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SOME CONVOLUTION ALGEBRAS AND THEIR MULTIPLIERS

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ABSTRACT

Let G be a locally compact Abelian group (nondiscrete and non compact) with dual group $\hat{G}$. For $1 \leq P < \infty$, $A_p(G)$ denotes the vector space of all complex-valued functions in $L^1(G)$ whose Fourier transforms $\hat{f}$ belong to $L^p(\hat{G})$. Research on the spaces $A_p(G)$ was initiated by Warner [20] and Larsen, Liu and Wang [14]. Later several generalizations of these spaces to the weighted case was given by Gürkanlı [6], Feichtinger and Gürkanlı [4] and Fischer, Gürkanlı and Liu [5]. One of these generalization is the space $A^p_{w,\omega}(G)$, [4]. Also the multipliers of $A_p(G)$ were discussed in some papers such as [14], [1], [13], [3], [9] and proved that the space of multipliers of $A_p(G)$ is the space of all bounded complex-valued regular Borel measures on G.

In the present paper we discussed the multipliers of the Banach algebra $A^p_{w,\omega}(G)$ and proved that under certain conditions for given any multiplier T of $A^p_{w,\omega}(G)$ there exists a unique pseudo measure σ such that $Tf = \sigma * f$ for all $f \in A^p_{w,\omega}(G)$.

1. INTRODUCTION

Let G be a locally compact Abelian group with dual group $\hat{G}$ and let dx and $d\hat{x}$ be Haar measures on these groups respectively. We denote by K(G) the vector spaces of continuous functions on G with compact support and $K_c(G)$ the subclass of thoso functions in K(G) whose supports are contained in C. For functions in $L^1(G)$ the Fourier Transform is denoted by $\hat{f}$ or Ff. It is known that $\hat{f}$ is continuous on $\hat{G}$ which, vanish at infinity and the inequality $\|\hat{f}\|_\infty \leq \|f\|_1$ is satisfied ([16], 1.2.4. Theorem). We will denote the space of pseudo-measures by $A'(G)$, ([11], pp.97).

We set for $1 \leq p < \infty$,

$$L^p_w(G) = \{f \mid f, w \in L^p(G)\},$$

where w is the Beurling's weight function on G, i.e. w is a continuous function satisfying $w(x) \geq 1$ and $w(x+y) \leq w(x).\ w(y)$ for all x, y $\in$ G. It is known hat $L^p_w(G)$ is a Banach space under the norm

$$\| f \|_{p,w} = \left[\int_G |f(x)|^p \ .\ w^p(x)dx \right]^{\frac{1}{p}}$$

$L^1_w(G)$ is called a Beurling algebra [15]. In some parts of the present paper it is used an extra condition on W: A weight w is said to satisfy the Beurling-Domar condition (Shortly. (BD)) if one has

$$\sum_{n\geq 1} n^{-2} \log (w(nx)) < \infty$$

for all $x \in G$, [2].

It is known that regular maximal ideal space of $L^1(G)$ can be identified with the space of all generalized characters η on G such that $\eta \in L^\infty_\omega(G)$ and $\eta \leq \omega(x1.a.c$, [19]. If w satisfies the (B.D) condition the regular maximal ideal space of $L^1(G)$ is equal to the dual group $\widehat{G}$. (c.g[2] pp.15 and Theorem 2.11).

Now we set

$$\Lambda^W_K(G) = \left\{ f \in L^1_w(G) \middle| \hat{f} \in K\left(\widehat{G}\right) \right\},$$

$$\Lambda^W_{K,\hat{L}}(G) = \left\{ f \in \Lambda^K_w(G) \middle| \hat{f} \in K_{\hat{L}}\left(\widehat{G}\right) \right\}$$

where $\hat{L} \subset \left(\widehat{G}\right)$. Again $A^1(G)$ will denote the linear subspace of $L^1(G)$ consisting of those $f \in L^1(G)$ such that $\hat{f} \in L^1\left(\widehat{G}\right)$. It is known by the proof of ([11] Th. 6.2.2) that $A^1(G) \subset A(G)$, where

$$A(G) = \left\{ \hat{f} \;\middle|\; f \in L^1\left(\widehat{G}\right) \right\}.$$

Since $\Lambda^W_K(G) \subset A^1(G)$, then we have $\Lambda^W_K(G) \subset A^1(G) \subset A(G)$.

Again the Banach algebra $A^p_{w,\omega}(G)$ is defined to be the set of functions $f \in L^1_w(G)$ such that $\hat{f} \in L^P_W\left(\widehat{G}\right)$ with the norm

$$\| f \|^p_{w,\omega} = \| f \|_{f,w} + \left\| \hat{f} \right\|_{p,\omega} \quad , \quad 1 \leq p < \infty \ ,$$

where w and ω are Beurling's weight functions on G and $\widehat{G}$, respectively [4]. It is known that if w satisfies (BD), then the regular maximal ideal space of $L^1_w(G)$ is homeomorphic to the one of $A^p_{w,\omega}(G)$, ([5], Theorem. 1.16). It is also known that if W satisfies (B.D) then the regular maximal ideal space of $L^1_w(G)$ is the dual group $\widehat{G}$ ([2], pp.15 and theorem 2.11). Then if W satisfies (BD), the regular maximal ideal of $A^p_{w,\omega}(G)$ is the dual space $\widehat{G}$.

