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ON DEFORMATIONS PRODUCT

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ABSTRACT

The product of deformations in Riemannian manifolds is defined. The product of infinitesimal isometric deformations (IID) is considered and some results of submanifolds of codimension ≥ 2 are established. We also study the second fundamental form on the product manifold and prove that the product $M_1 \times M_2$ of two hypersurfaces $M_1 \subset \overline{M}_1$ and $M_2 \subset \overline{M}_2$, where $\overline{M}_i$ is a Riemannian manifold, i = 1, 2, with a normal IID is a totally geodesic submanifold of $M_1 \times M_2$.

1. INTRODUCTION

1.1. Deformations of submanifolds

Goldstein and Ryan [1], formulated the theory of infinitesimal isometric deformations for submanifolds in a general Riemannian manifold. They defined infinitesimal rigid submanifolds of Riemannian manifolds and then specialized to the case of Euclidean spaces to obtain some results concerning infinitesimal rigidity of hyperspheres as dealing with submanifolds of arbitrary codimension is complicated.

Consider a submanifold S = (M,r), where $r: M \to \overline{M}$ is an immersion into a Riemannian manifold $(\overline{M}, \langle,\rangle)$ [1]. Let $I = [-\delta,\delta] \subset \mathbb{R}$, for some $\delta > 0$, then a map

$$\gamma: I \times M \to \overline{M} \tag{1.1}$$

is said to be a deformation of S if $\gamma_0 = r$ and γ_t is an immersion for each $t \in I$. Note that we write $\gamma(t,x) = \gamma_t(x)$ and $\gamma_t(M) = M_t$. In this way, each γ_t induces a Riemannian metric g_t on M. The map γ is said to be an isometric deformation (ID) of S if $g_t = g_0$ for each $t \in I$ and to be an infinitesimal isometric deformation (IID) of S if $g'(0) = 0$. In [1], the vector field z associated with a deformation is defined so as to determine the infinitesimal properties of the deformation γ. (See the interesting examples given in [1]).

Goldstein and Ryan established in [1] the main theorem about the characterization of infinitesimal isometric deformations as follows:

Theorem 1.1. A deformation γ is an IID if and only if for X,Y vector fields of M,

$$\langle \overline{\nabla}_X z, Y\rangle + \langle X, \overline{\nabla}_Y z\rangle = 0, \tag{1.2}$$

where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$.

To study the rigidity of the sphere, Goldstein and Ryan defined a trivial IID to be the one whose deformation field z coincides with that of a deformation induced by a curve $\phi(t)$ in $I(\overline{M})$, the group of isometries of $\overline{M}$. They also proved that a deformation of a submanifold of E^{n+1} is trivial if and only if $z_x = ar(x) + b$ for every $x \in M$, for some skew-symmetric matrix a and some constant vector b.

Finally they defined a submanifold S = (M,r) to be infinitesimally rigid (IR) if all IID are trivial. Hence they proved that the standard sphere of radius R in E^{n+1} is infinitesimally rigid as a hypersurface.

In this paper, we define deformations product and hence we get more room to deal with submanifolds of higher codimensions. In particular, we show that the product $S^n \times S^m \subset E^{n+m+2}$ of the unit spheres $S^n \subset E^{n+1}$ and $S^m \subset E^{m+1}$, is infinitesimally rigid as a submanifold of codimension 2 in E^{n+m+2}.

1.2. Manifolds product

Cartesian product of manifolds is a constructive method for using known manifolds as building blocks to form more manifolds. Pandey, [4], studied the Cartesian product $M = M_1 \times M_2$ of two Riemannian manifolds M_1 and M_2 where the Riemannian metric G of M is given by

$$G(X,Y) = g^1(X_1,Y_1) + g^2(X_2,Y_2) \tag{1.3}$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ and X_i, Y_i are vector fields on M_i, $i = 1,2$ and g^i is the Riemannian metric of M_i, for $i = 1, 2$. The connection ∇ is M is taken to be

$$\nabla_X Y = \left(\overset{1}{\nabla}_{X_1} Y_1 \ , \ \overset{2}{\nabla}_{X_2} Y_2\right), \tag{1.4}$$

where ∇^i is the Riemannian connection on M_i, $i = 1,2$.

