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HOMOLOGY ACTION ON REGULAR HYPERMAPS OF GENUS 2*

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ABSTRACT

In this paper, we find the homology representations and characters of the orientation-preserving automorphism groups of regular hypermaps of genus 2.

KEYWORDS

Homology Representation, Homology Character, Regular Map, Regular Hypermap.

1. INTRODUCTION

A classification of regular hypermaps of genus 2 have been completed by Azevedo and Jones in [2]. They have shown that there are 43 of them, of which 10 are maps. Another 20 regular hypermaps can be obtained from these regular maps. The remaining 13 regular hypermaps are not associates of these maps: they are 5 regular hypermaps and their associates.

It follows from the result obtained by Singerman in [22] that there are three possible lattices of triangle groups corresponding to genus 2 hypermaps, so that the orientation-preserving automorphism groups Aut H, one for each of the three Riemann surfaces which admit regular hypermap of genus 2. Thus we consider three lattices separately in our calculations. If we choose a homology basis on the hypermap we are let to a matrix representation ρ_1 under the action of the automorphism group on this homology basis. Then we can find corresponding homology character τ_1 of G which is faithful character.

In [2] Azevedo and Jones showed that all hypermaps of genus 2 are reflexible, that is, each has an additional orientation-reversing automorphism α. This allows us to choose a reflection line on the hypermap, so we obtain a matrix representing α with respect to a chosen homology basis on the hypermap. One can use these

* This work is originated from the Author's Ph.D. thesis.

representations and characters to work on finite abelian coverings and reflexibility of regular hypermaps of genus 2.

2. REGULAR HYPERMAPS OF GENUS 2

In this chapter, we will give a brief summary of the regular hypermaps of genus 2. The regular hypermaps on the sphere and torus have been determined by Corn and Singerman in [5]; in each case there are infinitely many regular hypermaps. On the other hand, it is known that the number of regular hypermaps of genus $g \geq 2$ is finite [15]. Moreover, all regular hypermaps H of genus 2 are known. If H is a map then the possibilities for its type $\{ m, n \}$ and automorphism group Aut H are given by Coxeter and Moser in [6, p.140]. If H is not a map then the possibilities for its type $\{ l, m, n \}$ and automorphism group Aut H are given by Corn and Singerman in [5]. In fact, Azevedo and Jones in [2], have completed full results by enumerating, describing and constructing all these hypermaps and specifying their full automorphism groups $\mathrm{Aut}^{*}\,\mathrm{H}$ (including orientation-reversing automorphisms).

Table 1 shows the regular maps M_0, M_1, M_2, M_3, M_4, M_5 of genus 2 together with the duals of M_1, M_3, M_4 and M_5 which are denoted by $\mathrm{M}_j^{(02)}$, $j = 1,3,4,5$ (to indicate a transposition of vertices and faces) [2].

The entries in each of the six rows are explained as follows: the first two columns give our notation for M and that in [6, Table 9]. The third column gives the type $\{ l, m, n \}$ of M as a hypermap. The next column gives the number σ of non-isomorphic associates M^{π} of M ($\pi \in S_3$, Machi's group of hypermap operations which transforms one hypermap H to another, called an associate H^{π} of H). The fifth column gives the number n_i ($i = 0,1,2$) of hypervertices, hyperedges and hyperfaces, respectively. The sixth column describes the orientation-preserving automorphism group Aut M, and the final column gives the full automorphism group $\mathrm{Aut}^{*}\,\mathrm{M}$ of order $2\,|\mathrm{Aut}\,\mathrm{M}\,|$, since each hypermap M is reflexible.

Table 1: The regular maps of genus 2.

	Map	Type	σ	n_0 n_1 n_2	Aut M	Aut* M
M_0	$\{8, 8\}_{1,0}$	8 2 8	3	1 4 1	C_8	D_8
M_1	$\{10, 5\}_2$	5 2 10	6	2 5 1	C_{10}	D_{10}
$M_1(02)$	$\{5, 10\}_2$	10 2 5		1 5 2		
M_2	$\{6, 6\}_2$	6 2 6	3	2 6 2	$C_6 \times C_2$	$D_6 \times C_2$
M_3	$\{8, 4\}_{1,1}$	4 2 8	6	4 8 2	<-2, 4 | 2>	$Hol(C_8)$
$M_3(02)$	$\{4, 8\}_{1,1}$	8 2 4		2 8 4		
M_4	$\{6, 4 \mid 2\}$	4 2 6	6	6 12 4	(4, 6 >2,2)	$D_3 \times D_4$
$M_4(02)$	$\{4, 6 \mid 2\}$	6 2 4		4 12 6		
M_5	{4+4, 3}	3 2 8	6	16 24 6	$GL_2(3)$	$GL_2(3)\square C_2$
$M_5(02)$	{3, 4+4}	8 2 3		6 24 16		

The regular maps M_0, M_1, M_2, M_3, M_4, M_5 are illustrated in Figure 1.