2. THE SPACES $E^W(G)$ AND THEIR PROPERTIES

Let G be a local compact abelian group, K and $\hat{L}$ be the compact subsets of G and $\widehat{G}$, respectively. We define the vector space $E^W_{K,\hat{L}}(G)$ as the space of all function u which can be represented as

$$u = \sum_{k=1}^{\infty} f_k * g_k \ , \ f_k \in K_K(G) \ , \ g_k \in L^1_w(G) \ , \ \hat{g}_k \in K_{\hat{L}}(\widehat{G}) \tag{1}$$

with

$$\sum_{k=1}^{\infty} \|f_k\|_{\infty} \cdot \|g_k\|_{1,w} < \infty$$

If one endows it with the norm

$$\|u\|^W_{K,\hat{L}} = \inf \sum_{k=1}^{\infty} \|f_k\|_{\infty} \cdot \|g_k\|_{1,w} < \infty$$

then it is easy to see that $E^W_{K,\hat{L}}(G)$ becomes a Banach space under this norm, where the infimum is taken over all representations of u as an element $E^W_{K,\hat{L}}(G)$. The proof is similar to that of Guadry [4] and Larsen [5]). Now we define the vector space $E^W(G)$ to be

$$E^W(G) = \bigcup_{K,\hat{L}} E^W_{K,\hat{L}}(G) \tag{3}$$

together with the internal inductive limit topology of the Banach spaces $E_{K,\hat{L}}(G)$.

Proposition 2.1.

If w satisfies the (B.D) condition then to every compact subset $\widehat{K} \subset \widehat{G}$ there is a constant $C_K > 0$ such that for every $f \in A^p_{w,\omega}(G)$ whose Fourier transform vanishes outside of $\widehat{K}$ satisfies

$$\| f \|^p_{w\omega} \leq C_{\widehat{K}} \cdot \| f \|_{1,w}. \tag{1}$$

Proof. Since the (B.D) condition is satisfied, then for given any compact subset $\widehat{K} \subset \widehat{G}$ one can find a function $g \in A^p_{w,\omega}(G)$ such that

$\hat{q}(x) = 1$ for all $x \in \hat{K}$. Take $f \in A^p_{w,\omega}(G)$ satisfying supp $\hat{F} \subset \hat{K}$. Hence we have $f * g \in A^p_{w,\omega}(G)$ and

$$\| f * g \|^p_{w\omega} \leq \| f \|_{1,w} \cdot \| g \|^p_{w,\omega} \tag{2}$$

because $A^p_{w,\omega}(G)$ is a module over $f \in L^1_w(G)$, ([3]). If we set $C_{\hat{K}} = \|g\|^p_{w,\omega}(G)$ then find

$$\| f * g \|^p_{w,\omega} \leq C_{\hat{K}} \cdot \| f \|_{1,w}. \tag{3}$$

Because the hypothesis, supp $\hat{F} \subset \hat{K}$ and $\hat{g}(\hat{x}) = 1$ over $\hat{K}$, we write $f \hat{*} g = \hat{f} \cdot \hat{g} = \hat{f}$. Hence combining (2) and (3) we have

$$\| f \|^p_{w,\omega} = \| f * g \|^p_{w,\omega} \leq C_{\hat{K}} \cdot \| f \|_{1,w}. \tag{4}$$

Lemma 2.2. If w satisfies the (B.D) condition, then the norms $\|\bullet\|_{1,w}$ and $\|\bullet\|^p_{w,\omega}$ are equivalent on $\wedge^W_{K,\hat{L}}(G)$.

Proof. It is easy to see that $\wedge^W_{K,\hat{L}}(G) \subset A^p_{w,\omega}(G)$ by the Theorem 4.2. in [2]. Let $f \in \wedge^W_{K,\hat{L}}(G)$ be given. Since supp $\hat{F} \subset \hat{K}$, by the proposition 2.1, one can find a constant $C_{\hat{L}} > 0$ such that

$$\| f \|^p_{w,\omega} \leq C_{\hat{L}} \cdot \| f \|_{1,w}.$$

It is also known that

$$\| f \|_{1,w} \leq \| f \|^p_{w,\omega}$$

Therefore these two norms are equivalent on $\wedge^W_{K,\hat{L}}(G)$

Theorem 2.3. If w satisfies (B.D) then

1) $E^W(G)$ is continuously embedded into $A^p_{w,\omega}(G)$,

2) $E^W(G)$ is everywhere dense in $\wedge^W_K(G)$ with respect to the norms $\|\bullet\|_{1,w}$ and $\|\bullet\|^p_{w,\omega}$.

3) $E^W(G)$ is everywhere dense in $A^p_{w,\omega}(G)$.

Proof.

1) Let $u \in E^W(G)$. Then $u \in E^W_{k,\hat{L}}(G)$ for a pair K, $\hat{L}$, where K and $\hat{L}$ are compact subsets of G and $\hat{G}$, respectively. Then u can be represent as

$$u = \sum_{k=1}^{\infty} f_k * g_k \ , \ f_k \in K_K(G) \ , \ \hat{g} \in K_{\hat{L}}(\hat{G}),$$

with

$$\sum_{k=1}^{\infty} \| f_k \|_{\infty} * \| g_k \|_{1,w} < \infty \tag{1}$$

Since $L^1_w(G)$ is a Banach convolution algebra then we write

$$\| u \|_{1,w} \le \sum_{k=1}^{\infty} \| f_k * g_k \|_{1,w} \le \sum_{k=1}^{\infty} \| f_k \|_{1,w} \cdot \| g_k \|_{1,w} \tag{2}$$

$$\le M \cdot \sum_{k=1}^{\infty} \| f_k \|_{\infty} \cdot \| g_k \|_{1,w} \tag{3}$$

where $M = \sup_{x \in K} |W(x)| \cdot \mu(K)$ and $\mu(K)$ is the measure of K. Also we have

$$\|\hat{u}\|_{p,\omega} = \left\| \sum_{k=1}^{\infty} \hat{f}_k \cdot \hat{g}_k \right\|_{p,\omega} \le \sum_{k=1}^{\infty} \left\{ \int_{\hat{L}} |\hat{f}_k(x) \cdot \hat{g}_k(x)|^p \cdot \omega^p(x)dx \right\}^{\frac{1}{p}}$$

