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SOME COMMENTS ON CONCENTRATION AND EXPANSION FUNCTIONS AS APPLIED TO BIVARIATE DEPENDENCE

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ABSTRACT

Potential utilization of concentration and expansion functions in the detection of dependence of two random variables is investigated. Also, a brief literature survey is explored. Pitfalls and drawbacks of such applications are emphasized.

1. INTRODUCTION

The exposition presented in the remainder of this work shall frequently refer to the immediate following definition and remark. The definition is based essentially on Raoult [5] (c.f., also, the monograph of Hengartner and Theodorescu [2]).

Definition 1. For a bi-measure space $(\Omega,\Im,\mu,\vartheta)$ (a measurable space $(\Omega,\Im)$ with a pair (μ,ϑ) of measures simultaneously defined on it), the measure ϑ is assumed to have Lebesgue decomposition with respect to the measure $\mu : \vartheta = \vartheta^a + \vartheta^s$, such that ϑ^a is absolutely continuous with respect to μ ($\vartheta^a << \mu$, in short) and ϑ^s is singular with respect to μ ($\vartheta^s \perp \mu$, in short). Furthermore, if Q^c is a subset of Ω, such that $Q^c = \{\omega \in \Omega : \mu(\omega) = 0\}$, the subspace $(Q,\mathcal{U})$ of $(\Omega,\Im)$ will be relevant for the following discussions, where $Q = \Omega - Q^c$ with $\mu(\Omega) = \mu(Q)$ and $\mathcal{U}$ is the $\sigma-$algebra of Q satisfying $\mathcal{U} \subseteq \Im$. A real-valued function $f_u(t)$ (resp., $g_u(t)$) with domain $I = [0,\mu(Q)]$ and range $[0,\vartheta(Q)]$ is called **concentration** (resp., **expansion**) of the measure ϑ with respect to the measure μ on the subalgebra $\mathcal{U}$, if, $\forall t \in I$ there exist some $A, B \in \mathcal{U}$, such that

$$f_u(t) = \vartheta^a(A) = \sup\{v : \mu(A) \le t \text{ and } \vartheta(A) = v \text{ for } A \in \mathcal{U}\} \quad (1.1)$$

$$(resp., g_u(t) = \vartheta^a(B) = \inf\{s : \mu(B) = t \text{ and } \vartheta(B) = s \text{ for } B \in \mathcal{U}\}. \quad (1.2)$$

Remark 1.

(i) The above definition can also be extended to a family of bi-measure spaces such as $((\Omega, \Im, \mu, \vartheta_j),\ j \in J)$ where J is an index set.

(ii) Existence of concentration function (the upper bound for (1.1)) is provided in Raoult [5] in terms of Neymann-Pearson test procedure (c.f., also, [2]). As a matter of fact, a typical example for application of concentration to a statistical area is Neymann-Hypothesis testing procedure, where A is rejection region, μ represents the measure under null hypothesis and ϑ (and hence ϑ^a) denotes the measure under alternative hypothesis.

(iii) If $\mathcal{U} = \Im$ or if the relevant algebra is clear from the context, the index $\mathcal{U}$ of f_u and g_u can be ignored.

(iv) Note that $\vartheta^a << \mu$ implies both $\vartheta^a(Q^c) = \mu(Q^c) = 0$ and $\vartheta^s(Q) = 0,$ so that $\vartheta(\Omega) = \vartheta^a(\Omega) + \vartheta^s(\Omega) = \vartheta^a(Q) + \vartheta^s(Q^c).$

(v) If $(\Omega, \Im, \mu, \vartheta)$ is a bi-probability space, i.e., if ϑ and μ are probability measures on $(\Omega, \Im)$ then $\vartheta(\Omega) = 1$ and $\mu(\Omega) = \mu(Q) = 1$ with $I = [0, \mu(Q)] = [0, \vartheta(Q)] = [0,1]$

(vi) Also, for the special case $B = A^c = \Omega - A$ with $\mu\{\omega \in A\} = t$ and $\mu\{\omega \in A^c\} = 1 - t$ in such a bi-probability space, we have $\vartheta(\Omega) = \vartheta(A) + \vartheta(A^c)$ $= f_u(t) + g_u(1-t) = 1$

Raoult [5] discusses the concepts of concentration and expansion (*étalement*) in connection with Lebesgue decompositions covering a general bi-measure space on a unidimensional basis. The concentration in (1.1) is also noted [2]. An alternate better-known notion of concentration is due to Levy [4], who uses the concept for sums of random variables. To discriminate it from the former, the latter type of concentration is generally labelled as *Lévy concentration functions*. Raoult [5] shows the relation between these two types of concentration – Further developments concerning Lévy concentration functions can be found in [2] – The expansion function in (1.2) is however less known and hence has apparently no statistical and probabilistic applications.

Within a different setting and in a relatively recent attempt to obtain a general formulation for the so-called Gini-type concentration indices, on the other hand, Gifarelli and Regazzini [1] re-dwell upon concentration, and to this end, the authors specify a function (c.f., Theorem 2.2 in [1]).

$$\varphi(t) = \vartheta^a(A) = \inf\{\vartheta(A) : A \in \mathcal{U} \text{ and } \mu(A) \ge t\}, \quad (1.3)$$

$\forall t \in \mathcal{I}$. The relevant measure space for (1.3) is again a bi-probability space such as $(\Omega, \Im, \mu, \vartheta)$. Essentially, $\varphi(x)$ corresponds to an **expansion function** in the sense of (1.2), and yet, it is called **concentration function** by the authors. In fact, except for (1.3), the remaining results of Cifarelli and Regazzini [1] pertain to **concentration** functions. It is thus possible to classify Cifarelli-Regazzini type of concentration functions within the general category of concentration functions given in Definition1.

