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ON THE SOME NEW CHARACTERISTIC PROPERTIES OF THE a-PEDAL HYPERSURFACE OF A HYPERSURFACE WITH THE CONSTANT SUPPORT FUNCTION IN (n+1)-DIMENSIONAL EUCLIDEAN SPACE E^{n+1}

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ABSTRACT

The purpose of this paper is, first, to present the definition of the a-pedal hypersurface of a given smooth, connected and orientable surface M in E^{n+1} (chapter I) and if M is taken as a hypersurface with constant support function, then we will give the some new characteristic properties of the a-pedal hypersurface of M related to support function, the area element, the higher order Gaussian curvatures, the characteristic polynom, the first, second and third fundamental forms and their coefficients of M (chapter II).

1. INTRODUCTION

In this section, we will give the basic concepts related with hypersurfaces and the definition of a-pedal hypersurface of a hypersurface in E^{n+1}.

All hypersurfaces under consideration shall be smooth (C^∞), connected and orientable. We begin by giving the following the proposition:

Proposition 1.1. Let M be a hypersurface in E^{n+1}. M is called the regular hypersurface if the following conditions are satisfied, [1]:

(I) The immersed hypersurface M has the Gauss-Kronecker curvature $K \neq 0$ everywhere.

(II) The origin O lies no tangent hyperplane to M. Such an origin will henceforth be called admissible for M. Clearly, admissible origin always exist locally for a given M: It is sufficient to pick O close enough to M.

Moreover, if (I) and (II) hold and $n \geq 2$, there exist an orientation of M. If we take the parameter system $\{u_1,u_2,...,u_n\}$ on M, for the unit normal vector N of M we can write

$$N = \frac{X_1 x X_2 x ... x X_n}{\|X_1 x X_2 x ... x X_n\|}$$

where X is the position vector with initial point of O of $P \in M$, $X_i = \frac{\partial X}{\partial u_i}$, $1 \leq i \leq n$. Also, the support function h of M is defined by

$$h = - \langle X,N \rangle \tag{1.1}$$

where X is the position vector of M, [4].

In this paper, the length of the position vector X of the point $P \in M$ will be denoted by ρ. We have $h > 0$ throughout by assumption (II) and the choice of orientation.

In this study, we will consider M as a regular hypersurface with the constant support function. The hypersurfaces with constant support function have been studied by Hasanis and Koutroufiotis and they have shown that if the support function of hypersurface M is identically 1, then $M = E^n$ or $M = S^n$ or $M = S^{n-t} \times E^t$, $1 \leq t \leq n$ [3].

Definition 1.1. Let M be a regular hypersurface and O be a arbitrary point in E^{n+1}. For a given real number a, the hypersurface $\overline{M}$ with position vector defined by

$$\overline{X} = - h^a N \tag{1.2}$$

with respect to O will be called a a-pedal hypersurface of M with respect to O, where h is the support function of M at $P \in M$, [4].

Proposition 1.2. For $n \geq 2$, $\overline{M}$ is a regular surface if assumptions (I) and (II) hold.

If we consider the local parameter system $\{u_1,u_2,...,u_n\}$ on M, then we can write $\overline{X} = \overline{X}(u_1,u_2,...,u_n)$ and the normal vector field $\overline{N}$ of $\overline{M}$

$$\overline{N} = \frac{\overline{X}_1 x \overline{X}_2 x ... x \overline{X}_n}{\|\overline{X}_1 x \overline{X}_2 x ... x \overline{X}_n\|} \tag{1.3}$$

where $\overline{X}_i$, $1 \leq i \leq n$, is the partial derivate with respect to the parameter u_i.

