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ON THE ORDER OF APPROXIMATION OF UNBOUNDED FUNCTIONS BY THE FAMILY OF GENERALIZED LINEAR POSITIVE OPERATORS

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ABSTRACT

In this study, a family of linear positive operators, which includes the sequence of linear positive operators built in a paper of A.D. Gadjiev and I.I. Ibragimov and later investigated by B. Wood and P. Radatz is defined and the order of convergence of these operators to continuous functions are studied.

We give the estimate of difference $|L_\lambda(f;\ x) - f(x)|$, where $L_\lambda(f;\ x)$ is a family of linear positive operators, in terms of modulus of continuity of the mentioned function f.

1. INTRODUCTION

Let $C([0,\ A];\ x^2)$ be the space of all functions f defined in $[0,\ \infty)$ and continuous in the interval $[0,\ A]$ for which the inequality

$$|f(x)| \leq M_f(1 + x^2),\ 0 \leq x < \infty, \qquad (1)$$

holds, where M_f is a positive constant depends on f.

For any positive A, we denote by $\omega_A(f;\ \delta)$ the modulus of continuity of function f on closed interval $[0,\ A]$, that is

$$\omega_A(f;\ \delta) = \sup\ \{|f(t) - f(x)|;\ x,\ t \in [0,\ A],\ |x - t| \leq \delta\} \qquad (2)$$

It is well known that the function $\omega_A(f;\ \delta)$ has the following properties:

(i) $|f(t) - f(x)| \leq \omega(f;\ |t - x|)$

(ii) $|f(t) - f(x)| \leq \left(\frac{|t - x|}{\delta} + 1\right) \omega(f;\ \delta)$

(iii) $\delta \leq C_f\ \omega(f;\ \delta)$, where C_f is a positive real constant, depends on f.

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2. GENERALIZED LINEAR POSITIVE OPERATORS

Let λ and A be positive real numbers, $\{\varphi_\lambda(t)\}$ and $\{\psi_\lambda(t)\}$ be the family of functions in C[0, A] such that $\varphi_\lambda(0) = 0$, $\psi_\lambda(t) > 0$, for each $t \in [0, A]$.

Let also $\{\alpha_\lambda\}$ be a family of positive numbers such that

$$\lim_{\lambda\to\infty} = \alpha_\lambda = \infty, \quad \lim_{\lambda\to\infty} = \frac{\alpha_\lambda}{\lambda} = 1, \quad \lim_{\lambda\to\infty} = \frac{1}{\lambda^2 \psi_\lambda(0)} = 0.$$

Assume that $\{K_\lambda(x, t\ u)\}$ $(x, t \in [0, A], -\infty < u < \infty, \lambda > 0)$ is a family of functions of three variables satisfying the following conditions:

1° Each function of this family is an entire analytic function with respect to u for fixed x and t of the segment [0, A].

2° $K_\lambda(x, 0, 0) = 1$ for any $x \in [0, A]$ and for any $\lambda > 0$.

3° $\left\{(-1)^\upsilon \frac{\partial^\upsilon}{\partial u^\upsilon} K_\lambda(x, t, u)\Big|_{\substack{u=u_1\\ t=0}}\right\} \geq 0 \ \left(\forall x \in [0, A], \lambda > 0, \upsilon = 0, 1, 2, \ldots\right).$

4° $\frac{\partial^\upsilon}{\partial u^\upsilon} K_\lambda(x, t, u)\Big|_{\substack{u=u_1\\ t=0}} = -\lambda x \left[\frac{\partial^{\upsilon-1}}{\partial u^{\upsilon-1}} K_{h(\lambda)}(x, t, u)\right]_{\substack{u=u_1\\ t=0}}$

$(\forall x \in [0, A], \lambda \in \mathbb{R}^+, \upsilon = 0, 1, 2, \ldots)$ where $h(\lambda)$ is a nonnegative function satisfying the condition $\lim_{\lambda\to\infty} = \frac{h(\lambda)}{\lambda} = 1$.

