Commun. Fac. Sci. Univ. Ank. Series A1
V. 48. pp. 33-37 (1999)

THE CHARACTERIZATION OF THE LUNE DOMAINS

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(Received June 11, 1998; Revised June 14,1999; Accepted Sept. 7,1999)

ABSTRACT

Let us consider a domain D that is bounded by two arcs of circles that intersect at points α and β at angle $\gamma = \frac{\pi}{m}$ This domain is called a lune. Let B(D) be an algebra of bounded analytic functions on D. In this paper taking complex algebra R, we give an algebric characterization of conformal mappings from D to U, by taking $\alpha \in R$ that satisfies certain conditions, where $U = \{w \in \mathbf{C} : |w| < 1\}$.

1. INTRODUCTION

This paper presents a solution to a problem in subject of rings of bounded analytic functions. In late 1940's it was shown that two domains, D_1 and D_2 in the complex plane, are conformally equivalent iff the rings $B(D_1)$ and $B(D_2)$ of all bounded analytic functions defined on them are algebraically isomorphic. Let R be a ring which is known to be isomorphic with the ring of bounded analytic functions on an annulus $A = \{z \in \mathbf{C} : \rho_1 < |z| < \rho_2\}$ where ρ_1 and ρ_2 are not known. From the ring R deduce the number ρ_1 / ρ_2 [2]. The characterization of Schwarz Theorem and Unit Disc has been done [5].

In our study we take the ring and give an algebraic characterization.

ALGEBRAIC CHARACTERIZATION

Let ϕ be an isomorphism from B(D) onto R. We will denote the elements of B(D) by f,g,h, ... and the elements of R by a,b,c,... Let e and 1 be multiplicative identities of R and B(D), respectively. Thus, $1 \in B(D)$ is the function identically equal to 1 on D. Since $\phi : B(D) \to R$ is an isomorphism, $\phi(1) = e$ Furthermore $\phi(n1) = ne$ so that $\phi(\pm(m/n).1) = \pm(m/n)e. -e$ has two square roots in R, one is the image of i.1, the other is the image of $-i.1$. It is algebraically impossible to distinguish between these, since R has an automorphism which takes one into the other (corresponding to the mapping $f \to \bar{f} \in B(D)$ Thus, we choose one root of $-e$ and make it correspond to i.1; denote it as ie.

Hencefort, we will denote the complex number field by **C** and the complex rational number field by $\mathbf{C}_r$ where a complex number, both real and imaginary parts are real rationals, is called a complex rational number. Clearly, $\mathbf{C}_r$ and **C** are subrings of B(D).

Lemma 2.1. For each $\alpha \in \mathbf{C}_r, \phi(\alpha) = \alpha$ (or $\bar{\alpha}$).

Proof. If $\alpha \in \mathbf{C}_r$, there are two rational number r_1 and r_2 such that $\alpha = r_1 + ir_2$. Since $\phi(1) = e$ and $\phi(i) = i$ (or $-i$), we get $\phi[(r_1 + ir_2)].1 = r_1e + r_2ie$ (or $r_1e - r_2ie$), ([3], [4]).

Lemma 2.2. For each real number $c, \phi(c1) = ce$.

Proof. If c is a rational number, by the lemma (2.1) $\phi(c1) = ce$. If c is an irrational number, for each rational number, number $c, c - r \neq 0$. Thus there exist $(c-r)^{-1} = \frac{1}{c-r}$. Then $\phi[(c-r).1] = \phi(c1) - re$ and $\phi\left[\left(\frac{1}{c-r}\right).1\right] = \frac{e}{\phi(c1) - re}$. Therefore $\phi(c1) = ce$.

Corollary 2.3. If $c \in \mathbf{C}$, $\phi(c1) = ce$ [2].

Lemma 2.4. Let $f \in B(D)$ and let $\bar{R}_f$ be the closed range of f. Then $\lambda \in \bar{R}_f$ iff $f - \lambda.1$ has no inverse in B(D).