$$\le \sum_{k=1}^{\infty} \|\hat{f}_k \cdot \hat{g}_k\|_{\infty} \cdot \left\{ \int_{\hat{L}} \cdot \ \omega^p(x)dx \right\}^{\frac{1}{p}} \le \sum_{k=1}^{\infty} \|f_k * g_k\|_1 \cdot \left\{ \int_{\hat{L}} \cdot \ \omega^p(x)dx \right\}^{\frac{1}{p}} \tag{4}$$

$$\le \sum_{k=1}^{\infty} \|f_k\|_{\infty} \cdot \|g_k\|_1 \cdot \left\{ \int_{\hat{L}} \cdot \ \omega^p(x)dx \right\}^{\frac{1}{p}} \cdot \mu(K) = N \sum_{k=1}^{\infty} \|f_k\|_{\infty} \cdot \|g_k\|_{1,w}$$

where $N = \left\{ \int_{\hat{L}} \cdot \ \omega^p(x)dx \right\}^{\frac{1}{p}} \cdot \mu(K)$

If one uses (3) and (4) obtains that $E^W(G) \subset A^p_{w,\omega}(G)$. Also by the Lemma 2.2 and (2), (4) the restriction of the identity map i from $E^W(G)$ into $A^p_{w,\omega}(G)$ to every subspace $E^W_{K,\hat{L}}(G)$ is continuous. Hence i is a continuous embedding from $E^W(G)$ into $A^p_{w,\omega}(G)$.

2) It is easy to see the inclusion $E^W(G) \subset \wedge^W_K(G)$. For the proof of denseness of $E^W(G)$ in $\wedge^W_K(G)$ with respect to the norm $\|\bullet\|_{1,w}$ take any function $h \in \wedge^W_K(G)$. Because the definition of $\wedge^W_K(G)$ there exists a compact subset $\hat{L} \subset \hat{G}$ such that $\hat{h} \in K_{\hat{L}}(\hat{G})$. Since w has (B.D) condition then $\wedge^W_K(G) \subset A^p_{w,\omega}(G)$ has an approximate identity $(e_\alpha)_{\alpha \in I}$ bounded in $L^1_w(G)$ with compactly supported Fourier transforms [2]. $L^1_w(G)$ also has another approximate identity $(u_\beta)_{\beta \in J}$ with compactly supported [6]. Hence

$$h * e_\alpha * u_\beta = u_\beta * h * e_\beta \in E^W(G),$$

for all $\beta \in J$ and

$$\|h * e_\alpha * u_\beta - h\|_{1,w} \leq \|h * e_\alpha * u_\beta - h * e_\alpha\|_{1,w} + \|h * e_\alpha - h\|_{1,w} \to 0.$$

Also since by the Lemma 2.2. the norms $\|\bullet\|_{1,w}$ and $\|\bullet\|^p_{w,\omega}$ are equivalent on $\wedge^W_{K,L}(G)$ for each pair (K, L), then it is easy to see that $E^W(G)$ is everywhere dense in $\wedge^W_K(G)$ with respect to the norm $\|\bullet\|^p_{w,\omega}$.

3) We know that $A^p_{w,\omega}(G)$ has an approximate identity bounded in the norm $L^1_w(G)$ ([2], Theorem 4.2). Using this approximate identity, a simple calculation shows that $\wedge^W_K(G)$ is everywhere dense in $A^p_{w,\omega}(G)$. If one combines this result with the first part of this theorem, observe that $E^W(G)$ is everywhere dense in $A^p_{w,\omega}(G)$.

Proposition 2.4. If $1 \leq p < \infty$ then

1) $L^1_w(G) \times L^p_\omega(\widehat{G})$ is a Banach space with the norm

$$\|(f,g)\| = \|f\|_{1,w} + \|g\|_{p,\omega}$$

where $(f,g) \in L^1_w(G) \times L^p_\omega(\widehat{G})$.

2) $A^p_{w,\omega}(G)$ is a closed subspace of the space $L^1_w(G) \times L^p_\omega(\widehat{G})$.

3) Every bounded linear functional F on $A^p_{w,\omega}(G)$ is represented by the formula

$$F(f) = \int_G f(x)\, \phi\,(x)dx + \int_G \hat{f}(y)\, \psi\,(y)dy$$

where $f \in A^p_{w,\omega}(G)$, $(\phi,\psi) \in L^\infty_{W^{-1}}(G) \times L^q_{\omega^{-1}}(\widehat{G})$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof of (1) is easy. For the proof of (2), define function $\phi_p(f) = (f, \hat{f})$ from $A^p_{w,\omega}(G)$ into $L^1_w(G) \times L^p_\omega(\widehat{G})$. ϕ_p is an isometry and $A^p_{w,\omega}(G) \hookrightarrow L^1_w(G) \times L^p_\omega(\widehat{G})$. This proves part (2).

Since $\frac{1}{p} + \frac{1}{q} = 1$, then the topological dual of $L^1_w(G) \times L^p_\omega(\widehat{G})$ is isomorphic to $L^\infty_{W^{-1}}(G) \times L^q_{\omega^{-1}}(\widehat{G})$ and every continuous linear functional on $L^1_w(G) \times L^q_\omega(\widehat{G})$ is represented by

$$F(f) = \int_G f(x)\, \phi\,(x)dx + \int_{\hat G} \hat f(y)\, \psi\,(y)dy \qquad (1)$$

$(\phi,\psi) \in L^{\infty}_{W^{-1}}(G) \times L^{q}_{\omega^{-1}}(\widehat{G})$. Because the fact (2) and by the Hahn Banach theorem, every continuous linear functional on $A^p{}_{w,\omega}(G)$ is also represented by the formula (1).

