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# ON THE RING OF THE HOLOMORPHIC FUNCTIONS OVER THE ALGEBRA

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## ABSTRACT

We prove that there exists a diffeomorphism between any subsets $G_m^1$ and $G_m^2$ of an algebra A of which nonzero elements are regular if there is an A-isomorphism between the rings $H(G_m^1)$ and $H(G_m^2)$.

## 1. INTRODUCTION

Working on conformal equivalence by means of the ring of analytic functions began in year 1940 [2].

Let $G_1$ and $G_2$ be two domains in the complex plane, and let $A(G_1)$ and $A(G_2)$ be the rings of analytic functions on them. If there exists a C-isomorphism between $A(G_1)$ and $A(G_2)$, then $G_1$ and $G_2$ are conformally equivalent, where C is the set of complex numbers [1]. The problem was generalized to open Riemann surfaces $G_1$ and $G_2$ [5]. It was shown that two domains $G_1$ and $G_2$ in the complex plane were conformally equivalent if the rings $B(G_1)$ and $B(G_2)$ of all bounded analytic functions defined on them were algebraically C-isomorphic [3]. When we discuss the rings $B(G_i)$ $(i=1,2)$, it is always assumed that $G_i$ is bounded and has the following property: for any $z \in \partial G_i$, boundary of $G_i$, there exists a function $f \in B(G_i)$ for which z is an unremovable singularity. It is proved that if there is a C-isomorphism between $A(G_1)$ and $A(G_2)$, then the sets $G_1$ and $G_2$ are conformally equivalent [7].

Now our aim is to investigate the above problem for the algebra A with finite dimensional.

## 2. THE HOLOMORPHIC FUNCTIONS OVER AN ALGEBRA

Let A be an associative commutative unital algebra of finite dimension m over the field R of real numbers. We have

$$e_\alpha e_\beta = C^\gamma_{\alpha\beta} e_\gamma, \quad (\alpha, \beta, \gamma = 1, \ldots, m)$$

such that the set $\{e_1, e_2, \ldots, e_m\}$ is a basis of the algebra A, where $C^\gamma_{\alpha\beta} e_\gamma$ is a new notation for $\sum_{\gamma=1}^{m} C^\gamma_{\alpha\beta} e_\gamma$, i.e., $C^\gamma_{\alpha\beta} e_\gamma = \sum_{\gamma=1}^{m} C^\gamma_{\alpha\beta} e_\gamma$ called the Einstein symbol. The coefficients $C^\gamma_{\alpha\beta}$ are called the structure constants of the algebra A. The structure constants are the components of the tensor field of type (1,2).

By using structure constants, in order to show that A is commutative, associative and unital algebra, we have

$$C^\gamma_{\alpha\beta} = C^\gamma_{\beta\alpha}$$

$$C^\gamma_{\alpha\beta} C^\theta_{\gamma\kappa} = C^\theta_{\alpha\kappa} C^\rho_{\beta\theta}$$

$$C^\gamma_{\alpha\beta} \varepsilon^\beta = C^\gamma_{\beta\alpha} \varepsilon^\beta = \delta^\gamma_\alpha$$

respectively. Where $\varepsilon^\beta$ is component of 1 which is the unit of A such that $1 = \varepsilon^\beta e_\beta$ and $\delta^\gamma_\alpha$ is Kronecker's symbol. In this paper, we assume that A is an associative commutative unital algebra.

Let $X = x^\alpha e_\alpha$, $\alpha = 1, \ldots, m$, be a variable in the algebra A, where $e_\alpha$ and $x^\alpha$ denote the basis units of A and real variables, respectively. Then the function

$$F = f^\alpha e_\alpha$$

defined over the algebra A is a function in X, where $f^\alpha = f^\alpha(x^1, \ldots, x^m)$ are real functions in all $x^\alpha$. We have $F = F(X)$. Let us define the differential in A by

$$dX = dx^\alpha e_\alpha \quad \text{ve} \quad dF = df^\alpha e_\alpha.$$

If the differential dF can be represented in the form $dF = F'(X)\,dX$, then $F = F(X)$ is said to be A-holomorphic (A-differentiable), where $F'(X)$ represents the derivative of F(X) [4, 6, 8].