Consider the family of linear operators;

$$L_\lambda(f; x) = \sum_{\upsilon=0}^{\infty} f\left(\frac{\upsilon}{\lambda^2 \psi_\lambda(0)}\right) \left\{\frac{\partial^\upsilon}{\partial u^\upsilon} K_\lambda(x, t, u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}\right\} \frac{\left(-\alpha_\lambda \psi_\lambda(0)\right)^\upsilon}{\upsilon!} \qquad (3)$$

acting on function $f \in C\left([0, A]; x^2\right)$.

Note that for $\lambda = n$ and $h(\lambda) = m + n$ $(m + n = 0, 1, 2, \ldots)$, the operators defined by (3) are reduced to the operators defined in [4].

Remark. By choosing $\lambda = n$ $(n = 1, 2, \ldots)$ we obtain, as in [4], some known sequences of linear positive operators.

By choosing

$K_n(x, t, u) = \left[1 - \frac{ux}{1 + t}\right]^n$, $\alpha_n = n$, $\psi_n(0) = \frac{1}{n}$, we have h(n) = n − 1 and the operators defined by (3) are transformed into Bernstein polynomials.

For

$$\alpha_n = n,\ \psi_n(0) = \frac{1}{nb_n}\left(\lim_{n\to\infty} b_n = \infty\ ,\ \lim_{n\to\infty}\frac{b_n}{n} = 0\right),$$

we obtain Bernstein-Chlodovsky polynomials.

By choosing

$$K_n(x,\ t,\ u) = e^{-n(t+ux)}\ ,\ \alpha_n = n,\ \psi_n(0) = \frac{1}{n}\ ,$$

we have h(n) = n and we get Szasz operators.

If $K_n(z)$ is entire analytic function and

$$K_n(x,\ t,\ u) = K_n(t + ux)\ ,\ \alpha_n = n,\ \psi_n(0) = \frac{1}{n}\ ,$$

then we obtain Baskakov [1] operators, and for

$$\alpha_n = n,\ \psi_n(0) = \frac{1}{\alpha_n} \text{ and } \frac{n^2}{\alpha_n} = \beta_n\ ,$$

we get the other Baskakov [2] operators.

The linear positive operators defined by (3) have the following properties: (See [3]).

$$L_\lambda(1;\ x) = 1 \tag{4}$$

$$L_\lambda(t;\ x) = \frac{\alpha_\lambda}{\lambda}\ x \tag{5}$$

$$L_\lambda(t^2;\ x) = \left(\frac{\alpha_\lambda x}{\lambda}\right)^2 \frac{h(\lambda)}{\lambda} + \frac{\alpha_\lambda x}{\lambda}\ \frac{1}{\lambda^2 \psi_\lambda(0)}\ . \tag{6}$$

Also from (4), (5) and (6), it is seen that

$$0 \le L_\lambda((t-x)^2;\ x) = \left(\frac{\alpha_\lambda^2}{\lambda^2}\frac{h(\lambda)}{\lambda} - 2\frac{\alpha_\lambda}{\lambda} + 1\right)x^2 + \frac{\alpha_\lambda x}{\lambda}\ \frac{1}{\lambda^2\psi_\lambda(0)} \tag{7}$$

since L_λ are positive operators.

Denote for each A > 0

$$\delta_\lambda = \left(\left\| \left(\frac{\alpha_\lambda^2}{\lambda^2} \frac{h(\lambda)}{\lambda} - 2\frac{\alpha_\lambda}{\lambda} + 1 \right) x^2 + \frac{\alpha_\lambda}{\lambda} \frac{1}{\lambda^2 \psi_\lambda(0)} x \right\|_{C[0,A]} \right)^{1/2} \tag{8}$$