Proposition 2.5. If $\gamma \in (A^p{}_{w,\omega}(G))'$ and $f,g \in A^p{}_{w,\omega}(G)$, then we have

$$\langle f * g, \gamma\rangle = \int_G f(y) \cdot \langle \tau_y\, g, \gamma\rangle\, dy ,$$

where τ_y is the translation operator defined by $\tau_y\, g(x) = g(x-y)$.

Proof. By the proposition 2.4. we write

$$\langle f*g,\gamma\rangle = \int_G (f*g)(x)\, \phi\,(x)dx = \int_G \widehat{(f*g)}\,(t)\, \psi\,(t)dt, \qquad (1)$$

where $(\phi,\psi) \in L^{\infty}_{W^{-1}}(G) \times L^{q}_{\omega^{-1}}(\widehat{G})$ and $\frac{1}{p} + \frac{1}{q} = 1$.A simple calculation shows that

$$\int_G (f*g)(x)\, \phi\,(x)dx = \int_G f(y)\, \langle \tau_y g,\phi\rangle dy \qquad (2)$$

and

$$\int_{\hat G} \widehat{(f*g)}\,(t)\, \psi\,(t)dt = \int_{\hat G} f(t)\langle \widehat{\tau_y g},\, \psi\rangle dy \qquad (3)$$

If one combines these results obtains

$$\langle f*g,\gamma\rangle = \int_G f(t)\, \langle \tau_y g,\, \phi\rangle dt + \int_{\hat G} f(t)\, \langle \widehat{\tau_y g},\psi\rangle dt =$$

$$= \int_G f(t)\, \left\{\langle \tau_y g,\, \phi\rangle + \langle \widehat{\tau_y g},\, \psi\rangle\right\} dt = \int_G f(t)\langle \tau_y g,\gamma\rangle dt.$$

Proposition 2.6. Let $h \in \wedge_K(G)$. If w is symmetric and $u \in E^W(G)$, then $u \to \tilde{h}*u$ is a continuous function from $E^W(G)$ into $E^W(G)$, where $\tilde{h}(x) = h(-x)$.

Proof. Let $u \in E^W(G)$. There is a pair $(K,\hat{L})$ such that $f_k \in K_K(G)$, $\hat{g}_k \in K_{\hat L}(\widehat{G})$,

$$u = \sum_{k=1}^{\infty} f_k * g_k \quad \text{and} \quad \sum_{k=1}^{\infty} \|f_k\|_{\infty} \cdot \|g_k\|_{1,w} < \infty . \qquad (1)$$

Since $\tilde{h}$, $f_k \in L_W(G)$ then we have

$$\tilde{h} * u = \sum_{k=1}^{\infty} f_k * (\tilde{h} * g_k)$$

and

$$\sum_{k=1}^{\infty} \|f_k\|_\infty \|\tilde{h} * g_k\|_{1,w} \le \|\tilde{h}\|_{1,w} \sum \|f_k\|_\infty \cdot \|g_k\|_{1,w} < \infty \tag{2}$$

Hence $\tilde{h} * u \in E^W(G)$. For the continuity, it is enough to show the restriction of the mapping $u \to \tilde{h} * u$ to each $E^W_{K,\hat{L}}(G)$ is continuous. But this is immediate because if $\|u_n - u\|^W_{K,\hat{L}} \to 0$ then we have

$$\|\tilde{h} * u_n - \tilde{h} * u\|^W_{K,\hat{L}} \le \|\tilde{h}\|_{1,w} \cdot \|u_n - u\|_{K,\hat{L}} \to 0. \tag{3}$$

The proof of the following proposition is clear because of the Theorem 2.3. and Proposition 2.1.

Proposition 2.7. If w satisfies the (B.D) then we have $(A^p_{w,\omega}(G))' \subset (E^W(G))'$, where $(A^p_{w,\omega}(G))'$ and $(E^W(G))'$ are topological duals of $A^p_{w,\omega}(G)$ and $E^W(G)$ respectively.

Definition 2.8. Let $f \in \Lambda^W_K(G)$, $\sigma \in (E^W(G))'$ and w be a symmetric Beurling's weight. We are going to define the convolution $\sigma * f$ to be

$$\langle u, \sigma * f\rangle = \langle \tilde{f} * u, \sigma\rangle \tag{1}$$

where $u \in E^W(G)$. It is easly seen that (1) is well defined because the Proposition 2.6.

Let w be a symmetric weight and $\upsilon \in (E^W(G))'$. Then the linear fnctional $\tilde{\upsilon} \in (E^W(G))'$ is defined to be $\langle u, \tilde{\upsilon}\rangle = \langle \tilde{u}, \upsilon\rangle$ for all $u \in E^W(G)$.

3. MULTIPLIERS ON THE SPACE $A^p_{w,\omega}(G)$.

Definition 3.1. A multipliers on $A^p_{w,\omega}(G)$ is a bounded linear operator T on $A^p_{w,\omega}(G)$ which commutes with translation operators, that is $T\tau_s = \tau_s T$ for each $s \in G$. The space of all multipliers on $A^p_{w,\omega}(G)$ will be denoted by $M(A^p_{w,\omega}(G))$.

Proposition 3.1. If $T \in M(A^p_{w,\omega}(G))$, then $T(f*g) = Tf*g$ for all f, $g \in A^p_{w,\omega}(G)$.