Note that vector fields and forms on the cartesian product $M=M_1 \times M_2$ of manifolds have a special behavior as we explain below. If $\lambda \in \mathbb{R}$, then we put

$$\lambda X = \lambda(X_1,X_2) = (\lambda X_1,\lambda X_2), \tag{1.5}$$

where X_i is a vector field on M_i, $i = 1, 2$. If $\alpha \in \mathbb{R}^2$, where $\alpha = (\alpha_1,\alpha_2)$ then we take

$$\alpha X = (\alpha_1,\alpha_2)(X_1,X_2) = (\alpha_1 X_1,\alpha_2 X_2). \tag{1.6}$$

If f: $M_1 \times M_2 \to \mathbb{R}$, then we take

$$Xf = (X_1,X_2)f = X_1 f + X_2 f. \tag{1.7}$$

To compute $X_1 f$ we consider the second variable x_2 in $f(x_1,x_2)$ as a constant. If f_i: $M_i \to \mathbb{R}$, $i = 1, 2$ and f: $M_1 \times M_2 \to \mathbb{R}^2$ is defined as $f(x_1,x_2) = (f_1(x_1),f_2(x_2))$, then we take

$$Xf = (X_1,X_2)(f_1,f_2) = (X_1 f_1,X_2 f_2) \tag{1.8}$$

and

$$fX = (f_1,f_2)(X_1,X_2) = (f_1 X_1,f_2 X_2). \tag{1.9}$$

We also have $X_1 f_2 = 0$ and $X_2 f_1 = 0$. Notice that for a general function f: $M_1 \times M_2 \to \mathbb{R}$, the term fX is not defined. If ω_i is a 1-form on M_i, $i = 1, 2$, we define an associated $\mathbb{R}$-valued 1-form ω on $M_1 \times M_2$ by

$$\omega(X_1,X_2) = \omega_1(X_1) + \omega_2(X_2). \tag{1.10}$$

We also define an associated $\mathbb{R}^2$-valued 1-form Ω on $M_1 \times M_2$ by

$$\Omega(X) = \Omega(X_1,X_2) = (\omega_1(X_1),\omega_2(X_2)). \tag{1.11}$$

If ω_i is a 2-form on M_i, $i = 1, 2$, we define an associated $\mathbb{R}$-valued 2-form ω on $M_1 \times M_2$ by

$$\omega((X_1,X_2),(Y_1,Y_2)) = \omega_1(X_1,Y_1) + \omega_2(X_2,Y_2). \tag{1.12}$$

(The metric G on $M_1 \times M_2$ is defined in this sense). Finally, we define an associated $\mathbb{R}^2$-valued 2-form Ω on $M_1 \times M_2$ as follows

$$\Omega((X_1,X_2),(Y_1,Y_2)) = (\omega_1(X_1,Y_1),\omega_2(X_2,Y_2)). \tag{1.13}$$

Using the pervious information about vector fields and forms, it is easy to see that the connection ∇ defined by (1.4) on $M = M_1 \times M_2$ is a Riemannian connection.

1.3. Second fundamental form of $M_1 \times M_2 \subset \overline{M}_1 \times \overline{M}_2$

We define the second fundamental form of the immersion r on the product manifold $M = M_1 \times M_2$ as follows: Let r_i: $M_i \to \overline{M}_i$ be the immersion of M_i into $\overline{M}_i$, for $i = 1, 2$. Let r: $M_1 \times M_2 \to \overline{M}_1 \times \overline{M}_2$ be the immersion of the product manifold defined as

$$r(x_1,x_2) = (r_1(x_1),r_2(x_2)) \tag{1.14}$$

where $x_i \in M_i$, for $i = 1,2$. Let the Riemannian connection on $\overline{M}=\overline{M}_1\times\overline{M}_2$ constructed according to equation (1.4) be denoted by $\overline{\nabla}$. Define

$$B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y, \tag{1.15}$$

where ∇ is the induced connection on $M_1 \times M_2$, and for local vector fields X, Y on M, and $\overline{X}$, $\overline{Y}$ local extensions to $\overline{M}$. Using equation (1.4) we have

$$\begin{aligned} B(X,Y) &= \left(\overset{1}{\overline{\nabla}}_{\overline{X}_1}\overline{Y}_1 , \overset{2}{\overline{\nabla}}_{\overline{X}_2}\overline{Y}_2\right) - \left(\overset{1}{\nabla}_{X_1}Y_1 , \overset{2}{\nabla}_{X_2}Y_2\right) \\ &= \left(\overset{1}{\overline{\nabla}}_{\overline{X}_1}\overline{Y}_1 - \overset{1}{\nabla}_{X_1}Y_1 , \overset{2}{\overline{\nabla}}_{\overline{X}_2}\overline{Y}_2 - \overset{2}{\nabla}_{X_2}Y_2\right) \\ &= \left(B^1(X_1,Y_1) , B^2(X_2,Y_2)\right), \end{aligned} \tag{1.16}$$

