Commun. Fac. Sci. Univ. Ank. Series A1
V. 46. pp. 135-141 (1997)

THE SHEAF OF THE HOMOLOGY GROUPS OF THE COMPLEX MANIFOLDS

CEMİL YILDIZ

Department of Mathematics, Faculty of Science and Arts, Gazi University, Ankara/TURKEY

(Received May 28, 1997; Accepted June 26, 1997)

ABSTRACT

Let X be a connected complex n-dimensional manifold with fundemental group $F_x \neq \{1\}$, H be the sheaf of the groups over X [7] and $\Gamma(X,H)$ be the group of the global sections of H over X.

In this paper, the sheaf of the Homology groups is constructed by means of the homology groups H_x/N_x of the connected complex n-dimension manifold and given some characterizations. Finally, it is shown that if two connected complex manifolds are topologically equivalent, then their corrosponding sheaves of the homology groups are isomorphic.

1. INTRODUCTION

Let X be a connected complex n-dimension manifold and $F \neq \{1\}$ be the fundemental group of X with respect to the base point x, for any $x \in X$, (X,x) pointed n-dimension complex manifolds which have the same homotopy type.

If P is any H-group, then there exists a sheaf $H = \bigvee_{x \in X} H_x$ over X which is formed by P H-group. For each $x \in X$,

$$\Pi^{-1}(x) = [(X,x);P] = H_x$$

is the stalk of the sheaf which has a discrete topology (where [(X,x);P] is set of homotopy classes of homotopic maps preserving the base points from (X,x) to (P,p_0) [6,7].

If $x \in X$ is an arbitrarily fixed point, then there is $W = W(x)$ an open neighborhood of x in X and mappings $s: W \to H$ such that s is continuous and $\Pi \circ s = 1_W$. Hence the mappings s is called a section of H over W.

Let us denote all of the sections of H over X by $\Gamma(W,H)$. The set $\Gamma(W,H)$ is a group [7].

The sheaf H satisfies the following properties:

1- Any two stalks of H are isomorphic with each other [6].

2- Let W_1, $W_2 \subset X$ be any two open sets $s_1 \in \Gamma(W_1,H)$ and $s_2 \in \Gamma(W_2,H)$. If $s_1(x_0) = s_2(x_0)$ for any point $x \in W_1 \cap W_2$, then $s_1 = s_2$ over the whole $W_1 \cap W_2$ [6].

3- Let $W \subset X$ be an open set. Every section over W can be extended to a global section over X. In other words, the sections over W are the restrictions of the sections over X, i.e., $s \mid W \in \Gamma(W,H)$, for every $s \in \Gamma(X,H)$.

4- Let $x \in X$ be any point and $W = W(x)$ be an open set. Then, $\Pi^{-1}(W) = \bigvee_{i\in I} s_i(W)$ for every $s_i \in \Gamma(W,H)$ and $\Pi|s_i(W) : s_i(W) \rightarrow W$ is a topological mapping for each $i \in I$. Thus, H is a covering space of X, such that to each point $\sigma_x = [f]_x \in H_x$ there corresponds a unique section $s \in \Gamma(W,H)$ such that $s(x) = \sigma_x$. Furthermore, H_x is isomorphic to $\Gamma(W,H)$. In particular, $H_x \cong \Gamma(W,H)$ [6].

5- A topological stalk preserving mapping of H onto itself is called a sheaf isomorphism or a cover transformation, and the set of all cover transformation of H is denoted by T.

Clearly, T is a group. T is isomorphic to the group $\Gamma(X,H)$. Hence $H_x \cong \Gamma(X,H) \cong T$. Thus, T is transitive and H is a regular covering space of X [6].

2. SUBSHEAVES OF H

Definition 2.1. Let H be the sheaf of the groups formed by H-group over X and $H' \subset H$ be an open set. Then H' is called a subsheaf of groups, if;

i) $\Pi(H') = X$

ii) For each point $x \in X$ the stalk H'_x is a subgroup of H_x [3].

Definition 2.2. Let H be sheaf of the groups formed by H-group over X and $N' \subset H$ be a subsheaf of groups. Then N′ is called a normal subsheaf, if the stalk $N'_x \subset H_x$ is a normal subgroup for each $x \in X$.