where $\|.\|_{C[0,A]}$ is a norm defined by

$$\|f(x)\|_{C[0,A]} = \max_{x \in [0,A]} |f(x)| \; . \tag{9}$$

Using (9) in (8), we obtain

$$\delta_\lambda = \left(\max_{x \in [0,A]} \left[\left(\frac{\alpha_\lambda^2}{\lambda^2} \frac{h(\lambda)}{\lambda} - 2\frac{\alpha_\lambda}{\lambda} + 1 \right) x^2 + \frac{\alpha_\lambda}{\lambda} \frac{1}{\lambda^2 \psi_\lambda(0)} x \right] \right)^{1/2} \tag{10}$$

and since

$$\lim_{\lambda \to \infty} \frac{\alpha_\lambda}{\lambda} = 1 \;, \; \lim_{\lambda \to \infty} \frac{h(\lambda)}{\lambda} = 1 \text{ and } \lim_{\lambda \to \infty} \frac{1}{\lambda^2 \psi_\lambda(0)} = 0 \;,$$

we can see that, for each $x \in [0, A]$,

$$\lim_{\lambda \to \infty} \delta_\lambda = 0 \; .$$

On the other hand, from (7) and (10) we get

$$L_\lambda((t - x)^2; x) \le \delta_\lambda^2 \; . \tag{11}$$

Now using the Cauchy-Bunyakovskii's inequality in $L_\lambda(|t - x|; x)$ and making the simplifications, we get

$$L_\lambda(|t - x|; x) \le [L_\lambda(t^2; x) - 2xL_\lambda(t; x) + x^2 L_\lambda(1; x)]^{1/2} \, L_\lambda(1; x) \tag{12}$$

Using (4) and the equality

$$[L_\lambda(t^2; x) - 2xL_\lambda(t; x) + x^2 L_\lambda(1; x)] = [L_\lambda((t - x)^2; x)].$$

in (12), we have

$$L_\lambda(|t - x|; x) \le [L_\lambda((t - x)^2; x)]^{1/2}$$

and by (11) we get

$$L_\lambda(|t - x|; x) \le \delta_\lambda \tag{13}$$

Now we can prove the following theorem.

Theorem 2.1. Let f be continous in $[0,\infty)$, satisfies (1) and $\omega_{2A}(f;\,\delta)$ be its modulus of continuity on the finite interval [0, 2A]. Then for the family of linear positive operators $\{L_\lambda\}$ given by (3) having the properties (4), (5), (6), the inequality

$$\|L_\lambda(f;x) - f(x)\|_{C[0,A]} \le \left[C_f\, M_f \left(5 + \frac{2}{A^2}\right) + 2\right] \omega_{2A}(f;\delta_\lambda) \tag{14}$$

holds for all sufficiently large λ, where M_f is a positive constant which depends on f, C_f is as in property (iii) of modulus of continuity and δ_λ is defined as in (8).

Proof. It is obvious that for $x \in [0, A]$ and $t \in [0, \infty)$ we can divide the line in two parts

$$E_1 = \{(x, t) : x \in [0, A];\ t > 2A\}$$

$$E_2 = \{(x, t) : x \in [0, A];\ t \le 2A\}$$

By using $|f(t)| \le M_f\,(1 + t^2)$, we obtain for $x \in [0, A]$ and $t \in [0, \infty)$ the inequality

$$|f(t) - f(x)| \le M_f(2 + (t - x)^2 + 2A|t - x| + 2A^2)$$

For $(x, t) \in E_1$ since $|t - x| > A$, we have

$$|f(t) - f(x)| \le M_f \left(2\, \frac{(t - x)^2}{A^2} + (t - x)^2 + 2(t - x)^2 + 2(t - x)^2\right).$$

and consequently

$$|f(t) - f(x)| \le M_f\,(t - x)^2 \left(\frac{2}{A^2} + 5\right) \tag{15}$$

Let now $(x, t) \in E_2$. Then $|t - x| \le 2A$ and using the properties (i) and (ii) for modulus of continuity, we can write

$$|f(t) - f(x)| \le \omega_{2A}(f;|t-x|) \le \omega_{2A}(f;\delta_\lambda) \left(\frac{|t - x|}{\delta_\lambda} + 1\right). \tag{16}$$