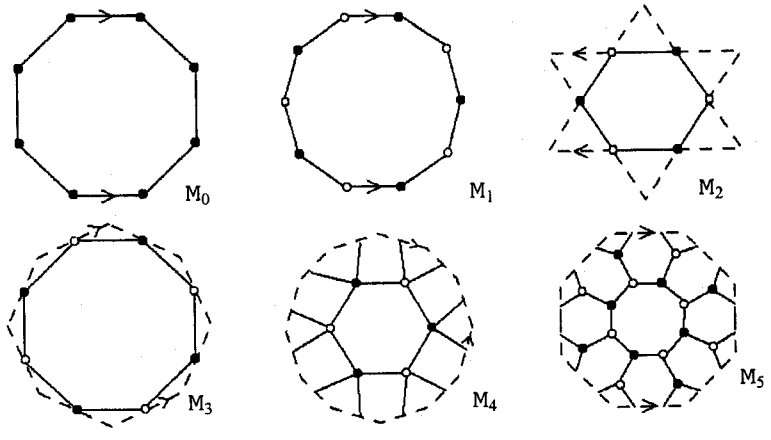

Figure 1: The regular maps of genus 2.

In each case we have indicated a pair of sides to be identified; the remaining identifications can be deduce by symmetry about the centre, since the maps are regular. For a more detailed account of these hypermaps, their orientations–preserving automorphism and full automorphism groups, see [2].

The remaining 13 regular hypermaps are described in Table 2, each row describes a representative $H_r(r = 1,\ldots,5)$ of an S_3-orbits of lengths σ.

Hypermap	Type	σ	$n_0\ n_1\ n_2$	Aut H	Aut* H
H_1	5 5 5	3	1 1 1	C_5	D_5
H_2	6 6 3	3	1 1 2	C_6	D_6
H_3	4 4 4	1	2 2 2	Q_8	$Q_8 \Upsilon C_4$
H_4	4 4 3	3	3 3 4	$\hat{D}_3$	(4,6 | 2,2)
H_5	3 3 4	3	8 8 6	$SL_2(3)$	$GL_2(3)$

Table 2: The regular hypermaps of genus 2.

We note that in the table, $Q_8 \Upsilon C_4$ shows a central product of Q_8 by C_4. The hypermaps $H_1, \ldots, H_5$ are illustrated in the following figure, Figure 2.

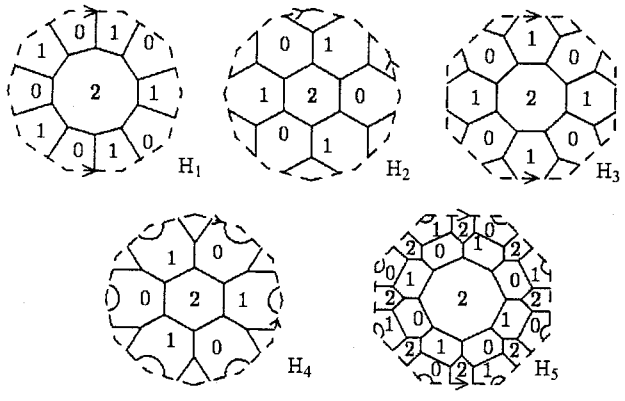

Figure 2: The regular hypermaps of genus 2.

Here we use the James model [13] of a hypermap: this is a trivalent graph imbedded in a surface, with hypervertices, hyperedges and hyperfaces represented by the regions labelled 0, 1 and 2 respectively. By contracting each edge separating

hypervertices and hyperedges to a point, we obtain the Cori model [5] of a topological hypermap.

3. INCLUSIONS OF AUTOMORPHISM GROUPS OF GENUS 2

In [22] Singerman gave all possible inclusions $\Delta_0 \leq \Delta_1$ between pairs of triangle groups. If Δ_0 is normal in Δ_1, then Table 3 gives all possible inclusions between hyperbolic triangle groups (they can arise for genus 2 regular hypermaps).

Δ_1	Δ_0	$[\Delta_1 : \Delta_0]$
$\Delta(3, 3, t)$, $t \geq 4$	$\Delta(t, t, t)$	3
$\Delta(2, 3, 2t)$, $t \geq 4$	$\Delta(t, t, t)$	6
$\Delta(2, t, 2u)$, $t \geq 4$, $t+u \geq 7$	$\Delta(t, t, u)$	2

Table 3: $\Delta_0 \trianglelefteq \Delta_1$.