Let $N \subset H$ be a subsheaf of groups and $W \subset X$ be an open set. Then, the set $\Gamma(W,N') \subset \Gamma(W,H)$ is a subgroup. Moreover, if $N' \subset H$ is a normal subsheaf, then $\Gamma(W,N') \subset \Gamma(W,H)$ is a normal subgroup. In particular, if we take $W = X$, then $\Gamma(X,N') \subset \Gamma(X,H)$ is a normal subgroup. Consequently, each subsheaf of group gives a subgroup of $\Gamma(X,H)$ and each normal subsheaf gives a normal subgroup of $\Gamma(X,H)$ [6].

Conversely, let us suppose that, $\Gamma(X,H)$ is group of global sections of H over X and $G \subset \Gamma(X,H)$ be a subgroup. Then, the set $\{s_i(x): s_i \in G\}$ is a subgroup of H_x over X for each $x \in X$.

Let us denote $\{s_i(x): s_i \in G\}$ by N'_x. Then $N' = \bigvee_{x \in X} N'_x$ is a set over X with the natural projection $\Pi' = \Pi \mid N'$ and $G = \Gamma(X,N')$.

Moreover, if $G \subset \Gamma(X,H)$ is a normal subgroup, then each stalk of N′ is a normal subgroup of H_x. (N',Π') is a subsheaf of the groups and (N',Π') is a normal subsheaf (or sheaf of normal subgroups) of H, if $G \subset \Gamma(X,H)$ is normal subgroup [1,6].

Then, we can state the following theorem:

Theorem 2.1. Let H be the sheaf of the groups formed by H-group over X and $\Gamma(X,H)$ be the group of global sections of H. Then, the subgroups of $\Gamma(X,H)$ define all the subsheaves of groups of H. In particular a normal subgroup of $\Gamma(X,H)$ defines a normal subsheaf of H.

It is easily seen that, subsheaves of groups of H (or normal subsheaves of H) have all the properties of H stated in Section 1. Thus, they are also regular covering spaces of X and $N'_x \cong \Gamma(X,N') \cong T'$ for each subsheaf of groups $N' \subset H$.

Definition 2.3. Let G be a commutator subgroup of $\Gamma(X,H)$. The normal subsheaf of H defined by G is called the commutator subsheaf of H and it is denoted by [H,H]. Moreover, $G = \Gamma(X,[H,H])$.

Now let $c \in X$ be an arbitrary fixed point and N_c be commutator subgroup of H_c. It is known that N_c is the smallest subgroup for which H_c/N_c is additive. H_c/N_c is called the homology group of the connected complex n-dimension manifold and it is denoted by $\overline{H}_c$. The sheaf N determined by the commutator subgroup is called homology covering space of X [1,5].

Let $x \in X$ be any point, $\overline{H}_x$ be the Homology group of X with respect to x i.e., $\overline{H}_x = H_x/N_x$, where $N_x \subset H_x$ is commutator subgroup. Let $\overline{H} = \bigvee_{x \in X} \overline{H}_x$. $\overline{H}$ is a set over X and the mapping $\overline{\Pi}(\sigma_x) = \overline{\Pi}([f]_x) = x$ for any $\overline{\sigma}_x = [f]_x \in \overline{H}_x$ is onto.

We introduce on $\overline{H}$ a topology as follows:

Let $c \in X$ be arbitrary fixed point and $\overline{H}_c$ be the homology group of X with respect to the point c. Then there exists an open neighbourhood $W = W(c)$ of c in X. If $\sigma_c = [h]_c \in [(X,c);P] = H_c$ is an arbitrary fixed element and $x \in W$ is any point, then there is a homotopy equivalance map $\psi : (X,x) \rightarrow (X,c)$. Hence the map $h \circ \psi : (X,x) \rightarrow (P,p_0)$ is continuous and base point preserving. $[f]_x \in [(X,x);P] = H_x$ is a homotopy class of map $f = h \circ \psi$. Therefore, we can define a mapping: $\overline{s}: W \rightarrow \overline{H}$ with $\overline{s}(x) = \overline{s(x)} = \overline{[f]}_x$, where $s \in \Gamma(W,H)$. Thus, $\overline{s}$ is well-defined,

$$\overline{\Pi} \circ \overline{s} = 1_w \text{ and } \bar{s}(c) = \overline{s(c)} \in \bar{s}(W) \subset \overline{H}.$$

Let us denote the set all of the mappings defined over W by $\Gamma(W,\overline{H})$.