Proof. Take any $T \in M(A^p_{w,\omega}(G))$, $f \in A^p_{w,\omega}(G)$ and $\gamma \in (A^p_{w,\omega}(G))'$. It is easy to prove that the map $f \to \langle Tf,\gamma\rangle$ is a continuous linear functional on $A^p_{w,\omega}(G)$. Then there exists $\psi \in (A^p_{w,\omega}(G))'$ such that $\langle f,\psi\rangle = \langle Tf,\gamma\rangle$ for all $f \in A^p_{w,\omega}(G)$. By the Proposition 2.5. one can write

$$\begin{aligned}\langle Tf*g,\gamma\rangle &= \int_G g(y)\,\langle \tau_y Tf,\ \gamma\rangle dy \\ &= \int_G g(y)\,\langle T\tau_y f,\gamma\rangle dy = \int_G g(y)\,\langle \tau_y f,\psi\rangle dy \\ &= \langle f*g,\psi\rangle = \langle T(f*g),\gamma\rangle.\end{aligned}$$

Using the Hahn Banach theorem we obtain $Tf*g = T(f*g)$ for every $f,g \in A^p_{w,\omega}(G)$.

Theorem 3.2. Let w be a symmetric weight on G satisfying (B.D). If $T \in M(A^p_{w,\omega}(G))$, then there exists a unique continous linear functional $\sigma \in (E^W(G))'$ such that $Tf = \sigma * f$ for all $f \in \Lambda^W_K(G)$.

Proof. If $u \in E^W_{K,\hat{L}}(G)$ then one writes

$$u = \sum_{k=1}^{\infty} f_k * g_k \tag{1}$$

for some $f_k \in K_K(G)$ and $g_k \in L^1_w(G)$ satisfying $\hat{g}_k \in K_{\hat{L}}(\hat{G})$. By the Proposition 2.1. we have

$$\begin{aligned}|(f_k * Tg_k)(0)| &\le \|f_k\|_\infty \cdot \|Tg_k\|_1 \le \|f_k\|_\infty \cdot \|Tg_k\|^p_{w,\omega} \\ &\le C_{\hat{L}} \cdot \|T\|\,\|f_k\|_\infty \cdot \|g_k\|_{1,w}.\end{aligned} \tag{2}$$

Hence the series

$$\sum_{k=1}^{\infty} f_k * Tg_k\ (0),$$

converges uniformly. If we set

$$\upsilon(u) = \sum_{k=1}^{\infty} f_k * Tg_k\ (0),$$

then it is easy to see that υ is well defined in the following means: If

$$\sum_{k=1}^{\infty} f_k * g_k$$

is a representation of 0 as an element of $E^W_{K,\hat{L}}(G)$ then

$$\sum_{k=1}^{\infty} f_k * Tg_k(0) = 0$$

Using the formula (2) one obtains

$$|\upsilon(u)| \le C_{\hat{L}} \|T\| \cdot \|u\|_{K,\hat{L}}$$

for all $\upsilon \in E^W_{K,\hat{L}}(G)$. Therefore $\upsilon \in (E^W(G))'$. Hence we have $\langle u, \tilde{\upsilon}*f\rangle = \langle \tilde{f}*u, \tilde{\upsilon}\rangle = \langle f*\tilde{u}, \upsilon\rangle = \tilde{u}*Tf(0) = \langle u, Tf\rangle$ for all $u \in E^W(G)$ and $f \in \Lambda^W_K(G)$. That means $Tf = \tilde{\upsilon} * f$ for each $f \in \Lambda^W_K(G)$. We set $\sigma = \tilde{u}$

Also since w satisfies (B.D), then there is a bounded approximate identity (e_α) in $L^1_w(G)$ ([2] Th. 4.2.). Let

$$h = \sum_{k=1}^{\infty} f_k * g_k \in E^W(G)$$

be given. Then there exists a pair $(K, \hat{L})$ such that $h \in E^W_{K,\hat{L}}(G)$. Since

$$\|e_\alpha * g_k - g_k\|_{1,w} \to 0,$$

using the equality

$$\|e_\alpha * h - h\|^W_{K,\hat{L}} = \left\|\sum_{k=1}^{\infty} f_k * [(e_a * g_k) - g_k]\right\|^W_{K,\hat{L}}$$

$$= \inf \sum \|f_k\|_\infty \cdot \|e_\alpha * g_k - g_k\|_{1,w}.$$

one easily shows that the set

$$\left\{f * h \mid f \in \Lambda^W_K(G),\ h \in E^W(G)\right\} \tag{3}$$

is dense in $E^W(G)$.

Assume that s is not unique. Then there exists $\sigma, \sigma' \in (E^W(G))'$ such that $Tf = \sigma * f = \sigma' * f$. Hence we have $\langle f * h, \sigma\rangle = \langle f * h, \sigma'\rangle$ for all $f \in \Lambda^W_K(G)$ and $h \in E^W(G)$. Using the denseness of (3) in $E^W(G)$ one obtains that $\sigma = \sigma'$. That means σ is unique.

We denote by A^ω the Banach algebra $H\left(L^1_\omega(\hat{G})\right)$ with its natural norm $\|\hat{f}\|^\omega = \|f\|_{1,\omega}$, [12].

Proposition 3.3. If w and ω satisfy (B.D) then $\Lambda^W_K(G)$ is dense in $A^\omega(G)$.

Proof. Since ω satisfies (B.D) then $\left(L^1_\omega(\widehat{G})\right)$ has a bounded Approximate identity $(u_j)_{j\in J}$ (shortly BAI) whose Fourier transforms have compact support ([3], Th. 4.2.). So, it is easily proved that the set $A^\omega_c(G) = A^\omega(G) \cap K(G)$ is dense in $A^\omega(G)$. Since W satisfies (B.D) then $L^1_w(G)$ also has a BAI $(e_\alpha)_{\alpha\in I}$ whose Fourier transforms have compact support. Suppose $\hat{f} \in A^\omega_C(G)$. Then $\left(e_\alpha * \hat{f}\right) \subset \wedge^W_K(G)$ for all $\alpha \in$ I. Again because the regularity of $F\left(L^1_\omega(\widehat{G})\right)$, given any compact subset $K_0 \subset (\widehat{G})$ there exists $g \in \wedge^W_K(G)$ such that $\hat{g}(x) = 1$ for all $x \in K_0$. Therefore we obtain