If Δ_0 is not normal in Δ_1, then it follows from the results of Singerman that there are 3 cases which can arise for genus 2 regular hypermaps, these cases are given in Table 4:

Δ_1	Δ_0	$[\Delta_1 : \Delta_0]$
$\Delta(2, 3, 4n)$, $n \geq 2$	$\Delta(n, 4n, 4n)$	6
$\Delta(2, 4, 2n)$, $n \geq 3$	$\Delta(n, 2n, 2n)$	4
$\Delta(2, 3, 2n)$, $n \geq 4$	$\Delta(2, n, 2n)$	3

Table 4: $\Delta_0 \leq \Delta_1$.

Now if $\Delta_0 \leq \Delta_1$ then every subgroup $M \leq \Delta_0$ is also in Δ_1, so every hypermap of type $\{l_0, m_0, n_0\}$ gives rise to another hypermap of type $\{l_0, m_0, n_0\}$ of the same genus (since M is the same group). Moreover, if $M \trianglelefteq \Delta_1$ (so $M \trianglelefteq \Delta_0$) then we obtain all possible inclusions between pairs of automorphism groups,

$$\text{Aut } H_0 = \Delta_0 / M \leq \Delta_1 / M = \text{Aut } H_1.$$

Thus from tables 1, 2, 3 and 4 we deduce all possible inclusions between triangle groups corresponding to genus 2 hypermaps:

$\Delta(2,8,8) \trianglelefteq \Delta(2,4,8) \leq \Delta(2,3,8)$, $\Delta(4,4,4) \trianglelefteq \Delta(2,4,8) \leq (2,3,8)$,

$\Delta(4,4,4) \trianglelefteq \Delta(3,3,4) \trianglelefteq \Delta(2,3,8)$;

$\Delta(3,3,6) \trianglelefteq \Delta(2,6,6) \trianglelefteq \Delta(2,4,6)$, $\Delta(4,4,3) \trianglelefteq \Delta(2,4,6)$;

$\Delta(5,5,5) \trianglelefteq \Delta(2,5,10)$,

so we get three possible lattices of triangle groups, and hence of automorphism groups of genus 2, namely;

I II III

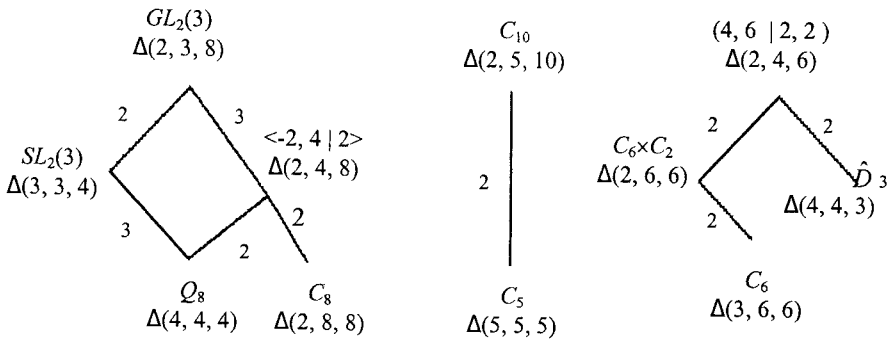

Figure 3: Lattices of Automorphism Groups and Triangle Groups.

These groups have the following relationships:

LATTICE I:

(I.1). $C_8 = < c \mid c^8 = 1 >$, where $c = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ *in* $GL_2(3)$.

(I.2). $Q_8 = < k, l \mid k^4 = 1,\ k^2 = l^2,\ klk = l >$, where $k = c^2 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$, $l = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

(I.3). $< -2,4 \mid 2 > = < g, h \mid g^8 = h^2 = 1,\ hgh = g^3 >$, $g = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $h = c^{-1} l = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$.

(I.4). $GL_2(3) = < a, b, c \mid a^2 = b^3 = c^8 = abc = (ac^4)^2 = 1 >$, where $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $c = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$.

(I.5). $SL_2(3) = < d, e \mid d^3 = e^4 = 1,\ ed^2e = ded >$, where

$d = (ac^2)^3 c^5 = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$, $e = c^2 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$.

LATTICE II:

(II.1). $C_5 = < a \mid a^5 = 1 >$.

(II.2). $C_{10} = < b \mid b^{10} = 1 >$, where $b^2 = a$.

LATTICE III:

(III.1). $C_6 = < a \mid a^6 = 1 >$.

(III.2). $C_6 \times C_2 = < a, b \mid a^6 = b^2 = 1,\ ab = ba >$.

(III.3). $(4,6 \mid 2,2) = < a, d \mid a^6 = d^4 = (da)^2 = (d^{-1}a)^2 = 1 >$.

(III.4). $\hat{D}_3 = < c, d \mid c^6 = 1,\ d^2 = c^3,\ d^{-1}cd = c^5 >$, where $c = a^2 d^2$ with $c^2 = b$.

