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ON A VARIATIONAL PROBLEM RELATED TO A MODEL OF BLACK AND WHITE PRINTING

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ABSTRACT

A variational problem, which might be considered as a case of shape optimisation, is studied characterising the existence of minimisers. The problem can be understood as a model for black and white printing on digitalised printers.

A direct construction of a minimising sequence is presented, based on measure theoretic tools. Another possible construction, using the relationship between the functional in questions and weak $*$ topology of L^∞, is given as well. The problem has the unique solution in a relaxed sense: every minimising sequence determines the unique Young measure.

1. INTRODUCTION

Let us consider a class $\mathcal{U}$ of measurable functions on a bounded open subset Ω of R^d with values in the segment [0, 1].

We study the problem of approximating a given function $u \in \mathcal{U}$ with functions υ from the class:

$$V := \{\upsilon \in \mathcal{U} : \upsilon(x) \in \{0, 1\} \ (\text{ae } x \in \Omega)\},$$

in the sense of minimisation of the functional (hereafter μ denotes Lebesgue measure on R^d)

$$(1)\quad J(\upsilon) := \int_0^1 g(r) \int_\Omega \left| \frac{1}{\mu(K(x, r) \cap \Omega)} \int_{K(x,r) \cap \Omega} (\upsilon(y) - \upsilon(y))dy \right| dxdr ,$$

where $g : [0, 1] \to R_0^+$ is a bounded function which satisfies

$$(2)\qquad (\forall\ r \langle 0, 1]) \ g(r) > 0.$$

Here and bellow $K(x, r)$ denotes the open ball centred at x with radius r.

This variational problem is closely related to black and white printing on digitalised printers. Imagine that Ω represents the rectangle on the paper, where the image is to be printed. Let $\lambda \in [0, 1]$ represent the darkness on the scale of grey, with 0 corresponding to pure white, and 1 to black. In this model the original picture will be represented by a function $u \in \mathcal{U}$ which we will try to represent as well as possible by black dots of ink and white patches of paper, represented by a function $\upsilon \in \mathrm{V}$. The innermost integral in (1) represents local mean difference of υ and u. This problem was proposed by Ball in [5].

More precisely, we study the minimisation problem for J on V. Our result is stated in the following theorem.

Theorem 1. With the notation introduced above, the following statements hold true:

(a) $\inf\{J(\upsilon) : \upsilon \in \mathrm{V}\} = 0$

(b) Each minimising sequence for J determines the unique Young measure

$$\nu_x = (1 - u(x))\, \delta_0 + u(x)\, \delta_1 \quad (\text{ae } x \in \Omega).$$

(c) The minimum is attained if and only if $u \in \mathrm{V}$.

Statement (c) says that for $u \in \mathcal{U} \setminus \mathrm{V}$ the minimum is not attained in V. Every minimising sequence exhibits a microstructure: while trying to satisfy the constraint on the range, the functions fluctuate over microscopic regions more and more rapidly. Nevertheless, in a broader sense, the problem has the unique minimiser, the Young measure ν. A detailed mathematical study of microstructures was performed by Ball and James [3,4].

Any function from the class V is essentially a characteristic function of a measurable subset of Ω. This interpretation connects our problem with shape optimisation (for more details on shape optimisation see Allaire et al. [1] and references there; for somehow related problems see Mumford and Shah [10] as well.)

2. YOUNG MEASURES

A sequence (υ_n) in $L^\infty(\Omega)$ is said to converge weakly * to a function $\upsilon \in L^\infty(\Omega)$, written

$$\upsilon_n \xrightarrow{*} \upsilon,$$

provided

$$\left(\forall\, u \in L^1(\Omega)\right) \int_\Omega \upsilon_n\, u\, dx \to \int_\Omega \upsilon\, u\, dx$$

Every weakly * convergent sequence is clearly bounded. The converse, of course, is not true. For $p \in [1, \infty]$, the spaces $L^p(\Omega)$ have the weak compactness property, which does not hold for $L^\infty(\Omega)$, where we have the weak * compactness instead. More precisely, if a sequence (υ_n) is bounded in $L^\infty(\Omega)$, then there exists a subsequence (υ_{n_k}), and a function $\upsilon \in L^\infty(\Omega)$, such that $\upsilon_{n_k} \xrightarrow{*} \upsilon$.