The presentation below is connected to the case where $(\Omega, \Im, \mu, \vartheta)$ is a bi-probability space. In the two-dimensional case involving families of distributions like Fréchet class, we shall also use a family of spaces $((\Omega, \Im, \mu, \vartheta_j), j \in J)$. Throughout the remainder of discussions, it is assumed that concentration and expansion functions exist. Within this setup, we let, $\forall t \in \mathcal{I}$,

$$\varphi_u = (1-t) = \vartheta^a(A^c) = \inf\{\vartheta(A^c) : A^c \in \mathcal{U} \text{ and } \mu(A^c) \geq 1-t\}, \tag{1.4}$$

denote an expansion function, and

$$\psi_u(t) = \vartheta^a(A) = \sup\{\vartheta(A) : A \in \mathcal{U} \text{ and } \mu(A) = t\}, \tag{1.5}$$

stand for the corresponding concentration function. If no confusion is expected to arise, the index $\mathcal{U}$ of φ_u in (1.4) and ψ_u in (1.5) is generally ignored for convenience of notation.

The functions in (1.4) and (1.5) can further be stated in convenient forms: In reality, the property $\vartheta^a \prec\prec \mu$ in the Lebesque decomposition $\vartheta = \vartheta^a + \vartheta^s$ implies, by Radon-Nikodym Theorem, existence of a nonnegative function $\ell \in R_+ = [0, \infty)$ such that $d\vartheta = \ell \cdot d\mu$. Thus, for A_t defined as $A_t = \{\omega : \ell(\omega) \leq c_t, \omega \in Q\} \in \mathcal{U}$ with $\mu(A_t) = t \neq 0, t \in \mathcal{I}$, we have

$$\psi(t) = \vartheta^a(A_t) = \int_{A_t} \ell(\omega) d\mu(\omega) = \int_0^{c_t} \ell dH_L(\ell). \tag{1.6}$$

The last integral is based on the existence of a distribution function $H_L(\ell)$ given by

$$H_L(\ell) = \mu\{\omega : L(\omega) \leq \ell, \omega \in Q\}.$$

Similarly, for $A_t^c = \{\omega : \ell(\omega) > c_t, \omega \in Q\}$ with $\mu(A_t^c) = 1 - t \neq 0$, we shall have

$$\varphi(1-t) = \vartheta^a(A_t^c) = \int_{A_t^c} \ell(\omega) d\mu(\omega) = \int_{c_t}^{\infty} \ell dH_L(\ell). \tag{1.7}$$

Note that when both ϑ and μ are dominated by a common measure ν with $d\vartheta = f \cdot d\nu$ and $d\mu = g \cdot d\nu$, then $d\vartheta = \ell \cdot d\mu = (\ell \cdot g) d\nu$ and $d\vartheta = f \cdot d\nu$. Thence, we have $f = \ell \cdot g$, so that

$$\ell(\omega) = \frac{d\vartheta(\omega)}{d\mu(\omega)} = \frac{f(\omega)}{g(\omega)}, \quad g(\omega) > 0$$

the last quotient being so-called likelihood ratio. $L = d\vartheta / d\mu$ with values $\ell(\omega) = d\vartheta(\omega) / d\mu(\omega)$ will otherwise be referred as generalized likelihood ratio below. Integrating out by parts, the last term of (1.6) becomes

$$\psi(t) = \int_0^{c_t} (t - H_L(\ell)) d\ell = \int_0^t c_s ds, \tag{1.8}$$

(c.f., also [1]), and, by substituting (1.8) in the relation $\varphi(1-t) + \psi(t) = 1$ given in Remark 1 above, the expansion in (1.7) on the other hand boils down to

$$\varphi(1-t) = 1 - \int_0^{c_t} (t - H_L(\ell)) dt = 1 - \int_0^t c_s ds = \int_t^1 c_s ds. \tag{1.9}$$

The quantity c_t in (1.6)-(1.7) and (1.8)-(1.9) is the so-called t-quantile for the population of L

$$c_t = \inf\{\ell \in R_+ : H_L(\ell) \geq t\} \tag{1.10}$$

$\forall t \in I$.

In a concluding remark, Cifarelli and Regazzini [1] point potentials of (1.3) for application to such probabilistic issues as homogeneity, association, etc. Upon the suggestion, Scarsini [6] attempts to extend the concept of concentration to the two-dimensional case and investigate its potential uses for ordering Fréchet class of bivariate distributions in terms of the degree of dependence they display. The conclusion reached by the author appears however to be hardly optimistic. As will be clear in the following pages, these nonpromissing results stem from the facts that (i), without (1.2) and/or (1.4), (1.1) and/or (1.5) alone provides only a partial, and often misleading, picture for dependence, especially in the presence of positive quadrant dependence (c.f., Lehmann [3], for the concept); (ii) as is also posed by Scarsini [6], the question that whether the concept of concentration and expansion does really coincide with the concept of dependence of random variables needs further investigations. These issues will be taken up next in a sequence of sections below. First, however, we summarize the known properties of these functions (Lemma 1 and Corollary to this lemma). We also prove some properties of expansion function (Lemma 2), which, in fact, do not seem to exist in literature. In order to be able to detect bivariate dependence in terms of concentration and expansion functions, we then set up a two-dimensional framework. The final two sections investigate the relationship of dependence to the concept of concentration-expansion functions.