4. HOMOLOGY REPRESENTATIONS AND CHARACTERS

We will find the representations ρ_1 and characters τ_1 on the first integer homology group $H_1(S, \square)$ of the orientation-preserving automorphism groups G of regular hypermaps of genus 2.

If H is a regular hypermap of type $\{k_0, k_1, k_2\}$ with hypermap subgroup $N \trianglelefteq \Delta = \Delta(k_0, k_1, k_2)$, then $G =$ Aut $\mathrm{H} \cong \Delta / N$ [5]. The natural action of G on the first integer homology group $H_1 = H_1(S, \mathbf{Z}) = N / N' \cong \mathbf{Z}^{2g}$ corresponds to the action (induced by conjugation) of Δ / N on N / N'. Thus we have a representation of G on the first homology group, i.e. we simply choose a homology basis on the hypermap and look at the action of G on this basis (or one can use the Reidemeister-Schreier method [14] to find a presentation for N, then work out the matrix representation of G on $H_1(S, \square)$). On the other hand, the homology character τ_1 is equal to

$$\tau_1(x) = \begin{cases} 2 - \phi(x), & \text{if } x \neq 1, \\ 2g, & \text{if } x = 1, \end{cases} \tag{1}$$

where $\phi(x)$ is the total number of hypervertices, hyperedges and hyperfaces invariant under x [16]. Let $\Delta(k_0, k_1, k_2)$ be the triangle group corresponding to a hypermap of type $\{k_0, k_1, k_2\}$, then in [18] Macbeath proved that

$$\phi(x) = \left| N_G(< x >) \right| \sum_{i=0}^{2} \frac{\varepsilon_i(x)}{k_i}$$

for $x \neq 1$, where $\varepsilon_i(x) = 1$ or 0 as x is not conjugate to a power of one of the generators x_i of G. Moreover, he also showed that if G is a cyclic group of order n and x has order d, then

$$\phi(x) = n \sum_{d \backslash k_i} k_i^{-1} \tag{2}$$

Furthermore, $\tau_1 = d_1\chi_1 + ... + d_k\chi_k$ for some non-negative integers $d_1,...,d_k$, where $\chi_1,...,\chi_k$ are the irreducible complex characters of G and

$$d_i = <\tau_1, \chi_1> = \frac{1}{|G|}\sum_{x\in G} \tau_1(x)\overline{\chi_i(x)},$$

the multiplicity of each irreducible character χ_1 of G in τ_1. For more details about the action on the first homology group, we refer to [1], [3], [4], [8], [16], [18], [19], [20].

From the previous chapter we have three lattices of automorphism groups of regular hypermaps of genus 2, so we will consider these lattices.

LATTICE I:

(I.1). Let $G = < c \mid c^8 = 1 >$ be the cyclic group of order 8 (the automorphism group of the map M_0), where c is the rotation in the anticlockwise direction by $2\pi/8$ about the central face. If we choose a homology basis e_1, e_2, e_3, e_4 on the map, see Figure 4, then c sends e_1 to $-e_4$, e_2 to e_1, e_3 to e_2, and e_4 to e_3.

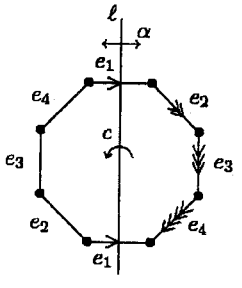

Figure 4

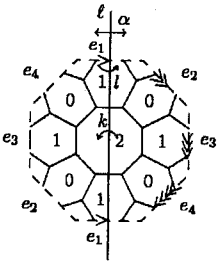

Figure 5

Thus we get the following matrix as a representation of G on the homology group $H_1(S, \mathbb{Z}) \cong \mathbb{Z}^4$:

$$\rho_1 : c \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = C \in GL_4(\mathbf{Z}).$$

If we apply the formula (2) to C_8, we see that $\phi(x) = 2,2,6$ as $x \in C_8$ has order 8, 4, 2 (for example, the vertex and face of M_0 are invariant under the elements

$x = c, c^3, c^5$ and c^7 of order 8, so $\phi(x) = 2$). Then (1) gives the character-values $\tau_1(x) = 4,0,0,-4$ as $x \in C_8$ has order 1, 8, 4, 2 (as we can see from the trace tr (ρ_1) of ρ_1). Then we get $\tau_1 = \chi_1 + \chi_3 + \chi_5 + \chi_7$ with values in the cyclotomic field $Q(e^{\pi i/4})$, where $\chi_1, \chi_3, \chi_5, \chi_7$ are irreducible (faithful) complex characters of G(i.e. $\chi_j(c) = w^j, w = e^{\pi i/4}$).