The Young measures were introduced by L. C. Young (v. [15]) as a tool for treating variational problems for which there does not exist a minimiser in ordinary sense. The following version of the fundamental theorem of existence and uniqueness for Young measures is due to Tartar [12]; a more general form was proved by Ball [2].

Theorem 2. Let $\Omega \subseteq R^d$ be a bounded open set, and (υ_n) a bounded sequence in $L^\infty(\Omega; R^r)$. Then there exists a subsequence (υ_{n_k}), and a family of Borel probability measures ν_x on R^r (the Young measure) depending measurably on x, such that for each $f \in C(R^r)$ we have

$$\overline{f \circ \upsilon_{n_k}} \xrightarrow{*} \overline{f}, \tag{3}$$

where $\overline{f}(x) := \langle \upsilon_x, f\rangle$ (ae $x \in \Omega$).

The Young measure $(\nu_n)_{x\in\Omega}$ is said to be associated with the subsequence (υ_{n_k}).

This theorem provide us with a concise measure - theoretic characterisation of the incompatibility of weak * convergence and nonlinear composition.

By making specific choices for a function $f \in C(R^r)$ we can read of some information regarding the structure of Young measures. For instance, if there exists a closed set $K \subseteq R^r$ such that $\upsilon_n(x) \in K$ (ae $x \in \Omega$), then supp $\nu_x \subseteq K$ (ae $x \in \Omega$). To verify this we need only consider functions f vanishing on K.

For a more detailed survey on Young measures see, for example, Evans [8] or Valadier [14].

3. PROOF OF STATEMENTS (b) AND (c)

Assuming (a), we shall first prove statements (c) and (b). Given a function $\omega \in L^\infty(\Omega)$, we define F_ω: $\Omega \times \langle 0, 1] \to R$ by the formula

$$F_\omega(x, r) := ⨍_{K(x,r) \cap \Omega} \omega(y)\, dy := \frac{1}{\mu\left(K(x, r) \cap \Omega\right)} \int_{K(x,r) \cap \Omega} \omega(y)\, dy \quad (4)$$

Having this defnition, we establish the continuity properties of F_ω.

Lemma 1. For each $\omega \in L^\infty(\Omega)$ the function F_ω is continuous on $\Omega \times \langle 0, 1]$. Moreover,

$$\lim_{r \to 0} F_\omega(x, r) = \omega(x) \text{ (ae } x \in \Omega).$$

Dem. For (x, r) and (x', r') in $\Omega \times \langle 0, 1]$ let us consider the difference

$$\begin{aligned} F_\omega(x,r) - F_\omega(x',r') = {} & \frac{1}{\mu\left(K(x, r) \cap \Omega\right)} \int_{K(x,r) \cap \Omega} \omega(y)\, dy \\ & - \frac{1}{\mu\left(K(x', r') \cap \Omega\right)} \int_{K(x',r') \cap \Omega} \omega(y)\, dy \\ = {} & \frac{1}{\mu\left(K(x, r) \cap \Omega\right)} \left(\int_{K(x,r) \cap \Omega} \omega(y)\, dy - \int_{K(x',r') \cap \Omega} \omega(y)\, dy \right) \\ & + \left(\int_{K(x,r) \cap \Omega} \omega(y)\, dy - \int_{K(x',r') \cap \Omega} \omega(y)\, dy \right) \int_{K(x',r') \cap \Omega} \omega(y)\, dy. \end{aligned}$$

After taking the absolute value on both sides, we obtain

$$\begin{aligned} \left|F_\omega(x, r) - F_\omega(x', r')\right| & \le \frac{\|\omega\|_{L^\infty} \mu(A)}{\mu\left(K(x, r) \cap \Omega\right)} + \|\omega\|_{L^\infty} \left| 1 - \frac{\left(K(x', r') \cap \Omega\right)}{\left(K(x, r) \cap \Omega\right)} \right| \quad (5) \\ & \le \frac{2\|\omega\|_{L^\infty} \mu(A)}{\mu\left(K(x, r) \cap \Omega\right)}, \end{aligned}$$

where A denotes the symmetric difference of sets $K(x, r) \cap \Omega$ and $K(x', r') \cap \Omega$. Since $\mu(A)$ tends to zero as (x', r') aproaches (x, r), we have the continuity of $F\omega$.

The second statement of the lemma is merely a reformulation of the Lebesgue-Besicovitch differentiation theorem (see, for example, Evans and Gariepi [9, Theorem 1.7.1]), after noting that for sufficiently small radii r one has $K(x, r) \subseteq \Omega$.