2. ANALYTICAL PROPERTIES OF CONCENTRATION AND EXPANSION FUNCTIONS

As explicitly given by Raoult [5], the concentration and expansion functions $\psi_u(t)$ and $\varphi_u(t)$ have the following analytical properties:

Lemma 1:

(i) $\psi_u(t)$ is an increasing (nondecreasing) and continuous concave function in I;

(ii) given two sub-algebras $\mathcal{U}_*$ and $\mathcal{U}_0$ of $\Im$, $\mathcal{U}_* \subset \mathcal{U}_0 \Rightarrow \psi_{u_*}(t) \le \psi_{u_0}(t), \ \forall t \in I$;

(iii) $\varphi_u(t)$ is also an increasing (nondecreasing) and continuous **convex** function in I;

(iv) the functions $\psi_u(t)$ and $\varphi_u(t)$ are related to each other through

$$\psi_u(t) = \vartheta(\Omega) - \varphi_u(\mu(\Omega) - t), \tag{2.1}$$

which, as noted in Remark 1 above, is equal to

$$\psi_u(t) = 1 - \varphi_u(1-t), \tag{2.2}$$

for a bi-probability space.

Proof: Noting that the functions are probability measures, the nondecreasing and continuity properties are easy to see. As for the concavity (convexity), this will be discussed below. See, also, [5].

Remark 2. Cifarelli and Regazzini (Theorem 2.3 in [1]) maintains without proof that concentration functions are convex. However, in connection with decomposition concentrations, a proof for the concavity of concentration functions is provided in Theorem 4.2.2 of [2].

In addition to the properties mentioned in Lemma 1, the expansion function φ_u displays some further features – the index $\mathcal{U}$ of φ_u will henceforth be ignored:

Lemma 2:

(i) $\varphi(1-t) = 1-t, \ \forall t \in I$, iff $\vartheta = \mu$ everywhere on $\mathcal{U}$;

(ii) $\varphi(1) = 1, \varphi_u(0) = 0$ and otherwise $1 - t < \varphi(1-t) < 1, \forall t \in I$;

(iii) $\varphi(1-t) = 0, \ \forall t \in I$, *iff* ϑ is singular with respect to μ, *i.e.*, $\vartheta = \vartheta^s$.

Proof:

(i) Note that, $(\forall \omega \in Q)$, $\vartheta(\omega) = \mu(\omega) \Rightarrow d\vartheta(\omega) = d\mu(\omega)$, $(\forall \omega \in Q) \Rightarrow \ell(\omega) = 1, \ (\forall \omega \in Q)$. Hence, from (1.7), we obtain

$$\varphi(1-t) = \vartheta^a(A_t^c) = \int_{A_t^c} \ell(\omega) d\mu(\omega) = \int_{A_t^c} d\mu(\omega) = 1 - t.$$

Conversely, $\varphi(1-t) = 1-t$ implies that $\ell(\omega) = 1, (\forall \omega \in Q)$. Hence, the result follows.

(ii) $t = 0 \Rightarrow A_0^c = \{\omega : \ell(\omega) \ge 0, \omega \in Q\} = Q$ and hence,

$$\varphi(1) = \vartheta^a(A_0^c) = \vartheta^a(Q) = \int_Q \ell(\omega) d\mu(\omega) = 1.$$

Likewise, $\lim_{t \to 1} A_t^c = \phi$ with $\lim_{t \to 1} \mu(A_t^c) = 0$ and

$$\varphi(0) = \lim_{t\to 1} \vartheta^a(A_t^c) = \int_{\phi} \ell(\omega) d\mu(\omega) = 0.$$

Finally, for $\ell \in (c_t, \infty)$ and $t \in (0,1]$, we have

$$(1-t) = \int_{A_t^c} d\mu(\omega) < \int_{A_t^c} \ell(\omega) d\mu(\omega) < \int_{\Omega} \ell(\omega) d\mu(\omega) = 1.$$

The left-hand side inequality follows from (1.4). To show this let $E_t = \{\omega : c_t \le \ell(\omega)\} \in \mathcal{U}$ with $\mu(E_t) = 1-t$ and $\vartheta^a(E_t) = \varphi(1-t)$. Consider the following collection of sets in $\mathcal{U}$

$$\mathcal{G} = \{E \in \mathcal{U} : 1-t = \mu(E_t) \le \mu(E)\}.$$

Obviously, by (1.4),

$$1-t = \mu(E_t) = \mu(\bigcap_{\mathcal{G}} E)$$

$$\varphi(1-t) = \vartheta^a(E_t) = \vartheta^a(\bigcap_{\mathcal{G}} E)$$

Now these two measures must satisfy $\mu(E_t) \le \vartheta^a(E_t)$, because the two simultaneous relations $\vartheta^a(E_t) = \mu(E_t)$ and $\vartheta^a(E_t) < \mu(E_t)$ are contradictory. In fact, if the latter would hold simultaneously, then there would be some E_* in $\mathcal{G}$ with $\vartheta^a(E_*) = \varphi(1-t) < \vartheta^a(E_t) = 1-t$, such that $E_* \subset E_t$. This however is against the initial assumption that there is no $E \in \mathcal{G}$ which is contained by E_t. Thence, the result follows.

(iii) When ϑ is singular with respect to μ, , so will be the measure ϑ^a and hence the integral in (1.7) will be zero. **QED**

Remark 3. The case where $\varphi(1-t) = \vartheta^a(A_t^c) = 1, \forall t \in (0,1]$, is interesting to note: In fact, for this case, we have $\vartheta^a(A_t^c) = \vartheta^a(Q) = 1, \forall t \in (0,1]$; that is, $\forall t \in (0,1]$,

$$\vartheta(A_t^c) = \int_{A_t^c} \ell(\omega) d\mu(\omega) = \int_{Q} \ell(\omega) d\mu(\omega) = 1.$$

This is tantamount to stating that $\ell(\omega)$ is equal to a constant ℓ_t^+ in A_t^c, such that $\ell_t^+ = 1/\mu(A_t^c)$. This case will be resumed in connection with the discussions on two-dimensional case (c.f., end of Section 3 below).