On the other hand, if we choose a reflection line ℓ on the map, see Figure 4, we see that the matrix α_1 of the orientation-reversing automorphism α with respect to the chosen homology basis is

$$\alpha_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(I.2). Let $G = Q_8 = < k, l \mid k^4 = 1, k^2 l^2, klk = l >$ be the quaternion group of order 8 (the automorphism group of the hypermap H_3), where k and l are the rotations by $\pi/2$ about the central face and the other face, respectively. Hence the six elements of order 4 in Q_8 are the quarter-turns fixing the centres of the two hypervertices, hyperedges, and hyperfaces, respectively, so the element k^2 rotates each of these through a half-turn.

We choose a homology basis $\{e_1, e_2, e_3, e_4\}$ on H_3 (see Figure 5), then k sends $e_1 \mapsto -e_3$, $e_2 \mapsto -e_4$, $e_3 \mapsto e_1$, and $e_4 \mapsto e_2$; and l sends $e_1 \mapsto -e_2 - e_3 - e_4$, $e_2 \mapsto e_1 + e_2 + e_3$, $e_3 \mapsto -e_1 - e_2 - +e_4$, and $e_4 \mapsto e_1 - e_3 - e_4$.

Thus we obtain the following homology representation of G

$$\rho_1 : k \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = K; \quad l \mapsto \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} = L.$$

Furthermore, it is not difficult to see from Figure 5 that $\phi(x) = 2,6$ *as* $x \in Q_8$ has order 4, 2, respectively. Then (1) gives $\tau_1(x) = 4,0,-4$ as x has order 1, 4, 2, (also from the representation ρ_1), respectively. Thus we have $\tau_1 = 2\chi_5$, where χ_5 is an irreducible complex character of Q_8 given by $\chi_5(1) = 2$, $\chi_5(k^2) = -2$, $\chi_5(k) = \chi_5(l) = \chi_5(lk) = 0$. Here $1, k^2, k, l, kl$ are the representatives of conjugacy classes of G.

If we choose a reflection line ℓ on the hypermap (see Figure 5) then we get the same matrix α_1 with respect to chosen homology basis as in (I.1).

(I.3). Let $G =< -2,4 \mid 2 >=< \mathrm{g}, h \mid \mathrm{g}^8 = h^2 = 1,\ hgh = \mathrm{g}^3 >$, of order 16 (the orientation-preserving automorphism group of M_3), where g is a rotation through $2\pi/8$ about the central face and h is a rotation about the midpoint of an edge, see Figure 6.

As usual, we choose a homology basis on this map, then we find the action of the automorphism group on these basis elements. It follows immediately from Figure 6 that g sends $e_1 \mapsto -e_4$, $e_2 \mapsto e_1$, $e_3 \mapsto e_2$, and $e_4 \mapsto e_3$, and similarly h sends $e_1 \mapsto -e_1 + e_3 - e_4$, $e_2 \mapsto e - e_2 + e_3 - e_4$, $e_3 \mapsto e_1 - e_2 + e_3$, and $e_4 \mapsto e_1 - e_2 + e_4$.

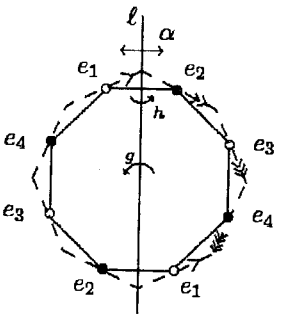

Figure 6

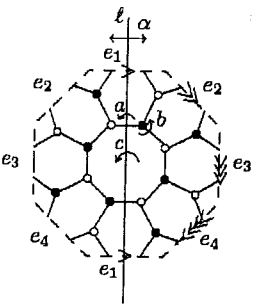

Figure 7

Then the homology representation ρ_1 of G follows:

$$\rho_1 : g \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = G \ ; h \mapsto \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} = H \in \mathrm{GL}_4(\square).$$

Moreover, the homology character is equal to $\tau_1 = \chi_6 + \chi_7$, where χ_6 and χ_7 are irreducible complex characters of G given by $\chi_6(1) = 2$, $\chi_6(\mathrm{g}^4) = -2$ $\chi_6(\mathrm{g}^2) = 0$, $\chi_6(\mathrm{g}) = \sqrt{-2}$, $\chi_6(\mathrm{g}^5) = -\sqrt{-2}$, $\chi_6(h) = 0$, $\chi_6(\mathrm{g}h) = 0$, and $\chi_7(1) = 2$, $\chi_7(\mathrm{g}^4) = -2$, $\chi_7(\mathrm{g}^2) = 0$, $\chi_7(\mathrm{g}) = -\sqrt{-2}$, $\chi_7(\mathrm{g}^5) = \sqrt{-2}$, $\chi_7(h) =$ $\chi_7(\mathrm{g}h) = 0$, where $1, \mathrm{g}^4, \mathrm{g}^2, \mathrm{g}, \mathrm{g}^5, h, \mathrm{g}h$ are the representatives of conjugacy classes of G.