From (15) and (16) for $x \in [0, A]$ and $t \in [0, \infty)$, we get

$$|f(t) - f(x)| \le M_f(t - x)^2 \left(\frac{2}{A^2} + 5\right) + \omega_{2A}(f;\delta_\lambda) \left(\frac{|t - x|}{\delta_\lambda} + 1\right). \tag{17}$$

Since $L_\lambda(f;\ x)$ is monotone increasing and linear in f, we have

$$L_\lambda(|f(t) - f(x)|;x) \le M_f\left(\frac{2}{A^2}+5\right)L_\lambda((t-x)^2;x)$$

$$+\ \omega_{2A}(f;\delta_\lambda)\left(\frac{1}{\delta_\lambda}L_\lambda(|t-x|;x)+L_\lambda(1;x)\right). \tag{18}$$

Since $\delta_\lambda \to 0$ we have, for sufficiently large λ, $\delta^2_\lambda \le \delta_\lambda$ and by the property (iii) of the modulus of continuity

$$\delta_\lambda \le C_f\,\omega_{2A}(f;\delta_\lambda)$$

Thus from (11) we can write

$$L_\lambda((t-x)^2;x) \le \delta_\lambda^2 \le C_f\,\omega_{2A}(f;\delta_\lambda)\ . \tag{19}$$

Using (4), (13) and (19) in (18) we obtain

$$L_\lambda(|f(t) - f(x)|;x) \le \left[C_f M_f\left(\frac{2}{A^2}+5\right)+2\right]\omega_{2A}(f;\delta_\lambda)\ . \tag{20}$$

We obtain the desired result.

3. A GENERALITAZION OF THE r-th ORDER OF THE FAMILY $\{L_\lambda\}$ DEFINED BY (3)

By $C^{(r)}[0,A]$, we denote the set of the functions $f:[0,A] \to \mathbb{R}$ having continuous r-th derivative $f^{(r)}$ $(f^{(0)}(x) = f(x))$ on the segment [0,A].

We consider a following generalization of the family of linear positive operators defined by (3)

$$L_\lambda^{[r]}(f;x) = \sum_{\upsilon=0}^{\infty}\sum_{i=0}^{r} f^{(i)}\left(\frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)\frac{\left(x-\frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)^i}{i!}\ \times$$

$$\times\left\{\frac{\partial^\upsilon}{\partial u^\upsilon}K_\lambda(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}\right\}\frac{\left(-\,\alpha_\lambda\psi_\lambda(0)\right)^\upsilon}{\upsilon!}\ . \tag{21}$$

Operators (21) we call the r-th order of the family (3).

Note that this definition for linear positive operators was given in [6].

We can prove the following proposition.

Theorem 3.1. Let $L_\lambda^{[r]}(f;x)$ be a family of operators defined by (21), L_λ is the family of linear positive operators defined in (3) and

$$\delta_\lambda = \max\left(\sup_{x\in[0,A]} L_\lambda(|x-t|^r;x)\, ,\ \sup_{x\in[0,A]} L_\lambda(|x-t|^{r+1};x)\, ,\ \sup_{x\in[0,A]} L_\lambda(|x-t|^{r+2};x)\right). \quad (22)$$

If $f \in C^{(r)}[0,2A]$, and

$$\left|f^{(r)}(t)\right| \le M_{f^{(r)}}(1 + t^2)\ ,\ 0 \le t < \infty\ ,$$

then the inequality

$$\left|L_\lambda^{[r]}(f;x) - f(x)\right| \le M_{f^{(r)}}\left(\frac{2}{A^2} + 5\right)\frac{2\delta_\lambda}{(r+2)!} + \frac{\delta_\lambda(r+1)+1}{(r+1)!}\,\omega_{2A}(f^{(r)};\delta_\lambda) \quad (23)$$

holds, where $\omega_{2A}(f^{(r)};\delta_\lambda)$ is the modulus of continuity of $f^{(r)}$ in $[0,2A]$.