Corollary: In view of the properties of $\varphi(1-t)$ mentioned in the foregoing lemma and remark, the concentration function $\psi(t) = 1-\varphi(1-t)$ in (1.5) satisfies further that:

(v) $\psi(t) = t, \forall t \in I$, *iff* $\varphi(1-t) = 1-t$, *i.e.*, *iff* $\vartheta = \mu$;

(vi) $\psi(t) = 1, \forall t \in I$, *iff* $\varphi(1-t) = 0$, *i.e.*, *iff* $\vartheta \perp \mu$;

(vii) $\psi(t) = 0, \forall t \in \mathbb{I}$, *iff* $\varphi(1-t) = 1$, *i.e.*, *iff* $\ell(\omega)$ is constant in its relative domain A_t and A_t^c, $\forall t \in (0,1]$;

(viii) otherwise, $0 < \psi(t) < t$, $\forall t \in \mathbb{I}$.

Remark 4. Through proved differently, excepting for (vii), the results of this corollary can also be found in *Theorem 2.3* of [1]. Part of the proof of (vii) is in *Remark 3*. The remainder of the proof can be found at the end of Section Three where we dwell on Fréchet bounds.

When concentration and expansion functions both exist, their convex and concave natures seem to render them dual to each other for potential applications in Statistics. For easy reference, this duality is emphasized below. In fact, let

$$d\vartheta(\omega) = d\mu(\omega), \forall \omega \in Q, \text{ such that } \ell(\omega) = 1, \forall \omega \in Q, \tag{2.3}$$

then, adopting $\psi_0(t)$ and $\varphi_0(1-t)$ for the respective concentration and expansion functions corresponding to (2.3), we have, $\forall t \in \mathbb{I}$,

$$\varphi_0(1-t) = 1 - \psi_0(t) = 1 - \int_{A_t} d\mu(\omega) = 1 - t, \ \psi_0(t) = \int_{A_t} d\mu(\omega) = t \ . \tag{2.4}$$

On the other hand, if, $\forall \omega \in Q$, we have $d\mu(\omega) < d\vartheta(\omega)$, then, $\forall \omega \in Q$, $1 < \ell(\omega)$, and hence

$$\varphi_0(1-t) = 1 - t = \int_{A_t^c} d\mu(\omega) < \int_{A_t^c} \ell(\omega) d\mu(w) = \varphi(1-t) < 1, \tag{2.5}$$

so that, for this case, expansion functions will be more appropriate. Conversely, if $\ell(\omega) < 1$, $\forall \omega \in Q$, so that

$$\psi(t) = \int_{A_t} \ell(\omega) d\mu(\omega) \le \int_{A_t} d\mu(\omega) = t = \psi_0(t), \tag{2.6}$$

and thus concentration functions will be more appropriate for this latter case.

Remark 5. When ϑ and μ are indexed and identified by some real-valued parameter θ with the respective values θ_ϑ and θ_μ concentration and expansion functions can meaningfully be associated with the well-known information integral of Kulbak: For $A_t = \{\omega \in Q : \ell(\omega) \le c_t\} \in \mathcal{U}$, $t \in (0,1)$, we obviously have

$$\int_{A_t} \log \ell(\omega) d\mu(\omega) = \int_{A_t} (\log d\vartheta(\omega)) d\mu(\omega) - \int_{A_t} (\log d\mu(\omega)) d\mu(\omega) \le \log \psi(t) \le \psi(t),$$

provided that they are all defined. When multiplied by (-1) and taken over the entire space $A_1 = \{\omega \in Q : \ell(\omega) < \infty\} = Q \in \mathcal{U}$, the integral on the left hand side is so-called Kulback's mean information or Kulback's information integral on A_t. In Wilks' [7] notation,

$$\{H(\theta_\mu, \theta_\mu \mid A_t) - H(\theta_\mu, \theta_\vartheta \mid A_t)\} \cdot \mu(A_t) = -\int_{A_t} \log \ell(\omega) d\mu(\omega) > 0$$

discriminates μ against ϑ on A_t, $t \in (0,1)$, where

$$H(\theta_\mu, \theta_\mu \mid A_t) = \frac{1}{\mu(A_t)} \int_{A_t} (\log d\mu(\omega)) d\mu(\omega)$$

$$H(\theta_\mu, \theta_\vartheta \mid A_t) = \frac{1}{\mu(A_t)} \int_{A_t} (\log d\vartheta(\omega)) d\mu(\omega)$$

represent H-functions of Boltzmann, conditional on A_t.

3. TWO-DIMENSIONAL SETUP

The discussions in both Raoult [5] and Cifarelli and Regazzini [1] run in a uni-dimensional setup. To provide a framework for applications of concentration and expansion functions to bivariate dependence, a two-dimensional setup must hence be introduced: Given a probability space $(\Omega, \Im, \pi)$ we define two distinct measurable functions (random variables) on Ω

$$\mathbf{Z} = \begin{pmatrix} X \\ Y \end{pmatrix} : \Omega \times \Omega \to \mathbb{R}_2 . \tag{3.1}$$