Remark. By Lemma 3 (see below), we have even more than stated in the provious lemma. For each $r_0 > 0$ the function F_ω is in fact uniformly continuous on $\Omega \times [r_0, 1]$.

Let us proceed by the proof of statement (c). For $u \in V$, the minimum is clearly attained by taking $\upsilon := u$. The goal is to prove the converse.

As the infimum of J is zero (which will be proved below), assume $J(\upsilon) = 0$. Therefore assumption (2), together with continuity of the function $F_{\upsilon - u}$, yields the conclusion

$$(\forall\ x \in \Omega)\ (\forall\ r \in \langle 0, 1])\quad F_{\upsilon - u}(x, r) = 0$$

Using the second part of Lemma 1 we obtain

$$0 = \lim_{r \to 0} F_{\upsilon - u}(x, r) = \upsilon(x) - u(x)\ \left(\text{ae } x \in \Omega\right).$$

Thus $\upsilon = u$ almost everywhere in Ω, so if the minimum is attained for some fnction $\upsilon \in V$, u must necessarily be in the given class V. This completes he proof of (c).

Let $(\nu_x)_{x \in \Omega}$ be the Young measure associated to a minimising sequence (υ_n). According to Theorem 2, there exists a subsequence (υ_{n_k}), such that

$$(\forall\ f \in C([0, 1]))\ f \circ \upsilon_{n_k} \xrightarrow{*} \overline{f}\,. \tag{6}$$

In particular $\upsilon_{n_k} \xrightarrow{*} \upsilon$, where

$$\upsilon(x) := \int_0^1 \lambda\ d\nu_x\left(\lambda\right)\left(\text{ae } x \in \Omega\right). \tag{7}$$

Since (υ_n) is a sequence in V, for each f in $C_c([0, 1])$ the convergence in (6) implies $\langle \nu_x, f \rangle = 0$ (ae $x \in \Omega$). It follows that supp $\nu_x \subseteq \{0, 1\}$ (ae $x \in \Omega$). This leads to the following expression for the Young measure

$$\nu_x = (1 - \omega(x))\, \delta_0 + \omega(x)\delta_1 \ (\text{ae } x \in \Omega),$$

for some $\omega \in L^\infty(\Omega)$. As $J(\upsilon) = 0$, and we have proved above that for such υ we have $\upsilon = u$ almost everywhere, (7) yields the desired formula for the Young measure, which proves statement (b).

4. CONSTRUCTION OF A MINIMISING SEQUENCE

Let us first construct a sequence of functions $\upsilon_n \in$ V, being equal on smaller and smaller cubes to the mean value of u. In order to do this, we decompose Ω in a disjoint countable collection of cubes, following Rudin [11].

For $\mathbf{a} \in \mathbf{R}^d$ and $r > 0$ we shall call the set $Q(\mathbf{a}, r) := \{x \in Rd : a^i \le x^i \le a^i + r, 1 \le i \le d\}$ the r-cube with corner at $\mathbf{a}$. For each $n \in \mathbf{N}$ let P_n be the set of all points in $\mathbf{R}^d$ whose coordinates are integral multiples of 2^{-n}. Denote by Q_n the collection of all 2^{-n}-cubes with corners at points of P_n, and by Q the union of all Q_n. The following lemma (id., p. 50) holds.

Lemma 2. Every nonempty open set $\Omega \subseteq R^d$ is a countable union of disjoint members of Q.

Next we construct a minimising sequence. Starting with the decomposition given by Lemma 2 (step 0), we define the sequence (υ_n) inductively, refining the decompositions as follows.

In the n-th step we divide all $2^{-(n-1)}$-cubes into 2^{-n}-cubes, by halving the edges. Thus we have Ω represented as a disjoint union

$$\Omega = \bigcup_{k \in N} Q_k^{(n)} \ ,$$

where for each $k \in \mathbf{N}$, $Q_k^{(n)} \in \cup_{m \ge n} Q_m$. Furhthermore, for each k let $I_k^{(n)}$ be a measurable subset of $Q_k^{(n)}$, such that

$$\mu\left(I_k^{(n)}\right) = m_k^{(n)}\ \mu\left(Q_k^{(n)}\right),$$

where $m_k^{(n)}$ denotes the average value of function u over the cube $Q_k^{(n)}$. A good choice for the sets $I_k^{(n)}$ is to take cubes centred at points in $Q_k^{(n)}$ of required size. Define

$$\Omega_n := \bigcup_{k\in N} I_k^{(n)} ,$$

and $\upsilon_n := \chi_{\Omega_n}$, completing the construction.