Now, if $\beta(x)$ is a basis of open neighbourhoods of x, then

$$\beta = \{\overline{s}(W) : W \in \beta(x), \overline{s} \in \Gamma(W,\overline{H})\}$$

is a topology base on $\overline{H}$. In this topology, the mappings $\overline{\Pi}$ and $\overline{s}$ are continuous and $\overline{\Pi}$ is a local homeomorphism.

Thus, $(\overline{H},\overline{\Pi})$ is a sheaf over X. $\overline{s}$ is called a section of the sheaf $\overline{H}$ over W. Let us denote the collection of all sections of $\overline{H}$ over W by $\Gamma(W,\overline{H})$ which is an abelian group [7]. The homology group $\overline{H}_x$ is called the stalk of the sheaf $\overline{H}$ over X, for each $x \in X$ [4].

Definition 2.4. The sheaf, $(\overline{H},\overline{\Pi})$ is called the sheaf of the Homology groups over the connected complex n-dimension manifold X. Moreover, $\overline{H}$ is a sheaf of abelian groups [3].

Then we can state,

Theorem 2.2. Let $\bar{s} \in \Gamma(W,H)$. Then $\overline{\Pi} \mid \bar{s}(W) : \bar{s}(W) \to W$ is a topological mapping and $(\overline{\Pi} \mid \bar{s}(W))^{-1} = \bar{s}$

Proof. If we consider the statement $\overline{\Pi} \circ \bar{s} = 1_w$, then for each

$x \in W$, $\bar{s} \circ (\overline{\Pi} \mid \bar{s}(W))\ (\bar{s}(x)) = (\bar{s} \circ \overline{\Pi} \circ \bar{s})(x) = \bar{s}(x)$,

Therefore $\bar{s} \circ (\overline{\Pi} \mid \bar{s}(W)) = 1_{\bar{s}(W)}$ [2].

Remark. It is easily seen that the sheaf $(\overline{H},\overline{\Pi})$ has all the properties of H stated in Introduction. Thus, it is also regular covering spaces of X and $\overline{H}_x \cong \Gamma(X,\overline{H}) \cong T'$, where T' is the set of all cover transformations of $\overline{H}$.

3. CHARACTERIZATIONS

In this section, we will explore the relation between connected complex manifolds and the sheaf of homology groups constructed over complex manifolds (see [1] for some definitions).

Theorem 3.1. Let P be any H-group and X_1, X_2 be connected complex manifolds of dimension n and $\overline{H}_1$, $\overline{H}_2$ be the corresponding sheaves of homology groups respectively. If the open map γ: $X_1 \to X_2$ is given as continuous and surjective then there exists a homeomorphism between the pairs $(X_1,\overline{H}_1)$ and $(X_2,\overline{H}_2)$.

Proof. $x_1 \in X_1$ be an arbitrary fixed point. Then $\gamma(x_1) \in X_2$ and $[(X_1,x_1):P] = H_{x_1} \subset H_1$, $[(X_2,\gamma(x_1));P] = H_{\gamma(x_1)} \subset H_2$ are the corresponding stalks.

If (X_1,x_1), $(X_2,\gamma(x_1))$ are pointed spaces and f_2, g_2 are base point preserving continuous maps from $(X_2,\gamma(x_1))$ to (P,p_0) then f_1, g_1 base point preserving continuous maps from (X_1,x_1) to (P,p_0) can be defined as

$f_1 = f_2 \circ \gamma$, $g_1 = g_2 \circ \gamma$, respectively. Furthermore if $f_2 \sim g_2$ rel $\gamma(x_1)$, then it can be easily shown that $f_1 \sim g_1$ rel x_1. Thus the correspondence $[f]_{\gamma(x_1)} \to [f \circ \gamma]_{x_1}$ is well-defined and the maps homotopy classes of base-point preserving continuous maps from $(X_2,\gamma(x_1))$ to (P,p_0) to the homotopy classes of base-point preserving continuous maps from (X_1,x_1) to (P,p_0). That is, to each element $[f]_{\gamma(x_1)}$ there corresponds a unique element $[f \circ \gamma]_{x_1}$. So $\overline{[f \circ \gamma]}_{\gamma(x_1)} \in \overline{H}_{\gamma(x_1)}$.

Since the point $x_1 \in X_1$ is arbitrary fixed, the above correspondence gives us a map $\overline{\gamma}^*\colon \overline{H}_1 \to \overline{H}_2$ such that $\overline{\gamma}^*(\overline{[f]}) = \overline{[f \circ \gamma]} \in \overline{H}_1$, for every $\overline{[f]} \in \overline{H}_2$. $\overline{\gamma}^*$ is a stalk preserving mapping and a homomorphism on each stalk.