A new probability space $(\mathbb{R}_2, \mathbb{B}_2, \mu) = (\mathfrak{X}, \mathfrak{B}, \mu)$ is hence induced, where, for convenience of notation, we let $\mathfrak{X}$ stand for $\mathbb{R}_2$ and $\mathfrak{B}$ for $\mathbb{B}_2$. As usual, $\mathbb{B}_2$ is the Borel algebra of $\mathbb{R}_2$ and μ is the probability measure induced under transformations $X(\omega)$ and $Y(\omega)$, *i.e.*, $\forall E \in \mathfrak{B}$,

$$\mu(E) = \pi(X^{-1}(E) \cap Y^{-1}E)). \tag{3.2}$$

Given $x, y \in \mathbb{R}_1$, for $C_x = \{X \le x\} \cap \mathbb{R}_1 \in \mathfrak{B}$ and $D_y = \{Y \le y\} \cap \mathbb{R}_1 \in \mathfrak{B}$, we have

$$\mu_X(C_x) = \pi(X^{-1}(C_x) \cap Y^{-1}(\mathbb{R}_1))$$

$$\mu_Y(D_y) = \pi(X^{-1}(\mathbb{R}_1) \cap Y^{-1}(D_y))$$

$$\mu_+(C_x \cap D_y) = \pi(X^{-1}C_x) \cap Y^{-1}(D_y))$$

$$\mu_0(C_x \cap D_y) = \mu_x(C_x) \times Y^{-1}(D_y)). \tag{3.3}$$

Accordingly, a bi-probability space $(\mathfrak{X}, \mathfrak{B}, \mu_0, \mu_+)$ can be obtained. The measure μ_+ cannot be singular with respect to the measure μ_0 everywhere in $\mathfrak{B}$, simply for the reason that marginal probabilities μ_X and μ_Y are obtained from μ_+ and that $\mu_0 = \mu_X \cdot \mu_Y$ (c.f., the final paragraph of Section 4 for further discussions). Therefore, μ_+ is assumed to have the Lebesgue decomposition

$$\mu_+ = \mu_+^a + \mu_+^s \tag{3.4}$$

with respect to μ_0, i.e.,

$$\mu_+^a << \mu_0 \text{ and } \mu_+^s \perp \mu_0,$$

where $\mu_+^a(E) = 0,$ whenever $\mu_0(E) = 0, \forall E \in \mathfrak{B}.$ Thus, if $\mathfrak{M}^c = \{\omega \in \mathfrak{X} : \mu_0(\omega) = 0\},$ with $\mathfrak{M} = \mathfrak{X} - \mathfrak{M}^c$ and $\mu_0(\mathfrak{X}) = \mu_0(\mathfrak{M}),$ then $\forall E \in \mathfrak{B},$ we can write

$$\mu_0(E) = \mu_0(E \cap \mathfrak{M}) + \mu_0(E \cap \mathfrak{M}^c) = \mu_0(E \cap \mathfrak{M})$$
$$\mu_+^a(E) = \mu_+(E \cap \mathfrak{M}), \text{ since } \mu_+^a << \mu_0. \tag{3.5}$$

By substituting μ_0 for μ in *Definition 1* above, μ_+ for ϑ, $\mathfrak{X}$ for Ω, $\mathfrak{M}$ for Q and the Borel algebra $\beta(\mathfrak{M})$ of subsets of $\mathfrak{M}$ for $\mathcal{U}$, a setup parallel to the one in *Definition 1* is obtained. Provided that the relevant distributions exist, we can set

$$\begin{aligned} &F_X(x) \ \textit{for} \ \mu_X(C_x) \\ &F_Y(y) \ \textit{for} \ \mu_Y(D_y) \\ &F(x,y) \ \textit{for} \ \mu_+(C_x \cap D_y) \\ &F_0(x,y) \ \textit{for} \ \mu_0(C_x \cap D_y). \end{aligned} \tag{3.6}$$

When no confusion is expected to arise, the probability measures in (3.3) and the distributions in (3.6) will interchangeably be used below.

As in the univariate case, the distribution $\mathrm{H_L}(\cdot)$ of the random variable L is obtained from

$$H_L(c_t) = \mu_0\{(x,y) \in \mathfrak{X} : \ell(x,y) \le c_t\} = t, \tag{3.7}$$

where

$$L : \mathfrak{X} \to \mathfrak{R}_+, \tag{3.8}$$

such that, $\forall (x,y) \in \mathfrak{X},$

$$\ell(x,y) = \frac{d\mu_+}{d\mu_0}, \tag{3.9}$$

with the differential being evaluated at $(x,y) \in \mathfrak{X}.$ As such, for $A_t = \{(x,y) \in \mathfrak{X} : \ell(x,y) \le c_t\}$

$$\psi(t) = \mu_+^a(A_t) = \iint_{A_t} \ell \, d\mu \tag{3.10}$$

gives the concentration function in (1.5), where, for some $t \in \mathcal{I}$ the t-quantile c_t is

$$c_t = \inf\{\ell \in \mathfrak{R}_+ : H_L(\ell) \ge t\}. \tag{3.11}$$

Similarly, $\forall t \in \mathcal{I},$ and for $A_t^c = \{(x,y) \in \mathfrak{X} : \ell(x,y) > c_t\}$

$$\begin{aligned} \varphi(1-t) &= \iint_{A_t^c} \ell \cdot d\mu_0 \\ &= \int_{c_t}^{\infty} \ell \cdot dH_L(\ell) = \int_t^1 c_s ds, \end{aligned} \tag{3.12}$$

yields the expansion function in (1.4) for the two-dimensional case.

An important point to note for the bivariate case is the fact that $\mu_+(C_x \cap D_y)$ and $\mu_0(C_x \cap D_y)$ cannot be singular with respect to each other. Actually, since $\mu_0(C_x \cap D_y) = 0$ if and only if $C_x = \varnothing$ or $D_y = \varnothing$ or both, then, for $C_x, D_y \in \mathfrak{B}$, implies that $\mu_+(C_x \cap D_y) = 0$ as well. Thus, we cannot have, $\forall C_x, D_y \in \mathfrak{B}$, $\mu_+(C_x \cap D_y) \perp \mu_0(C_x \cap D_y)$ unless of course $\mathfrak{X} = \varnothing$. In practice, this means that $\mu_+(E) = \mu_+^a(E)$, i.e., $\mu_+^s(F) = 0,\ \forall E \in \beta(\mathfrak{M})$ *and* $\forall F \in \mathfrak{B}$. In other words, the superscript a of μ_+^a is unrequired in the bivariate case.