By the construction of the sequence (υ_n), an easy computation gives that (indepentently of n) the following equality holds

$$\mu\left(\Omega_n\right) = \int_\Omega u(y)\ dy = m_{u,\ \Omega}\ \mu(\Omega) ,$$

where $m_{u,\Omega}$ denotes the mean value of the function u over Ω. Having the motivation given in the introduction in mind, we have a simple interpretation: A total quantity of ink used to represent the original picture is in every step proportional to its mean darkness.

In proving that the sequence (υ_n) constructed above is a minimising sequence for J we shall make use of the following lemma.

Lemma 3. For $r \in \langle 0, 1]$ and $m \in N$ the sets $E_m^{\ r} := \{x \in \Omega : \mu(K(x, r) \cap \Omega) \leq 2^{-m}\}$ are closed. The family $(E_m^{\ r})$ is decreasing with respect to both indices. Moreover,

$$(\forall\ r_0 \in \langle 0, 1]) \ (\exists\, m_0 \in N) \ (\forall\ r \geq r_0) \ (\forall\ m \geq m_0) \qquad E_m^{\ r} = \emptyset. \tag{8}$$

Dem. Each $E_m^{\ r}$ is closed being a preimage by the continuous function $x \to \mu(K(x, r) \cap \Omega)$ of a closed set $[0, 2^{-m}]$. The statements about the monotonicity are obvious.

Arguing by contradiction it can easily be seen that $\cap_{m\in N} E_m^{\ r} = \emptyset$. Now, for every $r_0 \in \langle 0, 1]$ we have a decreasing sequence of compact sets having empty intersection. Then, from some point on, it must necessarily consist of empty sets only. Combining this with the monotonocity in r we have (8).

Remark. Note that Lemma 3 asserts the uniform boundedness by 2^m, for some positive interger m, of the function $(x, r) \to \mu(K(x, r) \cap \Omega)^{-1}$ on the set $\Omega \times [r_o, 1]$.

For given $\varepsilon > 0$ we have to find $n \in N$ such that $J(\upsilon_n) < \varepsilon$. We decompose the integral into two parts

$$
\begin{aligned}
J(\upsilon_n) &:= I_1 + I_2 \\
&= \int_0^{r_0} g(r) \int_\Omega \left| ⨍_{K(x,r)\cap\Omega} (\upsilon_n(y) - u(y))\, dy \right| dx\, dr + \\
&\int_0^1 g(r) \int_\Omega \left| ⨍_{K(x,r)\cap\Omega} (\upsilon_n(y) - u(y))\, dy \right| dx\, dr.
\end{aligned}
$$

Taking $r_0 := \dfrac{\varepsilon}{2\|g\|_{L^\infty}\, \mu(\Omega)}$, we have the estimate $I_1 \leq \varepsilon/2$ (we assumed here that $\varepsilon < 2\|g\|_{L^\infty}(\Omega)$, since eventually ε is to be taken arbitrarily small). This was the easy part.

For $n \in N$ given, take the partition of Ω into cubes as above, and denote by $Q_n(x, r)$ the union of all such cubes contained in $K(x, r) \cap \Omega$. Furthermore, denote $R_n(x, r) := (K(x, r) \cap \Omega) \setminus Q_n(x, r)$. It is clear that, by increasing n, we can make $R_n(x, r)$ uniformly small in measure, more specifically, smaller than any prescribed $\delta > 0$. As $R_n(x, r)$ is contained in $K(x, r) \setminus Q_n(x, r)$, it is a simple matter to see that any $n \geq d(d + \log_2 (\theta_d/\delta))$ is good (θ_d denoting the volume of the unit ball in R^d).

By the construction of the sequence (υ_n), it follows that

$$
I_2 \leq \int_{r_0}^1 g(r) \int_\Omega \frac{1}{\mu(K(x, r) \cap \Omega)} \int_{R_n(x,r)} |\upsilon_n(y) - u(y)|\, dy\, dx\, dr.
$$

Applying the remark above, there exists some $m \in N$ such that $\dfrac{1}{\mu(K(x, r) \cap \Omega)} \leq 2^m$ for $r > r_0$, hence

$$
I_2 \leq \delta(1 - r_0)2^{m+1}\, \mu(\Omega)\, \|g\|_{L^\infty}.
$$

Taking any n which satisfies

$$
n \geq d\left(d + m + 1 + \log_2 \frac{\theta_d \left(2\|g\|_{L^\infty}\, \mu(\Omega) - \varepsilon\right)}{\varepsilon}\right),
$$

simple computations lead us to the estimate

$$\delta < \frac{\varepsilon}{2^{m+2}\left(1 - r_0\right) \mu\left(\Omega\right) \|g\|_{L^\infty}} ,$$

and thus yield the desired conclusion $J(\upsilon_n) < e$. This completes the proof of Theorem 1.