On the other hand, if we choose a reflection line ℓ on the map, see Figure 6, we find that the matrix α_1 corresponding to this reflection line and the homology basis is

$$\alpha_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(I.4). Let $G = GL_2(3) = < a, b, c \mid a^2 = b^3 = c^8 = abc = (ac^4)^2 = 1 >$ of order 48, the automorphism group of M_5.

From Figure 7, we get the homology representation and character of G as follows:

$$\rho_1 : a \mapsto \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} = A ; \quad b \mapsto \begin{pmatrix} -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} = B ;$$

$$: c \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = C \in \mathrm{GL}_4(\square),$$

and $\tau_1 = \chi_7 + \chi_8$, where χ_7 and χ_8 are irreducible complex characters of G given by $\chi_7(1) = 2$, $\chi_7(c^4) = -2$, $\chi_7(c^2) = 0$, $\chi_7(c) = \sqrt{-2}$, $\chi_7(c^7) = -\sqrt{-2}$, $\chi_7(a) = 0$, $\chi_7(b) = -1$, $\chi_7(c^4b) = 1$, and $\chi_8(1) = 2$, $\chi_8(c^4) = -2$, $\chi_8(c^2) = 0$, $\chi_8(c) = -\sqrt{-2}$, $\chi_8(c^7) = \sqrt{-2}$, $\chi_8(a) = 0$, $\chi_8(b) = -1$, $\chi_8(c^4b) = -1$ (where $1, c^4, c^2, c, c^7, a, b, c^4b$ are the representatives of conjugacy classes of G).

Moreover, if we choose a reflection line ℓ on the map (see Figure 7), then it is not difficult to see that the matrix α_1 of the orientation-reversing automorphism α is the same as in (I.1).

(I.5) Let $G = SL_2(3) = < d, e \mid d^3 = e^4 = 1, ed^2e = ded >$ be the special linear group of order 24, the automorphism group of H_5, where $d = (ac^2)^3c^5$ and $e = c^2$. Then it is easily verified from Figure 8 that the homology representation of G is

$$\rho_1 : d \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = D; \quad e \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = E,$$

and then $\tau_1 = 2\chi_5$, where χ_5 is an irreducible complex character of G given by $\chi_5(1) = 2$, $\chi_5(e^2) = -2$, $\chi_5(e) = 0$, $\chi_5(d) = -1$, $\chi_5(d^2) = -1$, $\chi_5(d^2e^2) = 1$, $\chi_5(de^2) = 1$, (where $1, e^2, e, d, d^2, d^2e^2, de^2$ are the representatives of conjugacy classes of G).

It can be seen that the matrix α_1 of the orientation-reversing automorphism α corresponding to a chosen reflection line ℓ (see Figure 8) on the hypermap with respect to the homology basis is the same as in (I.3).

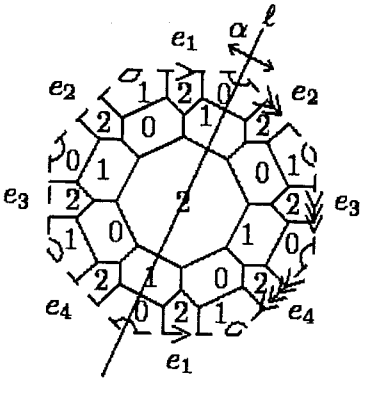

Figure 8

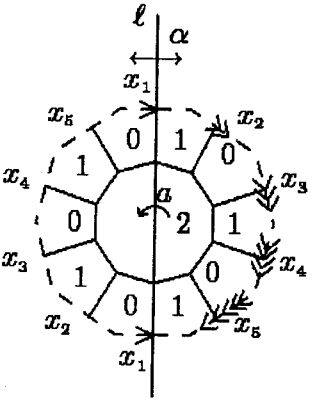

Figure 9

LATTICE II:

(II.1). Let $G = < a \mid a^5 = 1 >$ be the cyclic group of order 5, the automorphism group of the hypermap H_1, where a is a rotation through $2\pi/5$ about the centre.

Let $e_1 = x_1 + x_2$, $e_2 = -x_1 + x_5$, $e_3 = -x_4 - x_5$ and $e_4 = x_3 + x_4$ be a chosen homology basis on the hypermap, see Figure 9. Then we obtain the homology representation as

$$\rho_1 \; : \; a \; \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = A \in \mathrm{GL}_4(\square)$$

and then the homology character as $\tau_1 = \chi_1 + \chi_2 + \chi_3 + \chi_4$, where χ_1, χ_2, χ_3 and χ_4 are irreducible (faithful) complex characters of G.

