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ON THE (f, g) - LINEAR CONNECTIONS

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ABSTRACT

In this note, we present a method to determine the linear connections compatible with the (f, g)-structures defined by a tensor field f of type (1, 1), having the property $f^4 - f^2 = 0$ and a Riemannian structure g which satisfies a suplimentary condition.

BASIC CONCEPTS

Manifolds, mappings, tensor fields and connections we discuss are always assumed to be C^∞.

Let M be a manifold of dimension n and let F (M) be the algebra of all differentiable functions on M. We denote by $T^r{}_s(M)$ the F (M)-module of the tensor fields of type (r, s).

Definition 1. An f-structure on M is a non-null field f of tensors of $T^1{}_1(M)$, of constant rank everywhere, so that

$$f^4 - f^2 = 0.$$

If M is an f-manifold, that is, if M is an n-dimensional Riemannian manifold, equiped with an f-structure, then for

$$b = f^2,\ m = I - f^2 \text{ (I denoting identity operator)} \tag{1}$$

we have [Gadea P.M. and Cordero, (1974)]:

$$fb = bf = f^3,\ f^2b = bf^2 = b,\ fm = mf = f - f^3,\ f^2m = mf^2 = 0, \tag{2}$$

and

$$b + m = I,\ bm = mb = 0,\ b^2 = b,\ m^2 = m. \tag{3}$$

Thus the operators b and m are complementary operators on M.

The Riemannian structure g on M can be considered as a $T^1_0(M)$–valued differential 1–form meaning that g: $T^1_0(M) \to T^0_1(M)$, $g(_X) = g_X$, where $g_X(Y) = g(X, Y)$ for every X, Y $\in T^1_0(M)$. If $f \in T^1_1(M)$, then $^\tau f$ denotes the transpose of f, $^\tau f$: $T^0_1(M) \to T^0_1(M)$, $^\tau f(\eta) = \eta$ o f, $\forall \eta \in T^0_1(M)$.

Definition 2. An (f, g)–structure on M is a couple made up of an f–structure and a Riemannian structure g so that

$$g \text{ o } f = {}^\tau f \text{ o } g. \tag{4}$$

Theorem 1. If M is a paracompact differential manifold with an f–structure, then there is an (f, g)–structure.

Proof: In fact, if $\tilde{g}$ is a Riemannian metric fixed on M, then

$$g = \tfrac{1}{4}\left({}^\tau f^2 o \tilde{g} o f^2 + {}^\tau f^2 o \tilde{g} o f^3 + {}^\tau f^3 o \tilde{g} o f^2 + {}^\tau f^3 o \tilde{g} o f^3\right) \tag{5}$$

satisfies the condition (4).

Proposition 1. For an (f, g)–structure on M and b, m defined by the equation (1) we have

$$\begin{aligned} gof &= {}^\tau fog, \quad fog^{-1} = g^{-1}o\ {}^\tau f \\ gom &= {}^\tau mog, \quad mog^{-1} = g^{-1}o\ {}^\tau m. \end{aligned} \tag{6}$$

Definition 3. We call the Obata operators associated to f the maps A, A*: $T^1_1(M) \to T^1_1(M)$ defined by

$$A(w) = bowob - mowom, \qquad A^*(w) = bowom + mowob. \tag{7}$$

We also consider the Obata operators [Miron R. et Atanasiu Gh. (1986)] associated to g:

$$B(u) = \tfrac{1}{2}\left(u - g^{-1}o^\tau uog\right), \; B^*(u) = \tfrac{1}{2}\left(u + g^{-1}o^\tau uog\right) \tag{8}$$

Proposition 2. For an (f, g)–structure on M and for A, A* and B, B* defined by (7) and (8) we have:

1) A and A* complementary operators on $T^1_1(M)$:
2) B and B* commute pairwise with A and A*:
3) AoB and A*oB* are projections on $T^1_1(M)$:
4) Ker A $\cap$ Ker B = im (AoB).

In fact, by simple calculation, we obtain the result 1).

The assertion of 2) is true, because, taking into account the relations (6) we have:

$$(AoB - BoA)(u) =$$

$$\tfrac{1}{4}\ [(mog^{-1}o^{\tau}uog - g^{-1}o^{\tau}mo^{\tau}uog) + (g^{-1}o^{\tau}uogom - g^{-1}o^{\tau}uo^{\tau}mog)$$

$$-3\ (mog^{-1}o^{\tau}uogom - g^{-1}o^{\tau}mo^{\tau}uo^{\tau}mog)-$$

$$-({}^{\tau}f^2og^{-1}o\ {}^{\tau}uogof^2 - {}^{\tau}f^2o^{\tau}uo^{\tau}f^2og)\ ] = 0,\ \forall u \in T^1{}_1(M).$$

Thus from $AoB = BoA$ we obtain

$$AoB^* = B^*oA,\ A^*oB^* = B^*oA^*.$$

The above mentioned relations give us the possibility to formulate [Wilde A.C. (1987)]:

Proposition 3. The system of tensorial equations—

$$A^*(u) = a,\ B^*(u) = b \tag{9}$$

has a solution $u \in T^1{}_1(M)$, if and only if

$$A(a) = 0,\ B(b) = 0,\ A^*(b) = B^*(a). \tag{10}$$

If the conditions (10) are fulfilled, then the general solution of the system (9) is $u = a + A(b) + (AoB)(w)$ for every $w \in T^1{}_1(M)$:

In the following $\overset{\circ}{\nabla}$ will be a linear connection fixed on M and every tensor field $u \in T^1{}_1(M)$, may be considered as a field of $T^1{}_0(M)$–valued differential 1–forms. So, if ∇ is a linear connection on M, then we'll denote D and $\tilde{D}$ the associated connections, acting on the $T^1{}_0(M)$–valued differential 1–forms and on the differential 1–forms $g: T^1{}_0(M) \to T^0{}_1(M)$ by

$$D_X u = \nabla_X ou - uo\nabla_X \text{ and } D_X g = {}^{\tau}\nabla_X og - go\nabla_X,\ X \in T^1{}_0(M) \tag{11}$$

respectively, where

$$({}^{\tau}\nabla_X g)(Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z),\ X, Y, Z \in T^1{}_0(M). \tag{12}$$

Definition 4. A linear connection ∇ on M is called (f, g)–linear connection if

$$D_X f = 0,\ \tilde{D}_X g = 0,\ \forall X \in T^1{}_0(M). \tag{13}$$

Of course, for every (f, g)–linear connection, we have

$$D_X b = \nabla_X b - b\nabla_X = 0, \; D_X m = \nabla_X m - m\nabla_X = 0, \tag{14}$$

$D_X f^k = \nabla_X f^k - f^k \nabla_X = 0$, k being a natural number, $\forall\, X \in T^1_0(M)$.

We see that D and $\tilde{D}$ commute with the operators A, A*, B and B*.

We take $\nabla_X = \overset{o}{\nabla}_X + V_X$, where $V_X Y = V(X, Y)$ and $V \in T^1_2(M)$ for every X, $Y \in T^1_0(M)$ and we find the tensor field V so that it satisfies the conditions (13).

∇ will be an (f, g)–linear connection if and only if the field V satisfies the system

$$V_X \circ f - f \circ V_X = - \overset{o}{D}_X f, \; {}^{t}V_X \circ g + g \circ V_X = \overset{\tilde{o}}{D}_X\, g. \tag{15}$$

This system is equivalent with the system

$$A^* (V_X) = -m \circ \overset{o}{\nabla}_X \circ b,$$

$$B^* (V_X) = \tfrac{1}{2}\, g^{-1} \circ \overset{\tilde{o}}{D}_X g. \tag{16}$$

Applying the Proposition 3, it becomes evident that the system (16) has a solution and the general solution is

$$V_X = - m \circ \overset{o}{\nabla}_X \circ b +$$

$$+ \tfrac{1}{4}\, g^{-1} \circ [\overset{\tilde{o}}{D}_X g - (\overset{\tilde{o}}{D}_X g) \circ m - m \circ (\overset{\tilde{o}}{D}_X g) + 3\, m \circ (\overset{\tilde{o}}{D}_X g) \circ m +$$

$$f^2 \circ (\overset{\tilde{o}}{D}_X g) \circ f^2] + (A \circ B)(W_X), \; W \in T^1_2(M) \tag{17}$$

Then, we obtain the following

Theorem 2. There exist (f, g)–linear connections: one of them is

$$\nabla_X = \overset{o}{\nabla}_X + V_X, \tag{18}$$

where $\overset{o}{\nabla}$ is an arbitrary linear connection fixed on M and V_X is given by (17), W being an arbitrary tensor field.

If $\overset{o}{\nabla}$ is the Levi–Civita connection of g, then we have $\overset{\tilde{o}}{D}_X g = 0$ and Theorem 1 becomes

Theorem 3. For every (f, g)–structure, the linear connection

$$\overset{c}{\nabla}_X = \overset{o}{\nabla}_X + m o \overset{o}{\nabla}_X o b \tag{19}$$

has the following characteristics:

1) $\overset{c}{\nabla}$ is dependent uniquely of f and g,

2) $\overset{c}{\nabla}$ is an (f, g)–linear connection,

where ∇ is the Levi–Civita connection of g.

The linear connection $\overset{c}{\nabla}$ will be called the (f, g)–canonic connection.

Theorem 4. The set of all (f, g)–linear connections is given by

$$\overline{\nabla}_X = \nabla_X + (AoB)(W_X), \ W \in T^1{}_2(M), \tag{20}$$

where ∇ is an (f, g)– linear connection, in particular $\nabla = \overset{c}{\nabla}$.

Observing the fact that (20) can be considered as a transformation of (f, g)–linear connections, we have.

Theorem 5. The set of the transformations of (f, g)–linear connections together the multiplication of transformations is an abelian group. Furthermore, this group, denoted by G (g, f), is isomorfic to the additive group of the tensors $W \in T^1{}_2(M)$, which have the characteristic

$$W_X \in \operatorname{Im}(AoB) = \operatorname{Ker} A^* \cap \operatorname{Ker} B^*, \ X \in T^1{}_0(M).$$

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