On the other hand, for the discussion to follow and for future reference, it should be recalled that independence is defined in terms of

$$\mu_+(C_x \cap D_y) = \mu_0(C_x \cap D_y) \quad or \quad F(x,y) = F_0(x,y),\ \forall (x,y) \in \mathfrak{X},$$

such that negation of independence refers to dependence; negative complete dependence on the other hand corresponds to Fréchet lower bound, i.e.,

$$\mu_+(C_x \cap D_y) = \max\{\mu_X(C_x) + \mu_Y(D_y) - 1, 0\},\ or$$

$$F(x,y) = \max\{F_X(x) + F_Y(y) - 1, 0\},\ \forall (x,y) \in \mathfrak{X};$$

whereas positive complete dependence applies to Fréchet upper bound, i.e.,

$$\mu_+(C_x \cap D_y) = \min\{\mu_X(C_x), \mu_Y(D_y)\},\ \forall (x,y) \in \mathfrak{X},$$

which, in terms of distributions, can alternatively be re-expressed as

$$F(x,y) = \min\{F_X(x), F_Y(y)\},\ \forall (x,y) \in \mathfrak{X}.$$

As will be noted below, the Fréchet bounds for bivariate measures (distributions) are not necessarily identical with the respective lower and upper bounds for concentration and expansion functions. For practical reasons, this constitutes a drawback in detecting the phenomenon of complete dependence through these functions.

Returning to the comment made in *Remark 3* above, for some $t \in (0,1)$, we define

$$A_t^c = \{(x,y) \in \mathfrak{M} : \ell(x,y) > c_t\}$$

$$C_t^c = \{x \in \mathfrak{M} : \ell(x,y) > c_t, y \in \mathbb{R}_1\}$$

$$D_t^c = \{y \in \mathfrak{M} : \ell(x,y) > c_t, x \in \mathbb{R}_1\}$$

in $\mathfrak{B}$ with $A_t^c = C_t^c \cap D_t^c$. The constant value ℓ_t^+ of the generalized likelihood ratio L pointed out in *Remark 3* will correspond to

$$\ell_t^+ = \frac{1}{\mu_X(C_t^c) \cdot \mu_Y(D_t^c)},$$

such that

$$\frac{1}{\max\{\mu_X(C_t^c), \mu_Y(D_t^c)\}} \le \frac{1}{\mu_X(C_t^c) \cdot \mu_Y(D_t^c)} = \ell_t^+,\ \forall t \in (0,1)$$

in the two-dimensional case. Set

$$\varphi^+(1-t) = [\mu_+(A_t^c)]^+ = \ell_t^+ \cdot \iint_{A_t^c} d\mu_X d\mu_Y = \frac{1}{\mu_X(C_t^c)\cdot\mu_Y(D_t^c)} \iint_{A_t^c} d\mu_X d\mu_Y,$$

which, by hypothesis, is equal to 1, $\forall t \in (0,1)$. On the other hand, the Fréchet upper bound is

$$\begin{aligned}\overline{\varphi}(1-t) = \overline{\mu}_+(A_t^c) &= \min\left\{\mu_X(C_t^c)\cdot\mu_Y(D_t^c)\right\} \\ &= \frac{1}{\max\left\{\mu_X(C_t^c),\, \mu_Y(D_t^c)\right\}} \iint_{A_t^c} d\mu_X d\mu_Y \\ &\le \frac{1}{\mu_X(C_t^c)\cdot\mu_Y(D_t^c)} \iint_{A_t^c} d\mu_X d\mu_Y = [\mu_+(A_t^c)]^+ = \varphi^+(1-t) = 1,\end{aligned}$$

$\forall t \in (0,1)$. As such $\overline{\varphi}(1-t) < \varphi^+(1-t) = 1,\ \forall t \in (0,1)$ which means that there is no analytical necessity for the expansion function $\overline{\varphi}\ (1-t)$ of the upper Fréchet bound to reach the overall upper bound $\varphi^+(1-t) = 1$ for expansion functions, $\forall t \in (0,1)$.

For some $t \in (0,1)$. on the other hand, let us define the following sets in $\mathfrak{B}$

$$\begin{aligned}A_t &= \{(x,y) \in \mathfrak{M} : \ell(x,y) \le c_t\} \\ C_t &= \{x \in \mathfrak{M} : \ell(x,y) \le c_t, y \in \mathbb{R}_1\} \\ D_t &= \{y \in \mathfrak{M} : \ell(x,y) \le c_t, x \in \mathbb{R}_1\}\end{aligned}$$

with $A_t = C_t \cap D_t$. The lower bound for concentration functions is

$$\psi^-(t) = [\mu_+(A_t)]^- = \iint_{A_t} \ell d\mu_0 = 0,\ \mu_0(A_t) \neq 0,$$

$\forall t \in (0,1)$. For this to hold, we must have $\ell(x,y) = 0,\ \forall (x,y) \in \mathfrak{M}$. That is, $\mu_+(A_t) = 0$ or $\mu_+(A_t) = \alpha$ for some constant $0 < \alpha < 1,\ \forall t \in (0,1),$ so that

$$\ell = \frac{d\mu_+}{d\mu_0} = 0.$$

$\mu_+(A_t) = \alpha$ and $\ell = 0$ are however contradictory, and consequently we must have $\mu_+(A_t) = 0$. Now the Fréchet lower bound for $\mu_+(C_t \cap D_t)$ is however

$$\mu_+^*(C_t \cap D_t) = \max\{\mu_X(C_t) + \mu_Y(D_t) - 1, 0\}.$$

That is equal to zero everywhere in $\mathfrak{M}$, when

$$(1-\mu_X(C_t))(1-\mu_Y(D_t)) \ge \mu_X(C_t)\cdot\mu_Y(D_t),\ \forall (x,y) \in \mathfrak{M} \subset \mathfrak{X}.$$

Thus, a sufficient condition for $\ell(x,y) = 0,\ \forall (x,y) \in \mathfrak{M},$ is given by

$$\mu_+^*(C_t \cap D_t) = \max\{\mu_X(C_t) + \mu_Y(D_t) - 1, 0\} = 0,$$

$\forall (x,y) \in \mathfrak{M}$.