Proof. We can write

$$f(x) - L_\lambda^{[r]}(f;x) = \sum_{\upsilon=0}^{\infty}\left[\sum_{i=0}^{r} f^{(i)}\left(\frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)\frac{\left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)^i}{i!}\right]\times$$

$$\times\ \frac{\partial\upsilon}{\partial u^\upsilon} K_\lambda(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t\,=\,0}} \frac{(-\alpha_\lambda\psi_\lambda(0))^\upsilon}{\upsilon!} \quad (24)$$

Then from the Taylor's formula we get

$$f(x) - \sum_{i=0}^{r} f^{(i)}\left(\frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)\frac{\left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)^i}{i!} = \frac{\left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)^r}{(r-1)!}\times$$

$$\times\int_0^1 (1-t)^{r-1}\left[f^{(r)}\left(\frac{\upsilon}{\lambda^2\psi_\lambda(0)} + t\left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)\right) - f^{(r)}\left(\frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)\right]dt\ . \quad (25)$$

Using the similar arguments as when obtaining the inequality (17), we can write

$$\left| f^{(r)}\left(\frac{\upsilon}{\lambda^2\psi_\lambda(0)} + t\left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)\right) - f^{(r)}\left(\frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right) \right|$$

$$\leq \left(t\left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)\right)^2 \left(\frac{2}{A^2} + 5\right) + \left(\frac{\left| t\left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)\right|}{\delta_\lambda} + 1 \right) \omega_{2A}(f^{(r)};\delta_\lambda) \,. \qquad (26)$$

In view of the inequality (26), we have from (25)

$$\left| f(x) - \sum_{i=0}^{r} f^{(i)}\left(\frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right) \frac{\left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)^i}{i!} \right|$$

$$\leq \frac{\left| x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)} \right|^r}{(r-1)!} \left[\int_0^1 (1-t)^{r-1} M_{f^{(r)}}\, t^2 \left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)^2 \left(\frac{2}{A^2} + 5\right) dt \,\cdot \right.$$

$$\left. + \,\omega_{2A}(f^{(r)};\delta_\lambda) \left[\frac{1}{\delta_\lambda} \int_0^1 (1-t)^{r-1} t \left| x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)} \right| dt + \int_0^1 (1-t)^{r-1}\, dt \right] \right] . \qquad (27)$$

Since

$$\int_0^1 (1-t)^{r-1} dt = \frac{1}{r}\ , \quad \int_0^1 (1-t)^{r-1} t\ dt = \frac{1}{r(r+1)}\ , \quad \int_0^1 (1-t)^{r-1} t^2 dt = \frac{2}{r(r+1)(r+2)}$$

we have

$$\left| f(x) - \sum_{i=0}^{r} f^{(i)}\left(\frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right) \frac{\left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)^i}{i!} \right|$$

$$\leq \frac{\left| x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)} \right|^r}{(r-1)!} \left[M_{f^{(r)}} \left(\frac{2}{A^2} + 5\right) \left(x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)}\right)^2 \frac{2}{r(r+1)(r+2)} \right.$$

$$\left. + \,\omega_{2A}(f^{(r)};\delta_\lambda) \left(\frac{\left| x - \frac{\upsilon}{\lambda^2\psi_\lambda(0)} \right|}{\delta_\lambda r(r+1)} + \frac{1}{r} \right) \right] .$$

Using this inequality in (24) we obtain

$$\left|L_\lambda^{[r]}(f;x) - f(x)\right| \le M_{f^{(r)}} \left(\frac{2}{A^2} + 5\right) \frac{2}{(r+2)!} L_\lambda(|x-t|^{r+2};x) +$$

$$+ \left(\frac{L_\lambda(|x-t|^{r+1};x)}{(r+1)!\delta_\lambda} + \frac{L_\lambda(|x-t|^r;x)}{r!}\right) \omega_{2A}(f^{(r)};\delta_\lambda) \ . \tag{28}$$

Substituting the value of δ_λ defined by (22) in (28), we have (23) and thus theorem is proved.

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