Furthermore, if we choose a reflection line ℓ on the hypermap, see Figure 9, then we see that the matrix α_1 of the orientation-reversing automorphism α with respect to the homology basis is

$$\alpha_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(II.2). Let $G = < b \mid b^{10} = 1 >$ be the cyclic group of order 10, the automorphism group of the map M_1, where b is a rotation through $2\pi/10$ about the centre.

Let us choose a homology basis on the map, see Figure 10, such as $e_1 = x_1 + x_2$, $e_2 = -x_1 + x_5$, $e_3 = -x_4 - x_5$ and $e_4 = x_3 + x_4$. Then it is not difficult to see that the homology representation of G is

$$\rho_1 \; : \; b \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = B \in \mathrm{GL}_4(\square),$$

and $\tau_1 = \chi_1 + \chi_3 + \chi_7 + \chi_9$, where χ_1, χ_3, χ_7 and χ_9 are irreducible (all faithful) complex characters of G.

The matrix α_1 of the orientation-reversing automorphism α corresponding to the chosen reflection line ℓ with respect to the homology basis is the same as in (II.1).

LATTICE III:

(III.1). Let $G = C_6 = < a \mid a^6 = 1 >$ be the cyclic group of order 6, the automorphism group of the hypermap H_2, where a is a rotation through $2\pi/6$ about the central hyperface, see Figure 11.

Let us choose a homology basis on the hypermap such as $e_1 = x_1 + x_6$, $e_2 = x_1 - x_4$, $e_3 = -x_2 - x_5$, $e_4 = -x_2 - x_3$.

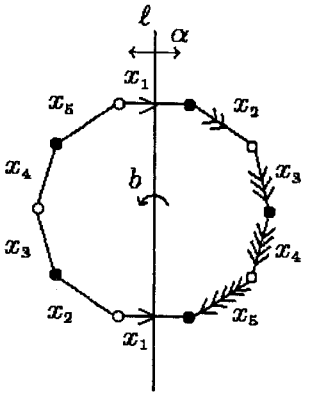

Figure 10

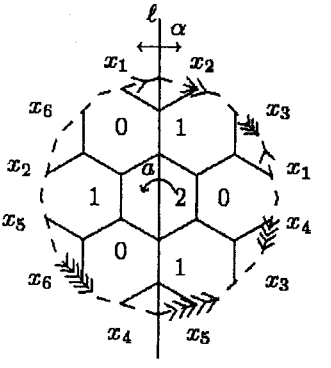

Figure 11

Then it follows that

$$\rho_1 : a \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = A \in \mathrm{GL}_4(\square),$$

and

$$\tau_1 = \chi_1 + \chi_2 + \chi_4 + \chi_5,$$

where χ_1, χ_2, χ_4 and χ_5 are irreducible complex characters of G.

Moreover, the matrix α_1 of the orientation-reversing automorphism α with respect to the chosen reflection line ℓ and the homology basis is

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(III.2). Let $G = C_6 \times C_2 = < a, b \mid a^6 = b^2 = 1,\ ab = ba >$ be the abelian group of order 12, the automorphism group of the map M_2, where a is a rotation of Figure 12 by $2\pi/6$ and b is a half-turn reversing each edge and transposing the two faces and the vertices.

Let us choose a homology basis on the map as $e_1 = x_1 - x_2$, $e_2 = x_5 - x_2$, $e_3 = x_6 - x_3$, $e_4 = x_4 - x_3$, see Figure 12.

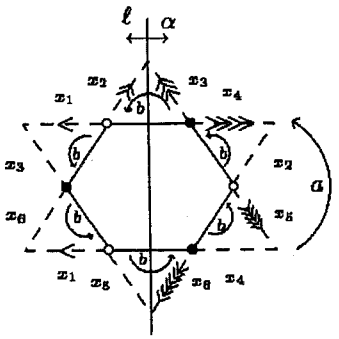

Figure 12

Then we get the following homology representation for G :

$$\rho_1 : a \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = A; \quad b \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = B ,$$

and then $\tau_1 = \chi_{1,1} + \chi_{2,1} + \chi_{4,1} + \chi_{5,1}$, where $\chi_{1,1}, \chi_{2,1}, \chi_{4,1}$ and $\chi_{5,1}$ are irreducible complex characters of G given by

g_i	:	1	a	a^2	a^3	a^4	a^5	b	ab	a^2b	a^3b	a^4b	a^5b
$\chi_{1,1}$	:	1	w	w^2	-1	w^4	w^5	-1	$-w$	w^2	1	$-w^4$	$-w^5$
$\chi_{2,1}$	:	1	w^2	w^4	1	w^2	w^4	-1	$-w^2$	$-w^4$	-1	$-w^2$	w^4
$\chi_{4,1}$	:	1	w^4	w^2	1	w^4	w^2	-1	$-w^4$	$-w^2$	-1	$-w^4$	$-w^2$
$\chi_{5,1}$	:	1	w^5	w^4	-1	w^2	w	-1	$-w^5$	$-w^4$	1	$-w^2$	$-w$

where $w = e^{\frac{2\pi i}{6}} = \dfrac{1+\sqrt{-3}}{2}$.