To complete the proof, let us show that $\overline{\gamma}^*$ is continuous and an open map. Since γ^* is continuous and an open map [7], it can be easily shown that $\overline{\gamma}^*$ is continuous and an open map. Thus, $(\overline{\gamma}^*,\gamma)$ is a homomorphism between the pairs $(X_1,\overline{H}_1)$ and $(X_2,\overline{H}_2)$ [7].

Then we can state the following theorem:

Theorem 3.2. Let the pairs $(X_1,\overline{H}_1)$, $(X_2,\overline{H}_2)$, $(X_3,\overline{H}_3)$ and the surjective, open and continuous maps $\gamma_1\colon X_1 \to X_2$, $\gamma_2\colon X_2 \to X_3$ be given. Then, there exists a homomorphism $(\overline{\gamma}^*,\gamma)\colon (X_1,\overline{H}_1) \rightleftarrows (X_3,\overline{H}_3)$ such that

$$\gamma = \gamma_2 \circ \gamma_1,\ \overline{\gamma}^* = \overline{\gamma}_1^* \circ \overline{\gamma}_2^*$$

Proof. Since $\gamma_2 \circ \gamma_1 : X_1 \to X_2$ is a surjective, open and continuous map, there exists a homomorphism

$$(\gamma^*,\gamma)\colon (X_1,H_1) \rightleftarrows (X_3,H_3)$$

(Theorem 3.1). To prove this theorem it is sufficient to show that $\overline{\gamma}^* = \overline{\gamma}_1^* \circ \overline{\gamma}_2^* = \overline{\gamma_1^* \circ \gamma_2^*}$. In fact, for any $\overline{[f]} \in \overline{H}_3$, we must show that

$$\overline{\gamma}^*\,(\overline{[f]}) = (\overline{\gamma_1^* \circ \gamma_2^*})(\overline{[f]}).$$

$$\overline{\gamma}^*\,(\overline{[f]}) = \overline{[f \circ \gamma]} = \overline{[f \circ (\gamma_1 \circ \gamma_2)]} = \overline{[(f \circ \gamma_2) \circ \gamma_1)]}$$
$$= \overline{\gamma}_1^*(\overline{[f \circ \gamma_2]}) = (\overline{\gamma_1^* \circ \gamma_2^*})(\overline{[f]})$$

Therefore $\overline{\gamma}^* = \overline{\gamma_1^* \circ \gamma_2^*}$.

Now, we can state the following theorems:

Theorem 3.3. There is a contravariant functor from the category of connected complex manifolds of dimension n and surjective open and continuous maps to the category of sheaves of homology grops and sheaf homomorphisms.

Theorem 3.4. Let the pairs $(X_1, \overline{H}_1)$ and $(X_2, \overline{H}_2)$ be given. If γ: $X_1 \rightarrow X_2$ is a topological map, then there exists an isomorphism between the pairs $(X_1, \overline{H}_1)$ and $(X_2, \overline{H}_2)$.

REFERENCES

[1] BALCI, S., "On the restricted sheaf" Commun. Fac. Sci. Univ. Ank. Series A1 V. 37, pp., 1-4 (1988).

[2] GRAUERT, H., FRITZSHE, K., Several Complex Variables, Springer Verlag (1976).

[3] GUNNING, R.C., ROSSI, H., Analytic Functions of Several Variables, Prentice-Hall, Inc., Englewood cliffs., N.T. 1965.

[4] ULUÇAY, C., "On Homology covering spaces and sheaf associated to the homology group." Commun. Fac. Sci. Univ. Ankara, Ser. A1, Mathematiques, Tom: 33, 1984, 22-28.

[5] ULUÇAY, C., "On the homology group of the complex analytic manifolds." Comm. Fac. Sci. Univ. Ankara, Ser. A1 Mathematiques, Tom: 30, 1981, 37-44.

[6] YILDIZ, C., İÇEN, I., "H-Gruplarının Oluşturduğu Demetlerin Karakterizasyonu." Erciyes Uni. Fen Bil. Enst. Dergisi (1991).

[7] YILDIZ, C., Öçal, A.A., "The sheaf of the groups formed by H-groups over pointed topological spaces" Pure and Applied Mathematika Sciences. Vol. XXII No: 1-2, September (1985).