Proof. If $\lambda \in \bar{R}_f$ there is a $z_0 \in D$ such that $f(z_0) = \lambda$. Then $(f - \lambda.1)(z_0) = 0$. Hence $f - \lambda.1$ has no inverse in B(D). Now we suppose that $f - \lambda.1$ has no inverse in B(D). Then at least for one point $z_0 \in D$, $(f - \lambda.1)(z_0) = 0$. It follows that $f(z_0) = \lambda$, i.e., $\lambda \in \bar{R}_f$.

Lemma 2.5. $\lambda \in \overline{R}_f$ iff $\phi(f) -$ has no inverse in R.

Proof. If $\lambda \in \overline{R}_f$, $f - \lambda.1$ has no inverse in B(D) by Lemma 2.4. Since ϕ is an isomorphism, $\phi(f - \lambda.1) = \phi(f) - \lambda e$ has no inverse in R, ([1], [2]).

Let $\sigma(f)$ and $\sigma(a)$ be the spectrums of $f \in B(D)$ and $a \in R$ respectively. If
$\rho(a) = \sup\{\, |\lambda| : \lambda \in \sigma(a) \,\}$,
then $\rho(a)$ is also the maximum modulus (Hereinafter abbreviated MM) of $\phi^{-1}(a)$.

In this paper, we always consider the complex algebra. Now we give our first theorem connected with algebraic characterization.

Lemma 2.6. Let $\mu = \dfrac{\alpha + \beta}{2}$. Suppose that $f \in B(D)$ satisfies the following conditions.

1. $f(\mu) = 0$,
2. $MM(f) = 1$,
3. f is schlicht.

Then,

$$f(z) = \frac{(z-\alpha)^m - (\beta - z)^m}{(z-\alpha)^m + (\beta - z)^m} \tag{2.5.1}$$

where $m \geq 2$ and $m \in N$.

Proof. $I_\mu = \{f \in B(D) : f(\mu) = 0\}$ is the maximal ideal of B(D). I_μ is generated by $h(z) = z - \mu$, i.e. $I_\mu = \langle z - \mu \rangle$. The function that we are looking for must be in I_μ. By means of the theorem of maximum modulus, $MM(z - \mu) \neq 1$ for any z in D. Therefore $f(z) \neq z - \mu$. If $f(z) = (z - \mu) g(z)$, $f(\mu) = 0$ and $MM(f) = 1$, then $g(z)$ must be

$$g(z) = \frac{2\left[(z-\alpha)^{m-1} + \cdots + (\beta - z)^{m-1}\right]}{(z-\alpha)^m + (\beta - z)^m},$$

Thus

$$f(z) = \frac{(z-\alpha)^m - (\beta - z)^m}{(z-\alpha)^m + (\beta - z)^m}$$

where $m \geq 2$ and $m \in N$. Furthermore if f is schlicht, $f(\mu)=0$ and $MM(f)=1$, then this function must be in the form of (2.5.1) [6].

Theorem 2.7. Let R be any algebra such that ϕ is an isomorphism from B(D) to R which satisfies $\phi(\alpha)=\alpha$ for every complex number α. Furthermore, suppose that the following condition are satisfied for some $a \in R$.

a) For each $\lambda_0 \in \sigma(a)=U$, there is only one point z_0.

b) For each $\mu \in \mathbf{C}$, $\langle b-\mu e\rangle$ is a maximal ideal of R. Furthermore, $\phi^{-1}(b)=z$ and $a \in \langle b-\mu e\rangle$, where $b \in R$.

c) $\rho(a)=MM(\phi^{-1}(b))=1$.