4. IMPLICATIONS FOR BIVARIATE DEPENDENCE

As is noted in earlier, concentration functions in (1.4) are appropriate for the case $\{\omega : \ell(\omega) < 1\}$ and expansion functions in (1.5) are in turn suitable for $\{\omega : 1 < \ell(\omega)\}$. These two respective cases can be matched with negative and positive dependencies in the bivariate setup: For two distinct random variables such as X and Y defined in (3.1) above, the concepts of negative and positive quadrant dependence are defined respectively as (c.f.,[3]), $\forall (x, y) \in \mathfrak{X}$

$$\mu_+(C_x \cap D_y) < \mu_0(C_x \cap D_y) = \mu_X(C_x) \cdot \mu_Y(D_y) \tag{4.1}$$

$$\mu_0(C_x \cap D_y) = \mu_X(C_x) \cdot \mu_Y(D_y) < \mu_+(C_x \cap D_y). \tag{4.2}$$

Obviously, the relations in (4.1)-(4.2) follow from the following respective relations of their derivatives:

$$d\mu_+ < d\mu_0$$

$$d\mu_0 < d\mu_+ .$$

Therefore, the corresponding generalized likelihood ratios become

$$\underline{\ell}(x, y) < 1 = \ell_0(x, y), \tag{4.3}$$

$$\ell_0(x, y) = 1 < \overline{\ell}(x, y), \tag{4.4}$$

$\forall (x, y) \in \mathfrak{X}$, where $\ell_0(x, y) = 1$ corresponds to the independence case with $\underline{\ell}(x, y)$ *and* $\overline{\ell}(x, y)$, standing for the respective ratios for (4.1) and (4.2). Thus, for $A_t = \{(x, y) \in \mathfrak{X} : \ell(x, y) \le c_t = \ell_0\} \in \mathfrak{B}$ and for the respective two cases (4.3) and (4.4), we have

$$\psi(t) = \mu_+^a(A_t) = \iint_{A_t} \underline{\ell} d\mu_0 < \iint_{A_t} \ell_0 d\mu_0 = \mu_0(A_t) = \psi_0(t) = t \tag{4.5}$$

$$1 - t = \varphi_0(1 - t) = \mu_0(A_t^c) = \iint_{A_t^c} \ell_0 \, d\mu_0 < \iint_{A_t^c} \overline{\ell} \, d\mu_0 = \mu_+^a(A_t^c) = \varphi(1 - t). \tag{4.6}$$

Application of concentration functions to positive quadrant dependence will thus result in an analytical inconsistency like

$$\mu_+^a(A_t) = \iint_{A_t^c} \overline{\ell} \, d\mu_0 < \iint_{A_t^c} \ell_0 \, d\mu_0 = \mu_0(A_t), \tag{4.7}$$

for $A_t^c = \{(x, y) \in \mathfrak{X} : \ell(x, y) > c_t = \ell_0\} \in \mathfrak{B}$. Such inconsistencies are often come across in literature dealing with positive dependence ordering.

To sum up the foregoing, we have:

Lemma 3. For all sets such as $A_t = \{(x, y) \in \mathfrak{X} : \ell(x, y) \le c_t = \ell_0\} \in \mathfrak{B}$ and for every nonnegative real t in [0,1],

$$0 \le \psi(t) \le t,$$

indicates that X and Y display negative quadrant dependence, such that, in the absence of singularity of the relevant measures, $\psi(t)=0$ is sufficient for a negative complete dependence and that $\psi(t)=t$ stands for independence of X and Y. Dually, for every $A_t^c=\{(x,y)\in\mathfrak{X}:\ell(x,y)>c_t=\ell_0\}\in\mathfrak{B}$ and for every positive $t\in I$,

$$1-t\le\varphi(1-t)\le 1$$

implies that X and Y have a positive quadrant dependence, such that $\varphi(1-t)=1-t$ stands for independence of X and Y, and $\varphi(1-t)=1$ corresponds to the case mentioned in Remark 3 above.

Corollary. Let ψ_i and ψ_j be two concentration functions for the respective distinct measures $\mu_{(i)+}$ *and* $\mu_{(j)+}$ (resp., distributions F_i and F_j) in a family $((\mathfrak{X},\mathfrak{B},\mu_0,\mu_{(\zeta)+}),\zeta\in J)$ of bi-probability spaces (resp., Fréchet class of bivariate distributions with a given pair of marginals). Also, assume that φ_i *and* φ_j are the corresponding expansion functions for these measures (resp., distributions). If $\forall t\in(0,1),\ \psi_i(t)\le\psi_j(t)$ and/or $\varphi_j(1-t)\le\varphi_i(1-t)$ and if, $\exists t\in(0,1),\ \psi_i(t)<\psi_j(t)$ and/or $\varphi_j(1-t)<\varphi_i(1-t)$ then $\mu_{(i)+}$ (resp., F_i) can be said to display a higher-order dependence than $\mu_{(j)+}$ (resp., F_j).