$$\left\|\hat{e}_\alpha - 1\right\|_{\infty,K_0} = \sup_{x\in K_0}\left|\hat{e}_\alpha(x) - 1\right| \le \left\|\hat{e}_\alpha \,.\, \hat{g} - \hat{g}\right\|_\infty$$
$$\le \left\|e_\alpha * g - g\right\|_{1,w} \to 0. \tag{1}$$

we let $C_0 = C + 1$ where $\|e_\alpha\| \le C$, for all $\alpha \in$ I. Since $\hat{f} \in A^\omega_c(G)$, then given $\varepsilon > 0$ there exists a compact subset $K \subset \widehat{G}$ such that

$$\int_{\widehat{G}-K} |f(x)|\omega\ (x)\ dx < \frac{\varepsilon}{2C_0}\,. \tag{2}$$

Moreover, because the formula (1) there exists an $\alpha_0 \in$ I such that if $\alpha > \alpha_0$ then

$$\left\|\hat{e}_\alpha - 1\right\|_{\infty,K} = \sup_{x\in K}\left|\hat{e}_\alpha(x) - 1\right| < \frac{\varepsilon}{2\| f \|_{1,\omega}}\,. \tag{3}$$

Using (2) and (3) we have

$$\left\|\hat{f} - e_\alpha * \hat{f}\right\|^\omega = \left\|f - \hat{e}\,.\,f\right\|_{1,\omega}$$
$$= \int_{\widehat{G}-K}\left|f(x) - \hat{e}_\alpha(x)f(x)\right|\omega(x)dx + \int_K\left|f(x) - \hat{e}_\alpha(x)f(x)\right|\omega(x)dx$$
$$\le \left(1 + \left\|\hat{e}_\alpha\right\|_\infty\right)\int_{\widehat{G}-K}|f(x)|\omega(x)dx + \left\|1 - \hat{e}_\alpha\right\|_{\infty,K}\| f \|_{1,\omega}$$
$$\le (1 + C)\int_{\widehat{G}-K}|f(x)|\omega(x)dx + \left\|1 - \hat{e}_\alpha\right\|_{\infty,K}\| f \|_{1,\omega}$$
$$\le C_0 \cdot \frac{\varepsilon}{2C_0} + \frac{\varepsilon}{2\| f \|_{1,\omega}} \cdot \| f \|_{1,\omega} = \varepsilon.$$

Since $A^\omega_c(G)$ is dense in $A^\omega(G)$, then given any $\hat{g} \in A^\omega(G)$ one can find $f \in A^\omega_c(G)$ such that $\left\|\hat{f} - \hat{g}\right\|^\omega < \varepsilon$. Then

$$\left\|\hat{g} - e_\alpha * \hat{f}\right\|^\omega \le \left\|\hat{g} - \hat{f}\right\|^\omega + \left\|\hat{f} - e_\alpha * \hat{f}\right\|^\omega . < 2\varepsilon \tag{4}$$

for all $\alpha \ge \alpha$. This completes the proof.

Corollary 3.4. If w and ω satisfy the conditions in Proposition 3.3. then $\Lambda_K^W(G)$ is dense in A(G).

Proof. Suppose $\hat{g} \in A(G)$. Since $K(\hat{G})$ is everywhere dense in $L^1(\hat{G})$, then given any $\varepsilon > 0$ there exists $h \in K(\hat{G}) \subset L^1_\omega(\hat{G})$ such that

$$\|\hat{g} - \hat{h}\|_A = \|g - h\|_1 < \frac{\varepsilon}{2}. \tag{1}$$

Hence by the Proposition 3.3. one can find $k \in \Lambda_K^W(G)$ such that

$$\|k - \hat{h}\|_A \leq \|k - \hat{h}\|^\omega < \frac{\varepsilon}{2}. \tag{2}$$

Combining (1) and (2) we have

$$\|\hat{g} - k\|_A \leq \|\hat{g} - \hat{h}\|_A + \|\hat{h} - k\|_A < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{3}$$

This proves our Corollary.

Now we recall that A(G) is a Banach algebra under pointwise multiplication operation with the norm $\|\hat{f}\|_A = \|f\|_1$. Every continuous linear functional on A(G) is called a pseudomeasure.

Proposition 3.5. If A′(G) denotes the algebra of all pseudomeasures on G, then $A'(G) \subset (E^W(G))'$.

Proof. Suppose that $u \in E^W(G)$. Then there exists a pair $(K, \hat{L})$ of compact sets such that $u \in E_{K,\hat{L}}(G)$. We also have

$$\|u\|_A = \left\|\sum_{k=1}^{\infty} f_k * g_k\right\|_A \leq \sum_{k=1}^{\infty} \|f_k * g_k\|_A = \sum_{k=1}^{\infty} \|\widehat{f_k * g_k}\|_1$$

$$= \sum_{k=1}^{\infty} \|\hat{f}_k \cdot \hat{g}_k\|_A \leq \sum_{k=1}^{\infty} \|\hat{f}_k\|_\infty \cdot \|\hat{g}_k\|_\infty \cdot \mu(\hat{L}). \tag{1}$$

If one combines the inequality

$$\|\hat{f}_k\|_\infty \leq \|f_k\|_1 \leq \|f_k\|_\infty \cdot \mu(K)$$

with (1), obtains

$$\|u\|_A \leq \sum_{k=1}^{\infty} \|f_k\|_\infty \cdot \|\hat{g}_k\| \cdot \mu(\hat{L}) \cdot \mu(K)$$

$$\leq \sum_{k=1}^{\infty} \|f_k\|_\infty \cdot \|g_k\|_1 \, \mu(\hat{L}) \cdot \mu(K) \leq \sum_{k=1}^{\infty} \|f_k\|_\infty \cdot \|g_k\|_{1,w} \, \mu(\hat{L}) \cdot \mu(K) < \infty. \tag{3}$$