Definition 1.2. Let M be a hypersurface in E^{n+1}. The curvatures $K_1, K_2, \ldots, K_n$ are called the higher order Gaussian curvatures and defined by

$$K_1 = \sum_{i=1}^{n} k_i \ , \ K_2 = \sum_{i<j}^{n} k_i k_j \ , \ \ldots \ , \ K_n = \prod_{i=1}^{n} k_i \tag{1.4}$$

where k_i, $1 \leq i \leq n$, is the i^{th}. principal curvature of M, [1]. Then, the curvatures K_1 and K_n are the mean and Gauss curvatures of M and denoted by H and K, [1].

Definition 1.3. Let M be a hypersurface in E^{n+1}. The polynom $P_S(\lambda)$ is called the characteristic polynom of M and defined by

$$P_S(\lambda) = \lambda^n + (-1)^1 K_1 \lambda^{n-1} + \ldots + (-1)^{n-1} K_{n-1} \lambda + (-1)^n K_n \tag{1.5}$$

where $K_1, K_2, \ldots, K_n$ and S are the higher order Gaussian curvatures and shape operator of M, [1].

II. ON THE CHARACTERISTIC PROPERTIES OF THE a-PEDAL HYPERSURFACE OF A HYPERSURFACE WITH THE CONSTANT SUPPORT FUNCTION

In this chapter, it will be studied the some characteristic properties of the a-pedal hypersurface of a regular hypersurface with the constant support function in the Euclidean space E^{n+1}.

Theorem 2.1. Let M be regular hypersurface with the constant support function in E^{n+1} and $\overline{M}$ be a a-pedal hypersurface of M. For the unit normal vector field $\overline{N}$ of $\overline{M}$, we have

$$\overline{N} = N \tag{2.1}$$

where N is the inner unit normal vector of M.

Proof. Because of the equation (1.1), we can write

$$\overline{X} = -\overset{a}{h} N \ .$$

Let us choose a local parameter system $\{u_1, u_2, \ldots, u_n\}$ in $P \in M$. If the vector field $\overline{X}$ is taken the partial derivate the parameter u_i, we get

$$\overline{X}_i = - h^a N_i \quad , \quad 1 \leq i \leq n \, . \tag{2.2}$$

Because of the equation (1.3), we can write

$$\overline{N} = \frac{\overline{X}_1 x \overline{X}_2 x \, ... \, x \overline{X}_n}{\|\overline{X}_1 x \overline{X}_2 x \, ... \, x \overline{X}_n\|} \, .$$

If the value of the vector field $\overline{X}_1 x \overline{X}_2 x \, ... \, x \overline{X}_n$ is calculated, we get

$$\overline{X}_1 x \overline{X}_2 x \, ... \, x \overline{X}_n = h^{na} K X_1 x X_2 x \, ... \, x X_n \tag{2.3}$$

If we consider a parameter system consisting of the lines curvature, we obtain

$$\overline{X}_1 x \overline{X}_2 x \, ... \, x \overline{X}_n = h^{na} K \|X_1 x X_2 x \, ... \, x X_n\| N. \tag{2.4}$$

If we calculate the norm of the vector $\overline{X}_1 x \overline{X}_2 x \, ... \, x \overline{X}_n$, we get

$$\|\overline{X}_1 x \overline{X}_2 x \, ... \, x \overline{X}_n\| = h^{na} K \|X_1 x X_2 x \, ... \, x X_n\| \, .$$

Substituting by the equations (2.3) and (2.4) into the equation (1.3), we get the result the theorem.

Thus, it can be said that the a-pedal hypersurface $\overline{M}$ of M is a parallel hypersurface of M.

Corollary 2.1. Let M be a regular hypersurface with the constant support function in E^{n+1} and $\overline{M}$ be the a-pedal hypersurface of M. For the support function $\overline{h}$ of $\overline{M}$, we have

$$\overline{h} = h^a \, . \tag{2.5}$$

Proof. Because of the equation (2.1) we can write

$$\overline{N} = N$$

Thus, the support function $\overline{h}$ of $\overline{M}$ can be written

$$\overline{h} = - \langle \overline{X}, \overline{N} \rangle \, . \tag{2.6}$$

Set the value $\overline{N}$ in the equation (2.6), we get the result of corollary 2.1.