where B^1, B^2 are the bilinear, symmetric mappings given by

$$B^i(X_i,Y_i) = \overset{i}{\overline{\nabla}}_{\overline{X}_i}\overline{Y}_i - \overset{i}{\nabla}_{X_i}Y_i , \tag{1.17}$$

for i = 1, 2. It is easy to show that the map B(X,Y) is bilinear and symmetric. We now define the second fundamental form of r at $p \in M_1 \times M_2$ along the normal vector η as

$$II_\eta(X) = H_\eta(X,X), \tag{1.18}$$

where

$$\begin{aligned} H_\eta(X,Y) &= G\left(B(X,Y),\eta\right) \\ &= G\left((\overset{1}{B}(X_1,Y_1),\overset{2}{B}(X_2,Y_2)),(\eta_1,\eta_2)\right) \\ &= \overset{1}{g}(\overset{1}{B}(X_1,Y_1),\eta_1) + \overset{2}{g}(\overset{2}{B}(X_2,Y_2),\eta_2). \\ &= \overset{1}{H}_{\eta_1}(X_1,Y_1) + \overset{2}{H}_{\eta_2}(X_2,Y_2). \end{aligned} \tag{1.19}$$

Hence the second fundamental form of r is the sum of the second fundamental forms of r_i, i = 1, 2, i.e.,

$$H_\eta(X,Y) = \overset{1}{H}_{\eta_1}(X_1,Y_1) + \overset{2}{H}_{\eta_2}(X_2,Y_2), \tag{1.20}$$

where η_i is a normal vector on M_i, i = 1, 2.

So if we have M_i totally geodesic submanifold of $\overline{M}_i$ for i = 1, 2, then the product manifold $M = M_1 \times M_2$ is totally geodesic in $\overline{M}_1 \times \overline{M}_2$ as well. If $M = M_1 \times M_2$ is totally geodesic then

$$H_\eta(X,X) = 0 = \overset{1}{H}_{\eta_1}(X_1,X_1) + \overset{2}{H}_{\eta_2}(X_2,X_2)$$

and since $\overset{1}{H}_{\eta_1}(X_1,X_1)$ is independent of $\overset{2}{H}_{\eta_2}(X_2,X_2)$ then each should be zero as well, hence each M_i, i = 1, 2, will be totally geodesic. This proves that

Proposition 1.2. Let M_1 and M_2 be two submanifolds of M_1 and M_2, respectively. Then $M_1 \times M_2$ is totally geodesic if and only if M_i is totally geodesic for i = 1, 2.

2. MAIN RESULTS

We now introduce what we mean by deformations product. Let M_1, M_2 be two immersed submanifolds of $\overline{M}_1$, $\overline{M}_2$, respectively, with immersions r_i: $M_i \to \overline{M}_i$, $i = 1, 2$. Let γ^i: $I \times M_i \to \overline{M}_i$ be a deformation of M_i, $i = 1, 2$. Then a deformation of the product manifold $M_1 \times M_2$ is a map

$$\gamma: I \times (M_1 \times M_2) \to \overline{M}_1 \times \overline{M}_2, \tag{2.1}$$

where

$$\begin{aligned} \gamma(t,x_1,x_2) &= \gamma_t(x_1,x_2) \\ &= (\gamma^1(t,x_1),\gamma^2(t,x_2)) \\ &= (\overset{1}{\gamma}_t(x_1),\overset{2}{\gamma}_t(x_2)) \end{aligned} \tag{2.2}$$

such that $\gamma_0 = r$ where r is an immersion of $M_1 \times M_2$ into $\overline{M}_1 \times \overline{M}_2$ defined by equation (1.14), and γ_t is an immersion for each $t \in I$. It is clear from the last equation (2.2) that the deformation of the product $M_1 \times M_2$ as an immersed submanifold of $\overline{M}_1 \times \overline{M}_2$ may be expressed as a product of two deformations of each component. Since each immersion $\overset{i}{\gamma}_t$ induces a Riemannian metric $\overset{i}{g}_t$ on M_i, $i = 1, 2$, then it is easy to see that γ_t will induce a metric G_t on the product manifold defined by

$$G_t = \overset{1}{g}_t + \overset{2}{g}_t . \tag{2.3}$$

Following Goldstein and Ryan [1], we define a deformation γ of $M_1 \times M_2$ to be an isometric deformation (ID) of $M_1 \times M_2$ if $G_t = G_0$ for all $t \in I$. A deformation product γ of $M_1 \times M_2$ is said to be an infinitesimal isometric deformation (IID) if $G'_t|_{t=0} = 0$. We now introduce the first result we have

Theorem 2.1. Let γ be a deformation of $M_1 \times M_2$. Then γ is an IID if and only if each of its components is an IID.