That means $E^W(G) \subset A(G)$. Now if $\sigma \in A'(G)$ and $u \in E^W_{K,\hat{L}}(G)$ then we have

$$|\langle u,\sigma\rangle| \leq \|\sigma\| \cdot \|u\|_A \leq \|\sigma\| \cdot \sum_{k=1}^{\infty} \|f_k\|_\infty \cdot \|g_k\|_{1,w} \, \mu(\hat{L}) \cdot \mu(K).$$

Therefore

$$|\langle u,\sigma\rangle| \leq \|\sigma\| \,.\, \|u\|^{W}_{k,\hat{L}}\ \mu(\hat{L})\mu(K)$$

Since σ is continuous on every $E^{W}_{K,\hat{L}}(G)$ then $\sigma \in (E^{W}(G))$. This completes the proof.

Theorem 3.6. Assume that w and ω satisfy (B.D) and W is symmetric. If $T \in M(A^{p}_{w,\omega}(G))$, then there exists a unique pseudo-measure $\sigma \in A'(G)$ such that $Tf = \sigma * f$ for all $f \in A^{p}_{w,\omega}(G)$.

Proof. Suppose that $T \in M(A^{p}_{w,\omega}(G))$. By the proposition 3.1, we have $T(f * g) = Tf * g$ for all $f, g \in A^{p}_{w,\omega}(G)$, since $A^{p}_{w,\omega}(G)$ is commutative it is easy to see that $Tf * g = f * Tg$ for all $f, g \in A^{p}_{w,\omega}(G)$. Then we write $(Tf)^{\wedge}.\hat{g} = \hat{f}\,.\,(Tg)^{\wedge}$. Since W satisfies (B.D) then $A^{p}_{w,\omega}(G)$ has an approximate identities ([4], Theorem 4.2). Also it is known that $A^{p}_{w,\omega}(G)$ is a Banach convolution algebra ([4], Theorem 2.1). Hence $A^{p}_{w,\omega}(G)$ is a commutative Banach algebra without order (i.e if for all $f \in A^{p}_{w,\omega}(G)$, $f * A^{p}_{w,\omega}(G) = 0$ then $f = 0$). Again since W satisfies (B.D) then the regular maximal ideal space of $L^{1}_{w}(G)$ is the dual group $\hat{G}$ ([2], pp.15 and Theorem 2.11). It is also known that in the case W satisfies (B.D) condition the regular maximal ideal space of $L^{1}_{w}(G)$ is homeomorphic to the one of $A^{p}_{w,\omega}(G)$, ([5], Th. 1.16), which implies that the regular maximal ideal space of $A^{p}_{w,\omega}(G)$ is the dual space $\hat{G}$. Then there exists a unique bounded continuous function Φ on $\hat{G}$ such that $(Tf)^{\wedge}.(y) = \Phi(y)\,.\,\hat{g}(y)$ for all $g \in A^{p}_{w,\omega}(G)$ by the Theorem 1.2.2. in [11]. If $f \in \wedge^{W}_{K}(G)$ then $Tf \in L^{1}_{w}(G)$ and $(Tf)^{\wedge} = \Phi.\hat{f} \in K(\hat{G})$. Therefore $\wedge^{W}_{K}(G)$ is invariant under T. Since every element of $\wedge^{W}_{K}(G)$ is continuous (see introduction) then we can define a linear functional on $\wedge^{W}_{K}(G)$ as $L(f) = Tf(0)$ for all $f \in \wedge^{W}_{K}(G)$. Also we write.

$$|L(f)| = |Tf(0)| \leq \|Tf\|_{\infty} \tag{1}$$

Since $Tf \in \wedge^{W}_{K}(G) \subset A(G)$ then there exists $g \in L^{1}(G)$ such that $\hat{g} = Tf$. If one uses the inequalities $\hat{\hat{g}} = \tilde{g}$ and $\|\tilde{g}\|_{1} = \|\hat{\hat{g}}\|_{1}$ writes

$$\|\widehat{Tf}\|_{1} = \|\hat{\hat{g}}\|_{1} = \|\tilde{g}\|_{1} = \|g\|_{1} \tag{2}$$

where $\widetilde{g}(x) = g(-x)$. Now if we combine (1) and (2) obtain

$$|L(f)| \leq \| Tf \|_\infty = \| \hat{g} \|_\infty \leq \| g \|_1 = \| \widehat{Tf} \|_1$$
$$= \| \Phi \hat{f} \|_\infty \leq \| \Phi \|_\infty . \| f \|_1 = \| \Phi \|_\infty . \| \hat{\tilde{f}} \|_1 =$$
$$= \| \Phi \|_\infty . \| \hat{\tilde{\hat{f}}} \|_A = \| \Phi \|_\infty . \| f \|_A. \tag{3}$$