Remark. Let us additionally assume that the boundary of Ω has d-dimensional Lebesgue measure zero. By the Lebesgue theorem, the characteristic function of Ω is Riemann integrable. In this case, instead of the construction by partition of Ω into cubes (which is in the spirit of the Lebesgue theory), we could have estimated I_2 above by noting that the function $|F_{\upsilon - u}|$ is uniformly continuous on $\Omega \times [r_0, 1]$, for any $r_0 \in \langle 0, 1]$, as stated in the remark following Lemma 1. For such a function the Riemann and the Lebesgue integral coincide, and the former can be approximated by a Riemann sum. More precisely, given any $\varepsilon' > 0$ there is a $\delta > 0$ such that for any mesh finer than δ the Riemann sum is ε' close to the value of the integral.

The above was valid for any $\upsilon \in V$. It is enough to select one such that for a mesh finer than δ the corresponding Riemann sum is zero. In order to do that, we choose any mesh finer than δ: $\{(x_j, r_j) : 1 \leq j \leq n\}$, and define $B_j := K(x_j, r_j) \cap \Omega$. For each of the atoms $E_1, \ldots , E_{N_n}$ (i.e. nonempty sets of the form $\bigcap_{j=1}^{n} A_j$, where A_j is either B_j or $\Omega \setminus B_j$) we define υ to be a characteristic function of some measurable set in E_k, such that:

$$\int_{E_k} \upsilon_n(y)\, dy = \int_{E_k} u(y)\, dy .$$

It is clear that $F_{\upsilon - u}(x_j, r_j) = 0$ for $1 \leq j \leq n$, and the Riemann sum is zero.

Remark. It might be of interest to note that the function F_ω, defined by (4), can be equally written as

$$F_\omega (x, r) = \langle \omega, e\rangle = \int_\Omega \omega(y)\, e(y)\, dy ,$$

where $e \in L^1(\Omega)$ is a function with norm one, defined by:

$$e := \frac{\chi_{K(x,\, r)\cap\Omega}}{\mu\left(K(x, r) \cap \Omega\right)}$$

For a countable dense subset $\mathcal{G} := \{(x_j, r_j) : j \in N\}$ of $\Omega \times \langle 0, 1]$ we obtain, by the above definition, a sequence (e_j) having a linear hull dense in $L^1(\Omega)$.

The weak * topology on the closed unit ball $K_{L^\infty(\Omega)}[0, 1]$ in $L^\infty(\Omega)$ is equivalent to the topology generated by the following bounded metric (the proof of this fact follows the lines* of Dunford and Schwartz [7, Theorem V.5.1]):

$$d(u, \upsilon) := \sum_{j=1}^{\infty} \frac{1}{2j} \left|\left\langle \upsilon - u, e_j \right\rangle\right| .$$

As any $u \in \mathcal{U}$ can be approximated by functions from $\mathcal{V}$ in the weak * topology of $L^\infty(\Omega)$, we have another construction of the minimising sequence, using Riemann sums in an analogous manner as in the previous remark (of course, under the same additional assumption on the boundary of Ω).

More precisely, for a given δ-mesh consisting of the points from dense set $\mathcal{G}$, with the largest index n, we can find a function $\upsilon \in \mathcal{V}$ such that $d(\upsilon - u) < \frac{\varepsilon}{2^n}$. This, in particular, gives us that

$$\left(\forall\, j \leq n\right) \quad F_{\upsilon - u}\left(x_j, r_j\right) \left|\left\langle \upsilon - u, e_j \right\rangle\right| < \varepsilon .$$

Thus, the Riemann sum is bounded by $\varepsilon \|g\|_{L^\infty(\Omega)} \, \mu(\Omega)$, which furnishes yet another construction of a minimising sequence.

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* Note that Dunford and Schwartz call *sphere* what we prefer to call a *ball*.

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