Proof. Let γ^i, $i = 1, 2$, be IID. Then by definition, $\overset{i}{g}_t = \overset{i}{g}_0 + o(t^2)$, for $i = 1, 2$, i.e., $\overset{i}{g}_t$ is of second order in the deformation parameter. Since G_t on the product manifold $M_1 \times M_2$ at t is of the form $G_t = \overset{1}{g}_t + \overset{2}{g}_t$, then we have

$$\begin{aligned} G_t &= \overset{1}{g}_0 + o(t^2) + \overset{2}{g}_0 + o(t^2) \\ &= \overset{1}{g}_0 + \overset{2}{g}_0 + o(t^2), \end{aligned} \tag{2.4}$$

i.e., G_t is of second order in the deformation parameter and so the deformation γ on $M_1 \times M_2$ is an IID.

On the other hand, if γ on $M_1 \times M_2$ is an IID then $G_t = G_0 + o(t^2)$. In particular, we have $G_t((X_1,0),(Y_1,0)) = g_t^1(X_1,Y_1)$ which implies that $g_t^1 = g_0^1 + o(t^2)$ and so γ is an IID. Similarly, $G_t((0,X_2),(0,Y_2)) = g_t^2(X_2,Y_2)$ which in turn implies that $g_t^2 = g_0^2 + o(t^2)$. Consequently, γ^2 is an IID.

So far we were dealing with the metric G_t of the deformation γ. Now when it comes to considering the associated field z, we will have to deal with the connection on the product manifold $\overline{M}_1 \times \overline{M}_2$ and apply the theorem of Goldstein and Ryan.

Introducing the deformation field $z = (z_1,z_2)$ of γ we can give a second proof to the previous result in the following way.

Suppose γ is an IID, this is equivalent to saying that for vector fields $X = (X_1,X_2)$ and $Y = (Y_1,Y_2)$ on $M_1 \times M_2$, and for a deformation field $z = (z_1,z_2)$ along $M_1 \times M_2$,

$$G(\overline{\overline{\nabla}}_X z,Y) + G(X,\overline{\overline{\nabla}}_Y z) = 0. \tag{2.5}$$

By definitions (1.3) of G and (1.4) of $\overline{\overline{\nabla}}$ we have,

$$g^1(\overline{\nabla}^1_{X_1} z_1,Y_1) + g^2(\overline{\nabla}^2_{X_2} z_2,Y_2) + g^1(X_1,\overline{\nabla}^1_{Y_1} z_1) + g^2(X_2,\overline{\nabla}^2_{Y_2} z_2) = 0. \tag{2.6}$$

Since each metric g^i is independent of the other then

$$g^i(\overline{\nabla}^i_{X_i} z_i,Y_i) + g^i(X_i,\overline{\nabla}^i_{Y_i} z_i) = 0, \quad i = 1, 2. \tag{2.7}$$

So we conclude that each γ^i, i = 1, 2, is an IID. Now we want to show that if each γ^i is an IID, then γ is an IID as well. Since γ^i is an IID the $g^i(\overline{\nabla}^i_{X_i} z_i,Y_i) + g^i(X_i,\overline{\nabla}^i_{Y_i} z_i) = 0, \quad i = 1, 2.$ Hence

$$\begin{aligned} G(\overline{\overline{\nabla}}_X z,Y) + G(X,\overline{\overline{\nabla}}_Y z) &= g^1(\overline{\nabla}^1_{X_1} z_1,Y_1) + g^2(\overline{\nabla}^2_{X_2} z_2,Y_2) \\ &\quad + g^1(X_1,\overline{\nabla}^1_{Y_1} z_1) + g^2(X_2,\overline{\nabla}^2_{Y_2} z_2) \\ &= 0 \end{aligned} \tag{2.8}$$

which implies that γ is an IID.

Theorem 2.2. Let S^n, S^m be the standart hyperspheres in E^{n+1} and E^{m+1}, respectively. Then the product $S^n \times S^m$ is an infinitesimally rigid submanifold of codimension 2 in E^{n+m+2}.