**Theorem 2.1.** The function F=F(X) is A-holomorphic if and only if

$$C_\alpha D = DC_\alpha \tag{1}$$

where $C_\alpha = (C^\gamma_{\alpha\beta})$ are structure constants matrix and $D = \left(\dfrac{\partial f^\alpha}{\partial x^\beta}\right)$ is real Jacobian matrix such that $\gamma$ and $\beta$ represent row and column, respectively [4].

The equality (1) are called Schaffers conditions[4]. In particular, if $A = C$ is the complex number algebra $(m = 2)$, the Schaffers conditions coincide with the Cauchy-Riemann conditions: Let us consider the algebra $C = R(i)$, $i^2 = -1$. The dimension of the algebra C is 2. The basis of the algebra C is the set $\{e_1, e_2\}$ such that $e_1 = 1, e_2 = i$. If the equality

$$e_i e_j = C^1_{ij} e_1 + C^2_{ij} e_2$$

is considered, we have the structure constants matrices

$$C_1 = \begin{pmatrix} C^1_{11} & C^1_{12} \\ C^2_{11} & C^2_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} C^1_{21} & C^1_{22} \\ C^2_{21} & C^2_{22} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Jacobian matrix of the function $f(z) = u(x, y) + iv(x, y)$ is that

$$D = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Using the Schaffers conditions $C_\alpha D = DC_\alpha$, $\alpha = 1,2$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or shortly

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

or shortly

$$\begin{pmatrix} -v_x & -v_y \\ u_x & u_y \end{pmatrix} = \begin{pmatrix} u_y & -u_x \\ v_y & -v_x \end{pmatrix}.$$

Therefore from the equalities $C_1D = DC_1$ and $C_2D = DC_2$, , we obtained

$$u_x = v_y,\ u_y = -v_x$$

known as the Cauchy-Riemann conditions.

Note that, generally, A-holomorphic functions and analytic functions are different in algebra A [8].

## 3. DIFFEOMORPHISM BETWEEN $G^1_m$ AND $G^2_m$

Let $G^i_m$, $(i = 1,2)$ be subsets of the algebra A. If there exists a bijective function $f : G^1_m \to G^2_m$ such that the functions f and $f^{-1}$ are A-differentiable, then the function f is called an A-diffeomorphism. If f is an A-diffeomorphism, the determinant of the Jacobian matrix $D_f$ of the function f is not zero. That is, $|D_f| > 0$ or $|D_f| < 0$, where $|D_f|$ is determinant of the Jacobian matrix $D_f$. Note that, if we consider the set of the complex numbers C, then a diffeomorphism, generally, is not a conformal mapping. Let $F : G^1_m \to A$ be a A-holomorphic mapping and $\varphi : G^1_m \to G^2_m$ be A-diffeomorphism. Since

$$C_\alpha D_{F\circ\varphi^{-1}} = D_{F\circ\varphi^{-1}} C_\alpha,$$

$F\circ\varphi^{-1} : G^2_m \to A$ is a A-holomorphic mapping, where $D_{F\circ\varphi^{-1}}$ is the Jacobian matrix of $F\circ\varphi^{-1}$.

On the other hand the sets

$$H(G^i_m) = \{F : G^i_m \to A : F \text{ is a } A-\text{holomorphic function}\} \quad (i = 1,2)$$

become a ring under the operations

$$(F+G)(X) = F(X) + G(X) \text{ and } (FG)(X) = F(X)G(X).$$

**Theorem 3.1.** If $\varphi : G^1_m \to G^2_m$ is a diffeomorphism, then $\Phi : H(G^1_m) \to H(G^2_m)$, $\Phi(F) = F\circ\varphi^{-1}$ is an A-isomorphism, i.e, the isomorphism $\Phi$ satisfies $\Phi(\alpha) = \alpha$ for every $\alpha \in A$.