Theorem 2.2. For the area elements dA and $d\overline{A}$ of M and $\overline{M}$, we have

$$d\overline{A} = h^{na} K \, dA \tag{2.7}$$

where K is the Gauss curvature of M.

Proof. For the area element $d\overline{A}$ of $\overline{M}$, we can write

$$d\overline{A} = \|\overline{X}_1 x \overline{X}_2 x \ \dots \ x \overline{X}_n\| \, du_1 du_2 \dots du_n \tag{2.8}$$

Since $\|\overline{X}_1 x \overline{X}_2 x \ \dots \ x \overline{X}_n\| = h^{na} K \|X_1 x X_2 x \ \dots \ x X_n\|$, we get

$$d\overline{A} = h^{na} K \, \|\overline{X}_1 x \overline{X}_2 x \ \dots \ x \overline{X}_n\| \, du_1 du_2 \dots du_n$$

or

$$d\overline{A} = h^{na} K \, dA \ .$$

Theorem 2.3. For the coefficients $\overline{g}_{ij}$ and g_{ij} of $\overline{M}$ and M, we have

$$\overline{g}_{ij} = h^{2a} n_{ij} \tag{2.9}$$

where n_{ij} is the coefficient of the third fundamental form of M.

Proof. For the coefficient $\overline{g}_{ij}$ of the first fundamental form of M, we can write

$$\overline{g}_{ij} = \langle \overline{X}_i, \overline{X}_j \rangle \tag{2.10}$$

Using the equation (2.2), we get the result the theorem.

Thus, we can be given the following the corollary.

Corollary 2.2. For the first fundamental form $\overline{I}$ of $\overline{M}$, we have

$$\overline{I} = h^{2a} III \tag{2.11}$$

where III is the third fundamental form of M.

Theorem 2.4. For the coefficient $\overline{b}_{ij}$ of the second fundamental form of $\overline{M}$, we have

$$\overline{b}_{ij} = h^{a} n_{ij} \ . \tag{2.12}$$

Proof. For the coefficient $\bar{b}_{ij}$ of $\overline{M}$, we can write

$$\bar{b}_{ij} = -\langle \overline{X}_i, \overline{N}_j \rangle . \tag{2.13}$$

Since $\overline{N} = N$, we get

$$\overline{N}_j = N_j \ , \ 1 \leq j \leq n. \tag{2.14}$$

Using the equations (2.2) and (2.14) we can obtain the result of the theorem 2.4. Thus we can give the following corollary.

Corollary 2.3. For the second fundamental form $\overline{\mathrm{II}}$ of $\overline{M}$, we can write

$$\overline{\mathrm{II}} = h^a \ \mathrm{III} . \tag{2.15}$$

Proof. The proof is clear.

Theorem 2.5. For the coefficient $\bar{n}_{ij}$ of $\overline{M}$, we have

$$\bar{n}_{ij} = n_{ij} \quad , \ 1 \leq i,j \leq n . \tag{2.16}$$

Proof. If we take derivate of the vector $\overline{N}$ with respect to the parameters u_i and u_j, we get $\overline{N}_i = N_i$ and $\overline{N}_j = N_j$. For the coefficient $\bar{n}_{ij}$ of $\overline{M}$, we can write

$$\bar{n}_{ij} = \langle \overline{N}_i, \overline{N}_j \rangle$$

Since $\overline{N}_i = N_i$ and $\overline{N}_j = N_j$, we get the result of the theorem.

Corollary 2.4. For the third fundamental forms $\overline{\mathrm{III}}$ and III of $\overline{M}$ and M, we can write

$$\overline{\mathrm{III}} = \mathrm{III}. \tag{2.17}$$

Proof. The proof is clear.

