } r \, dr = \int f_1 \, dr$

$f_2 := \text{simplify(subs}(H(r) = -a * r/(\text{sqrt}(1 - r^2 - a^2 * r^2) * (r^2 - 1)), f_1))$;
$\text{Vol} := \text{int}(f_2 , r = 0..t)$;
$V_1 := \text{simplify(subs}(t = 1/\text{sqrt}(1 + a^2), \text{Vol}))$;
$V_2 := \text{simplify(subs}(t = \text{sqrt}(1 - 1/(1 + a^2)), \text{Vol}))$;

This program provides the following output:

$$f_2 := \frac{I(1 + a^2)r}{\sqrt{-1 + r^2 + a^2 r^2}} \quad , \text{ where } I = \sqrt{-1}$$

$$\text{Vol}(t) := 1 + \sqrt{1 - t^2 - a^2 t^2}$$

The asymptote of Vol(t) is $t = 1/\sqrt{1 + a^2}$. So,

$$\text{Vol}(\frac{1}{\sqrt{1 + a^2}}) = 1$$

Since S^3 containes two solid tori, we have

$$F_0(a) = \frac{-ar}{(r^2 - 1)\sqrt{1 - r^2 - a^2 r^2}} \text{ and } F_0(\frac{1}{a}) = \frac{-\frac{1}{a}r}{(r^2 - 1)\sqrt{1 - r^2 - \left(\frac{1}{a}\right)^2 r^2}}$$

with the corresponding radii

$$r_1 = \frac{1}{\sqrt{1 + a^2}} \text{ and } r_2 = \frac{1}{\sqrt{1 + b^2}} \quad .$$

i.e. The foliation of S^3, $F_R(a)$, is made up of $F_0(a)$ and $F_0\left(\frac{1}{a}\right)$, in the two complimentary solid tori which form S^3. For each of these, we have

$$\text{Vol}(r_1) = 1 \text{ and } \text{Vol}(r_2) = 1.$$

Thus, the volume of $F_0(a)$ is

$$\text{Vol}(F_0(a)) = \int_0^{2\pi}\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{1+a^2}}} \text{INT } r \, dr \, d\theta \, dz$$

$$= \int_0^{2\pi}\int_0^{2\pi} 1 \, d\theta \, dz$$

$$= 4\pi^2 \quad .$$

This is the half of the foliation. So, the total volume is $8\pi^2$. Since the volume of the three-sphere, S^3, is $2\pi^2$, it follows that

$$\text{Vol}(F_R) = 4 \, \text{Vol}(S^3)$$

as claimed.

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