Proof. To show that $S^n \times S^m$ is infinitesimally rigid, we taken an IID field $z = (z_1,z_2)$ on $S^n \times S^m$ and prove that it is trivial. Since $z = (z_1,z_2)$ is an IID field along $S^n \times S^m$ then Theorem (2.1) guarantees that each z_i, $i = 1, 2$ is an IID field as well. Since S^n is infinitesimally rigid [1] in E^{n+1} then z_1 is a trivial IID field. Also for S^m, z_2 is a trivial IID field, By [1], we may consider each z_i, for $i = 1, 2$, in the form

$$z_i = a_i r_i + b_i, \tag{2.9}$$

where a_i is a skew-symmetric matrix and b_i is some constant vector. So if we can write z in this form, i.e., $z = Ar + B$, then the proof will be complete. Since

$$\begin{aligned} z &= (z_1,z_2) \\ &= (a_1 r_1 + b_1\ ,\ a_2 r_2 + b_2) \end{aligned} \tag{2.10}$$

then we can write z in the form,

$$z = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{2.11}$$

As each a_i is skew-symmetric, then

$$\begin{aligned} A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} &= \begin{bmatrix} -a_1^t & 0 \\ 0 & -a_2^t \end{bmatrix} \\ &= - \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}^t = - A^t . \end{aligned} \tag{2.12}$$

Hence A is skew-symmetric. So we conclude that the IID field $z = (z_1,z_2)$ is trivial and hence $S^n \times S^m$ is infinitesimally rigid.

Remark 2.3. In the previous theorem, we have taken an arbitrary IID field to show that it is trivial. Note that we can identify any deformation of $M_1 \times M_2$ in the Riemannian manifold $\overline{M}_1 \times \overline{M}_2$ with the deformation product by using the projection maps π_1 and π_2 onto the manifolds $\overline{M}_1$ and $\overline{M}_2$, respectively.

Remark 2.4. The previous theorem shows that we may discuss the infinitesimal rigidity problem of submanifolds with codimension 2 without complexification of manifolds under consideration. Considering the Cartesian product of more than two manifolds we may obtain similar results about submanifolds of arbitrary codimension.

Theorem 2.5. If $M = M_1 \times M_2$ is a submanifold of the Cartesian product $\overline{M} = \overline{M}_1 \times \overline{M}_2$ of two Riemannian manifolds $\overline{M}_1$ and $\overline{M}_2$, with each $M_i \subset \overline{M}_i$ a hypersurface, and $z = (z_1, z_2)$ is a normal IID of M, then M is a totally geodesic submanifold of $\overline{M}$.

Proof. For any vector fileds X, Y in M, we have

$$G(\overline{\nabla}_X z, Y) + G(X, \overline{\nabla}_Y z) = 0. \tag{2.13}$$

As z is a normal vector field of M we may write $z = fN$ where $N = (N_1, N_2)$ and N_i is a unit normal field along $M_i \subset \overline{M}_i$, i = 1, 2, and $f = (f_1, f_2)$ is any $\mathbb{R}^2$-valued function on M. Then we have

$$\begin{aligned} G(\overline{\nabla}_X z, Y) + G(X, \overline{\nabla}_Y z) &= G(\overline{\nabla}_X fN, Y) + G(X, \overline{\nabla}_Y fN) \\ &= g^1(\overline{\nabla}^1_{X_1} f_1 N_1, Y_1) + g^1(X_1, \overline{\nabla}^1_{Y_1} f_1 N_1) \\ &+ g^2(\overline{\nabla}^2_{X_2} f_2 N_2, Y_2) + g^2(X_2, \overline{\nabla}^2_{Y_2} f_2 N_2) \\ &= 2f_1 \overset{1}{H}_{N_1}(X_1, Y_1) + 2f_2 \overset{2}{H}_{N_2}(X_2, Y_2) = 0. \end{aligned} \tag{2.14}$$

From equation (2.14), as $\overset{1}{H}_{N_1}(X_1, Y_1)$ is independent of $\overset{2}{H}_{N_2}(X_2, Y_2)$ we conclude that $\overset{i}{H}_{N_i}(X_i, Y_i) = 0$, i = 1,2. Since $H_N(X,Y) = \overset{1}{H}_{N_1}(X_1, Y_1) + \overset{2}{H}_{N_2}(X_2, Y_2)$, then we have that $H_N(X,Y) = 0$ for every normal field N on M. Hence M is totally geodesic.

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