Theorem 2.6. Let M be regular hypersurface with the constant support function in E^{n+1} and $\overline{M}$ be a a-pedal hypersurface of M. For the i^{th}-principal curvature $\bar{k}_i$ of $\overline{M}$, we have

$$\bar{k}_i = \frac{1}{h^a} . \tag{2.18}$$

Proof. We consider a parameter system which has curvature lines on .
For the i^{th}-principal curvature $\overline{k}_i$ of $\overline{M}$, we can write

$$\overline{k}_i = \frac{\overline{b}_{ii}}{\overline{g}_{ii}}, \quad 1 \leq i \leq n. \tag{2.19}$$

Using the equations (2.9) and (2.12), we get the result the theorem.

Because of the equation (2.18), for the shape operator $\overline{S}$ of $\overline{M}$ we can write $\overline{S} = \frac{1}{h^a} I_n$ where I_n is a unit matrix. Then, it can be said that $\overline{M}$ has umbilical points.

Theorem 2.7. For the q^{th}-higher order Gaussian curvature $\overline{K}_q$ of $\overline{M}$, we have

$$\overline{K}_q = \binom{n}{q} \frac{1}{h^{qa}}, \quad 1 \leq q \leq n. \tag{2.20}$$

Proof. For the q^{th}-higher order Gaussian curvature $\overline{K}_q$ of $\overline{M}$, we can write

$$\overline{K}_q = \sum_{i_1<i_2<\ldots<i_q}^{n} k_{i_1} k_{i_2} \ldots k_{i_q}, \quad 1 \leq q \leq n. \tag{2.21}$$

Substituting by the equation (2.19) into the equation (2.21), we obtain

$$\overline{K}_q = \sum_{i_1<i_2<\ldots<i_q}^{n} \frac{1}{h^a} \frac{1}{h^a} \ldots \frac{1}{h^a}$$

$$= \binom{n}{q} \frac{1}{h^{qa}}.$$

If we take the q = n and q = 1, we get the following results:

1-) For $q = 1$, we can write

$$\overline{K}_1 = \binom{n}{1} \frac{1}{h^a}$$

$$= \frac{n}{h^a}.$$

The curvature $\overline{K}_1$ is called the mean curvature of $\overline{M}$ and denoted by $\overline{H}$.

2-) For $q = n$, we can write

$$\overline{K}_n = \binom{n}{n} \frac{1}{h^{na}} = \frac{1}{h^{na}}.$$

The curvature $\overline{K}_n$ is called the Gauss curvature of $\overline{M}$ and denoted by $\overline{K}$.

Theorem 2.8. For the characteristic polynom $P_{\overline{S}}(\lambda)$ of $\overline{M}$, we have

$$P_{\overline{S}}(\lambda) = \lambda^n + \sum_{q=1}^{n} (-1)^n \binom{n}{q} \frac{1}{h^{qa}} \lambda^{n-q} \tag{2.22}$$

where $\overline{S}$ is the shape operator of $\overline{M}$.

Proof. For the characteristic polynom $P_{\overline{S}}(\lambda)$ of $\overline{M}$, we can write

$$P_{\overline{S}}(\lambda) = \lambda^n + (-1)^1 \overline{K}_1 \lambda^{n-1} + \dots + (-1)^{n-1} \overline{K}_{n-1} \lambda + (-1)^n \overline{K}_n \tag{2.23}$$

Substituting by the equation (2.20) into the equation (2.23), we can obtain

$$P_{\overline{S}}(\lambda) = \lambda^n + (-1)^1 \binom{n}{1} \frac{1}{h^a} \lambda^{n-1} + \dots + (-1)^{n-1} \binom{n}{n-1} \frac{1}{h^{(n-1)a}} \lambda + (-1)^n \binom{n}{n} \frac{1}{h^{na}}$$

$$P_{\overline{S}}(\lambda) = \lambda^n + \sum_{q=1}^{n} (-1)^n \binom{n}{q} \frac{1}{h^{qa}} \lambda^{n-q}.$$

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