Then $\phi^{-1}(a)$ is a conformal mapping from D to U and

$$\phi^{-1}(a)(z)=\frac{(z-\alpha)^m-(\beta-z)^m}{(z-\alpha)^m+(\beta-z)^m}.$$

Proof. Since $a \in \langle b-\mu e\rangle$, there is an element $c \in R$ such that $(b-\mu e)c=a$. Since ϕ is an isomorphism, we can write $\phi^{-1}(a)=\phi^{-1}(b-\mu e)\phi^{-1}(c)$ and $\phi^{-1}(a)=[\phi^{-1}(b)-\phi^{-1}(\mu e)]\,\phi^{-1}(c)$. Thus we find

$$\phi^{-1}(a)=(z-\mu)\,\phi^{-1}(c).$$

By lemma (2.6), $MM(\phi^{-1}(a))=1$ and hence

$$\phi^{-1}(c)=\frac{2[(z-\alpha)^{m-1}+(z-\alpha)^{m-2}(\beta-z)+\cdots+(z-\alpha)(\beta-z)^{m-2}+(\beta-z)^{m-1}]}{(z-\alpha)^m+(\beta-z)^m}$$

Clearly, as $\phi^{-1}(c) \in B(D)$ we obtain

$$c=\frac{\phi\{2[(z-\alpha)^{m-1}+\cdots+(\beta-z)^{m-1}]\}}{\phi[(z-\alpha)^m+(\beta-z)^m]}$$

$$=\frac{\phi(2)\left[\phi(z-\alpha)^{m-1}+\cdots+\phi(\beta-z)^{m-1}]\right]}{\phi[(z-\alpha)^m]+\phi[(\beta-z)^m]}$$

$$=\frac{2e\left[(be-\alpha e)^{m-1}+\cdots+(\beta e-be)^{m-1}]\right]}{(be-\alpha e)^m+(\beta e-be)^m}$$

from the equality, and hence $c \in R$. Thus

$$a=(b-\mu e)\ \frac{2e\left[(be-\alpha e)^{m-1}+\cdots+(\beta e-be)^{m-1}]\right]}{(be-\alpha e)^m+(\beta e-be)^m}$$

and we deduce the mapping

$$\phi^{-1}(a)=\frac{(z-\alpha)^m-(\beta-z)^m}{(z-\alpha)^m+(\beta-z)^m}$$

It is well known that this is the mapping form D onto U. At the same time, the mapping $\phi^{-1}(a)$ is unique. Because, $\lambda_0 \in \overline{R}_{\phi^{-1}(a)}$, by $\lambda \in \sigma(a)$. Since to each point of $\overline{R}_{\phi^{-1}(a)}$ there corresponds a unique z_i by Lemma 2.4. and part a) of the theorem, $\phi^{-1}(a) \in B(D)$ is one-to-one. Since ϕ is an isomorphism and $\langle b-\mu e\rangle$ is a maximal principal ideal in R, $\phi(z-\mu e)$ is a maximal principal ideal in B(D). This maximal principal ideal generated by the $\phi(b)-\phi(\mu e)=z-\mu$ then $\phi^{-1}(a)\in\langle z-\mu\rangle$ by (b), $\phi^{-1}(a)$ is schlicht. Thus

$$\phi^{-1}(a)=\frac{(z-\alpha)^m-(\beta-z)^m}{(z-\alpha)^m+(\beta-z)^m}$$

by Lemma 2.6.

REFERENCES

[1] BAYRAKTAR, M., Üst yarı düzlemin birim daireye konform tasvirinin cebirsel karakterizasyonu, Journal of Mathematics and Statistics of the Faculty of Arts and Sciences, Gazi University, Volume 2, (1989), 111-116.

[2] BECK, A., On Rings On Rings, Proc. Amer. Math. Soc. 15 (1964), 350-353.

[3] BERS, L., On Rings of Analytic Functions, Bull. Amer. Math. Soc. 54(1984), 311-315.

[4] KAKUTANI, S., Rings of Analytic Functions, Proc.Michigan Conference on Functions of a Complex Variable, (1955), 71-84.

[5] KAMALİ, M., BAYRAKTAR, M., The Characterization of Schwarz Theorem and Unit Disc, Commun. Fac. Sci. Univ. Ank. Series A1, V. 43 (1994) 49-55.

[6] KODAIRA, K., Introduction to Complex Analysis, Cambridge University Press-Cambridge, (1997). P. 120-154.