Thus L is a continuous linear functional on $\Lambda_K^W(G)$. Since $\Lambda_K^W(G)$ is dense in A(G) by the Corollary 3.4., then L can be extended uniquely as a continuos linear functional on A(G). Hence there exists a unique pseudo-measure σ such that

$$L(f) = Tf(0) = \langle f, \widetilde{\sigma} \rangle \tag{4}$$

for all $f \in \Lambda_K^W(G)$. Then $Tf = \sigma * f$ for all $f \in \Lambda_K^W(G)$. An examination proof of Theorem 3.2 and proposition 3.5 show that σ is a pseudo measure and is unique. Hence to complete the proof of this theorem it remains to show that $Tf = \sigma * f$ holds for all $f \in A^p_{w,\omega}(G)$. Let f be any element of $A^p_{w,\omega}(G)$. If $(e_\alpha)_{\alpha \in I}$ is a bounded approximate identity for $A^p_{w,\omega}(G)$ choosen from $\Lambda_K^W(G)$ ([4], Th. 4.2) then for each $f \in A^p_{w,\omega}(G)$ the net $(e_\alpha * f)$ is Cauchy net in $\Lambda_K^W(G)$ and since $T(e_\alpha * f) = \sigma * (e_\alpha * f)$, we have

$$\| \sigma * (e_\alpha * f) - \sigma * (e_\beta * f) \|^p_{w,\omega}$$
$$\leq \| T(e_\alpha * f) - T(e_\beta * f) \|^p_{w,\omega} \leq \| T \| \, \| e_\alpha * f - e_\beta * f \|^p_{w,\omega} \tag{5}$$

which implies that $(\sigma * (e_\alpha * f))_{\alpha \in I}$ is a Cauchy net in $A^p_{w,\omega}(G)$ and converges to a function $F \in A^p_{w,\omega}(G)$. That means

$$\| F - \sigma * (e_\alpha * f) \|^p_{w,\omega} \to 0. \tag{6}$$

Again it is clear that $\sigma * f \in A'(G)$ because $f \in L^1(G)$ and $\sigma \in A'(G)$. If we use (6) and the relation

$$\| \hat{F} - \widehat{\sigma f} \|_\infty \leq \| \hat{F} - \hat{\sigma} (\hat{e}_\alpha \hat{f}) \|_\infty + \| \hat{\sigma} . (\hat{e}_\alpha \hat{f}) - \widehat{\sigma f} \|_\infty$$
$$\leq \| F - \sigma * (e_\alpha * f) \|_1 + \| \hat{\sigma} \|_\infty . \| e_\alpha * f - f \|_1$$
$$\leq \| F - \sigma * (e_\alpha * f) \|^p_{w,\omega} + \| \hat{\sigma} \|_\infty . \| e_\sigma * f - f \|^p_{w,\omega} \tag{7}$$

find that $\widehat{F} = \widehat{\sigma} \, . \, \hat{f}$. From the inversion theorem we write $F = \sigma * f$. Also we have

$$\|Tf - \sigma * (e_\alpha * f)\|^p_{w,\omega} = \|Tf - T\,(e_\alpha * f)\|^p_{w,\omega}$$
$$\leq \|T\| \, \|f - e_\alpha * f\|^p_{w,\omega} \to 0. \tag{8}$$

Consequently it follows from (6), (7) and (8) that $Tf = F = \sigma * f$ for all $f \in A^p_{w,\omega}(G)$. This completes the proof.

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REFERENCES

[1] BURNHAM, J.T., KROGSTAD, H.E., LARSEN, R., Multipliers and the Hilbert distribution, Nanta Mathematica, Vol. VIII, No 2., 95-103, 1975.

[2] DOMAR, Y. Harmonic analysis based on certain commutative Banach algebras, Acta Math. 96(1956), I-56.

[3] FEICHTINGER, H.G., Results on Banach ideals and spaces of multipliers, Math, Sacand, 41., (1977), (315,324).

[4] FEICHTINGER, H.G. and GÜRKANLI, A.T., On a family of weighted convolution algebras, Internat. J. Math. and Math. Sci. Vol. 13, No.3, (1990).

[5] FISCHER, R.H., GÜRKANLI, A.T. and LIU, T.S., On a family of weighted spaces, Math. Slovaca, 46(1996), No.1, 71-82.

[6] GÜRKANLI A.T., Some results in the weighted $A_p(IR^n)$ spaces, Demonstratio Mathematica Vol. 19/4 (1986), 825-830.

[7] GAUDRY, G.I., Multipliers of weighted Lebesque and measure spaces, Proc. Lon. Math. Soc. (3), 19, (1969).

[8] GAUDRY G.I., Quasimesures and operators commuting with convolution, Pacific Journal of Mathematics, Vol. 18, No.3, 1966.

[9] QUANGZHONG, Q., Multipliers of Segal Algebras, Proceedings of the Analysis Conference 1986, Elsevier Science Publishers B.V. (North Holland) 1988.

[10] KESAVA MURTY G.N. and UNNI, K.R., Multipliers on weighted spaces, Functional Analysis and its Applications International Conference, Madras, 1973, Lecture Notes in Mathematics, Springer Verlag 399.

[11] LARSEN, R., Introduction to the Theory of Multipliers. Springer Verlag, 1971.

[12] LARSEN, R., Tensor product factorization and multipliers, Preprint Series, Matematisk Intitutt, Universitetet i Oslo, ISBN 82-533-0180-1, 1974.

[13] LARSEN, R., The multipliers for functions with Fourier transfroms in L_p, Math. Scand. 28 (1971), 215-225.

[14] LARSEN, R., LIU, T.S. and WANK, J.K., On function with Fourier transforms in L_p, Michigan Math. J., 11, 369-378, 1964.

[15] REITER, W., Classical Harmonic Analysis and Locally Compact Groups, Oxford University Press, Oxford, (1968).

[16] RUDIN, W., Fourier Analysis on Groups, Inter science Publishers, New York, 1962.

[17] UNNI, K.R., A note on multipliers on Segal algebra, Studia Mathematica, TXLIX. (1974), 125-127.

[19] UNNI, K.R., Parameasures and multipliers of Segal algebras, Lecture notes in Mathematics # 399, Functional Analysis and its Applications International conference, Madras, 1973.

[19] WANG, H.C., Homogeneous Banach Algebras, Marcell Dekker Inc., New York - Basel, 1977.

[20] WARNER, C.R., Closed ideals in the group algebra $L^1(G) \cap L^2(G)$, Trans. Amer. Math. Soc. 121 (1996), 408-423.