If we choose a reflection line ℓ, see Figure 12, it can be seen that the corresponding matrix α_1 is the same as in (III.1).

(III.3) Let $G = (4, 6 \mid 2, 2) = < a, d \mid a^6 = d^4 = (da)^2 = (d^{-1}a)^2 = 1 >$ of order 24, the automorphism group of the map M_4.

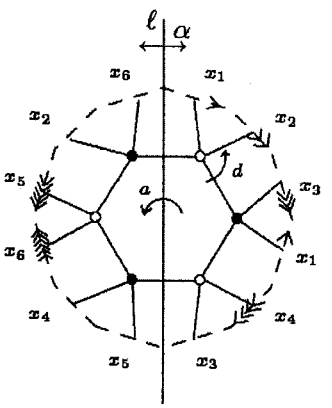

Figure 13

We choose a homology basis on the map as $e_1 = x_1 - x_2 - x_5 - x_6$, $e_2 = x_1 - x_2 - x_3 - x_4$, $e_3 = x_6 - x_2 - x_4 - x_5$ and $e_4 = x_6 - x_1 - x_2 - x_3$, see Figure 13.

Then the matrix representation ρ_1 of G with respect to the chosen basis is

$$\rho_1 : a \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = A ; \quad d \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = D ,$$

and $\tau_1 = \chi_6 + \chi_7$, where χ_6 and χ_7 are irreducible complex characters of G given by $\chi_6(1) = 2$, $\chi_6(b^2) = -2$, $\chi_6(a) = \sqrt{-3}$, $\chi_6(a^3) = 0$, $\chi_6(a^5) = -\sqrt{-3}$, $\chi_6(a^2) = -1$, $\chi_6(a^2b^2) = 1$, $\chi_6(b) = 0$, $\chi_6(ab) = 0$, and $\chi_7(1) = 2$, $\chi_7(b^2) = -2$, $\chi_7(a) = -\sqrt{-3}$, $\chi_7(a^3) = 0$, $\chi_7(a^5) = \sqrt{-3}$, $\chi_7(a^2) = -1$, $\chi_7(a^2b^2) = 1$, $\chi_7(b) = 0$, $\chi_7(ab) = 0$, where $1, d^2, a, a^3, a^5, a^2, a^2d^2, d, ad$ are the representatives of conjugacy classes of G.

If we choose a reflection line ℓ on the map, see Figure 13, we see that the matrix α_1 of the orientation-reversing automorphism α is the same as in case (III.1).

(III.4). Finally, let $G = \hat{D}_3 = < c, d \mid c^6 = 1, d^2 = c^3, d^{-1}cd = c^5 >$ be the binary dihedral group of order 12, the automorphism group of the hypermap H_4.

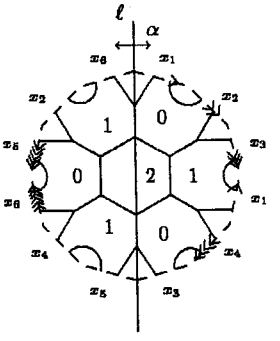

Figure 14

Since $\hat{D}_3 \trianglelefteq (4, 6 \mid 2, 2)$ with $c = a^2 d^2$ and $c^2 = b$, then the homology representation follows:

$$\rho_1 : c \mapsto \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = C; \quad d \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = D,$$

and $\tau_1 = 2\chi_5$ where χ_5 is an irreducible complex character of G given by, $\chi_5(1) = 2$, $\chi_5(c^3) = 2$, $\chi_5(c) = -1$, $\chi_5(c^2) = -1$, $\chi_5(d) = 0$, $\chi_5(cd) = 0$, where $1, c^3, c, c^2, d, cd$ are the representatives of conjugacy classes of G.

Let us choose a homology basis such as $e_1 = x_1 - x_2 - x_5 - x_6$, $e_2 = x_1 - x_2 - x_3 - x_4$ $e_3 = x_6 - x_2 - x_4 - x_5$, $e_4 = x_6 - x_1 - x_2 - x_3$, and a reflection line ℓ on the hypermap, see Figure 14. Then the matrix α_1 corresponding to this orientation-reversing automorphism α is the same as in case (III.1).

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