Remark 6. Note that the foregoing discussions are valid for sets $A_t=\{(x,y)\in\mathfrak{X}:\ell(x,y)\le c_t=\ell_0\}\in\mathfrak{B},\ \forall t\in I$. For sets E_t in $\mathfrak{B}$ such as

$$\begin{aligned}E_t&=\{(x,y)\in\mathfrak{X}:\ell(x,y)\le c_t\},\ t\in(0,1)\\&=\{(x,y)\in\mathfrak{X}:\ell(x,y)\le\ell_0\}\cup\{(x,y)\in\mathfrak{X}:\ell(x,y)>\ell_0\}\\&=F_t\cup(E_t-F_t),\end{aligned}$$

however a convex combination of concentration and expansion functions can be suggested, i.e.,

$$\phi(t)=\alpha_\psi\cdot\psi(t)+\alpha_\varphi\cdot\varphi(1-t)$$

where $\alpha_\psi=\mu_0(F_t\mid E_t)$ and $\alpha_\varphi=\mu_0(E_t-F_t\mid E_t)=1-\alpha_\psi$ provided that the latter are known.

Using both of concentration and expansion functions at the same time, the interval $\mathbb{I}=[0,1]$ can thus be reduced by its half, i.e., $\left[0,\frac{1}{2}\right]$. The inequalities in (4.5) and (4.6) indicate on the other hand that, given some $t\in\left[0,\frac{1}{2}\right]$, neither concentration nor expansions functions are appropriate to discriminate μ_+ $(resp.,F(x,y))$ against μ_0 $(resp.,F_0(x,y))$ in the interval $(t,1-t)$. This setback however can obviously be overcome by scanning all values of $t\in\left[0,\frac{1}{2}\right]$, which actually is the optimal way to screen out such a detection. Since the larger t is the smaller the interval $(t,1-t)$ will

be, checking large t-values may be tempting for practical reasons. Obviously, for $t = \frac{1}{2}$ this interval is zero.

Apart from such practical problems, there does not seem to exist any problem for detecting dependence in terms of concentration and expansion functions. We have to keep in mind however the fact that concentration and expansion functions are designed merely to distinguish μ_+ $(resp., F(x,y))$ from μ_0 $(resp., F_0(x,y))$ for all $(x,y) \in \mathfrak{X}$. And this is accomplished for those $\mu_{(\zeta)+}$ $(resp., F(x,y))$ that belong to the family $((\mathfrak{X}, \mathfrak{B}, \mu_0, \mu_{(\zeta)+}), \zeta \in J)$ (resp., the so-called Fréchet Class of joint distributions) with some given pair of marginals. How much information on dependence and independence that $F(x,y)$ and $F_0(x,y)$ can contain is completely another problem, as will be observed in the following extremal, but illuminating, case.

Illustration: Consider the Bernoulli case where the random variables $x = 0, 1$ and $y = 1 - x = 0,1$ are completely and negatively dependent on each other with correlation $\rho(X,Y) = -1$. The probability functions of $F(x,y)$ and $F_0(x,y)$ are respectively $f(x,y) = p^x(1-p)^y$ and $f_0(x,y) = p^x(1-p)^{1-x}p^{1-y}(1-p)^y = p^{1+x-y}(1-p)^{1+y-x}$ with $F_0(1,1) = (p+q)^2 = 1$. Clearly,

$$F(0,0) = 0,\ F(0,1) = q,\ F(1,0) = p \ \ and \ \ F(1,1) = 1$$

$$F_0(0,0) = pq,\ F_0(0,1) = q,\ F_0(1,0) = p \ \ and \ \ F_0(1,1) = 1$$

Thus $F(x,y) \le F_0(x,y),\ \forall (x,y) \in \mathfrak{X}$. Consequently, the corresponding concentration function must lie in $[0,t],\ \forall t \in \mathcal{I}$ (c.f., *Corollary (vi) to Lemmas 1 and 2*).

Straightforward computations however show that concentration values are $\psi(0) = 0, \psi(q) = q, \psi(p) = p \ \ and \ \ \psi(1) = 1,$ say, for $q < p$. For this case, the concentration function $\psi(t)$ is capable of detecting the artificial discrepancy between $F(x,y)$ and $F_0(x,y)$ In fact, though they are different functionally, as far as dependence of X and Y is concerned, $F(x,y)$ and $F_0(x,y)$ are inherently not different from each other, because $F_0(x,y)$ is a product of the marginal distributions of two random variables assumed initially and intentionally to be completely dependent. Thus, the concentration function $\psi(t)$ has in this instance been able to detect whether, $\forall (x,y) \in \mathfrak{X}$, $F(x,y)$ and $F_0(x,y)$ display the same structure of dependence or not.

5. CONCLUDING REMARKS

We end the foregoing discussions on concentration and expansion functions by re-emphasizing the following:

(i) If they are to be used for statistical applications on bivariate dependence, concentration and expansion functions must be both be used, because concentration is seemingly appropriate for negative dependence, and expansion appears on the contrary to be suitable for positive dependence. These functions are nonetheless incapable of detecting complete dependences, especially positive complete dependence.

(ii) Clearly, the concepts of concentration and expansion are intrinsically not identical with the concept of dependence bearing on two random variables. The former relate to detection of continuity or singularity of two probability measures. The latter bears on the question whether joint probabilities (distributions) are formed by the product of their marginal measures (distributions) or not. Therefore, care should be taken in applications of concentration and expansion to dependence.

(iii) Since both the marginals and the joint measures are defined on a common measurable space, joint measures (distributions) cannot be singular with respect to the product of their marginals (marginal distributions) everywhere in the relevant space, unless we are confronted with an empty probability space.

(iv) The foregoing exposition comprises some initial results of an ongoing research on the topic. The authors intend to present full results in a separate paper in near future.

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