**Proof.** It is easily shown that $\Phi$ is bijective and $\Phi(F+G) = \Phi(F) + \Phi(G)$ and $\Phi(FG) = \Phi(F)\Phi(G)$. Thus $\Phi$ is an isomorphism. On the other hand $\Phi(\alpha) = \alpha$ for all $\alpha \in A$. Thus $\Phi$ is an A-isomorphism.

**Definition 3.2.** Let $\alpha$ be a nonzero element of the algebra A. If there exists $\beta \in A$ such that $\alpha\beta = 1$, then $\alpha \in A$ is said to be a regular element.

For each $\alpha \in G^i_m$ $(i = 1,2)$, we consider the set

$$M(\alpha) = \{F \in H(G^i_m) : F(\alpha) = 0\}.$$

**Lemma 3.3.** $M(\alpha)$ is a principal ideal of $H(G^i_m)$ $(i = 1,2)$ generated by the function $X - \alpha$.

**Proof.** The proof is clear.

Now, suppose that nonzero elements of the algebra A are regular. We can write the following lemma such that $G^i_m$ $(i = 1,2)$ is a subset of $A$.

**Lemma 3.4.** $M(\alpha)$ is a maximal ideal of $H(G^i_m)$.

**Proof.** For instant, suppose that $M(\alpha)$ is not a maximal ideal. In that case there exists an ideal $I$ which contains $M(\alpha)$. There exists a function G such that $G \in I$ and $G \notin M(\alpha)$. Thus, $G(\alpha) \neq 0$. If $H(X) = G(X) - G(\alpha)$, then $H \in M(\alpha) \subset I$. Hence, we have $G(\alpha) = G(X) - H(X)$. Since $G(\alpha) \in A$ is a regular element, we have $I = H(G^i_m)$. Hence, the assertion holds.

**Definition 3.5.** $M(\alpha)$ is called a fixed maximal ideal of $H(G^i_m)$. All other maximal ideals of $H(G^i_m)$ are called free maximal ideals.

**Theorem 3.6.** If $\Phi : H(G^1_m) \to H(G^2_m)$ is an A-isomorphism, then there exists an A-diffeomorphism between $G^1_m$ and $G^2_m$.

**Proof.** Let $\Phi : H(G^1_m) \to H(G^2_m)$ be an A-isomorphism. Then, to every fixed maximal ideal $M(\alpha)$ of $H(G^1_m)$ corresponds to a fixed maximal ideal $M'(\alpha')$ of $H(G^2_m)$. If we put $\alpha' = \varphi(\alpha)$, then $\varphi : G^1_m \to G^2_m$ is a bijective mapping. In order to prove that $\varphi : G^1_m \to G^2_m$ is an A-diffeomorphism, let us put $G_o(X) = X$ on $G^2_m$. Then $G_o \in H(G^2_m)$. Since $\Phi$ is an A-isomorphism, there exists $F_o \in H(G^1_m)$ such that $\Phi^{-1}(G_o) = F_o$. It is then easy to see that, for any $\alpha \in G^1_m$, $F_o(X) - F_o(\alpha) \in M(\alpha)$ and $\Phi(F_o(X) - F_o(\alpha)) \in M'(\alpha')$ hence

$$G(X)-F_o(\alpha)=X-F_o(\alpha)\in M'(\alpha')=M'(\varphi(\alpha)) .$$

This shows that $\varphi\in H(G_m^2)$, i.e. $\varphi$ is a A-holomorphic function. Similarly, we can also show that $\varphi^{-1}$ is an A-holomorphic function. Hence, $\varphi$ is an A-diffeomorphism.

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