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# ON GENERALIZED NÖRLUND SUMMABILITY FACTORS OF INFINITE SERIES

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**ABSTRACT**

In this paper we have proved three theorems on generalized Nörlund summability factors of infinite series which generalizes various known results.

## 1. INTRODUCTION

1. Definitions and Notations: Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with $\{s_n\}$ its $n^{th}$ partial sum.

The $(N, p, \lambda)$ transform[1] of $s_n = \sum_{\nu=0}^{n} a_\nu$ is defined by

$$\tau_n = \frac{\sum_{\nu=0}^{n} p_{n-\nu} \lambda_\nu S_\nu}{r_n}$$

where

$$r_n = \sum_{\nu=0}^{n} p_n \lambda_{n-\nu} \quad (p_{-1} = \lambda_{-1} = r_{-1} = 0)$$

$$\neq 0 \quad \text{for} \quad n \geq 0 .$$

The series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be summable $(N, p, \lambda)$ to s,

[1] BORWEIN [3], this is called the generalized Nörlund transform.

if $\tau_n \to s$ as $n \to \infty$, and is said to be absolutely summable $(N, p, \lambda)$ if $\tau_n$ is of bounded variation and when this happens, we shall write symbolically $\{s_n\} \in |N, p, \lambda|$.

The method $(N, p, \lambda)$ reduces to the method $(N, p_n)$ when $\lambda_n = 1$ ([8], $p.64$); to the Euler Knopp method (E, $\delta$ ) when $P_n = \frac{\alpha_n \delta_n}{n!}$, $\lambda_n = \frac{\alpha_n}{n!}$ $(\alpha > 0, \delta > 0)$ ([8], p.178); to the method (C, $\alpha, \beta$ ) [2] when $p_n = \binom{n+\alpha-1}{\alpha}$, $\lambda_n = \binom{n+\beta}{\beta}$. We write

$$\varepsilon_n = p_n - p_{n-1} = \Delta p_n$$

$$\mu_n = q_n - q_{n-1} = \Delta q_n$$

and

$$\xi_n = \delta_n^{\,\alpha} \text{ furthermore } \delta_n = \sum_{v=0}^{n} \lambda_v .$$

We note that

$$r_n = \sum_{v=0}^{n} p_{n-v} \lambda_v = \sum_{v=0}^{n} \varepsilon_{n-v} \delta_v$$

and

$$\sum_{v=0}^{n} p_{n-v} \lambda_v s_v = \sum_{v=0}^{n} (p_{n-v} - p_{n-v-1}) \sum_{i=0}^{v} \lambda_i s_i$$

$$= \sum_{v=0}^{n} \varepsilon_{n-v} t_v s_v$$

where

$$t_v = \frac{1}{\delta_v} \sum_{i=0}^{v} \lambda_i s_i$$

$$= \frac{1}{\delta_v} \sum_{i=0}^{v} (\delta_i - \delta_{i-1}) a_i$$

Here $\{t_v\}$ is the $\left(\bar{N}, \lambda\right)$ mean ([8], $p.57$) which is equivalent to $(R^*, \delta_{n-1}, 1)$ mean ([8], $p.113$).

Rewriting $\tau_n$ in terms of the simplification, given above, we now have

$$\tau_n = \frac{\sum_{\nu=0}^{n}(p_{n-\nu}-p_{n-\nu-1})t_\nu\delta_\nu}{\sum_{\nu=0}^{n}(p_{n-\nu}-p_{n-\nu-1})\delta_\nu}$$

and this form suggests that we can have the following extension of the $(N, p, \lambda)$ method.

We now write, for any $\{\varepsilon_n\}$

$$\tau_n^{(a)} = \frac{\sum_{\nu=0}^{n}\varepsilon_{n-\nu}t_\nu\delta_\nu^\alpha}{\sum_{\nu=0}^{n}\varepsilon_{n-\nu}\delta_\nu^\alpha} = \frac{\sum_{\nu=0}^{n}\varepsilon_{n-\nu}t_\nu^\alpha\xi_\nu}{\sum_{\nu=0}^{n}\varepsilon_{n-\nu}\xi_\nu} \tag{1.1}$$

where

$$t_n^{(a)} = \frac{1}{\delta_n^a}\sum_{\nu=0}^{n}(\delta_\nu - \delta_{\nu-1})^\alpha a_\nu$$

we denote this mean by $G(N, p, \lambda)$ [7] when $\alpha = 1$.

$\tau_n^{(1)} = (N, p, \lambda)(s_n)$ the $G(N, p, \lambda)$ method reduces to $(N, p, \lambda)$ method.

We say that the $G(N, p, \lambda)$ method is applicable to the given infinite series $\sum a_n$, if

$$\mu_n \sum_{k=n}^{\infty}\frac{\varepsilon_{k-n}a_k}{r_k} \tag{1.2}$$

exists for each $n \geq 0$. If further, $\sum b_n = s$, then we say that $\sum a_n$ is summable by $G(N, p, \lambda)$ method to the sum s, and if $\sum|b_n| < \infty$ then $\sum a_n$ is said to be absolutely summable by $G|N, p, \lambda|$ method.

2. Concerning Nörlund summability factors of infinite series, DAS [6] has proved the following theorem:

THEOREM A: Let $\{p_n\} \in M$, $q_n \geq 0$, then if $\sum a_n$ is summable $|N, p, q|$, it is summable $|\bar{N}, q|$.

In 2000 SINGH AND SHARMA [11] extended the theorem of DAS to $|\bar{N}, q|$ summability. They established the following theorems:

THEOREM B: Let $\{p_n\} \in M$, $q_0 > 0$, $q_n \geq 0$, and let $\{q_n\}$ be monotonic non-increasing sequence for $n \geq 0$. The necessary and sufficient conditon that $\sum a_n \varepsilon_n$ Should be summable $|\bar{N}, q|$, whenever

$$\sum a_n = O(1)(N, p, q) \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\varepsilon_n| < \infty \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\Delta \varepsilon_n| < \infty \tag{2.3}$$

$$\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} |\Delta^2 \varepsilon_n| < \infty \tag{2.4}$$

is that

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |s_n| |\varepsilon_n| < \infty$$

THEOREM C: Let $\{p_n\} \in M$, $q_0 > 0$, $q_n \geq 0$ and let $\{q_n\}$ be monotonic non-increasing sequence for $n \geq 0$. The necessary and sufficient conditon that $\sum a_n \varepsilon_n$ should be summable $|\bar{N}, q|$, whenever

$$\sum a_n = O(\mu_n)(N, p, q) \tag{2.5}$$

where $\{\mu_n\}$ is positive and monotonic non-decreasing and $\{\varepsilon_n\}$ is such that

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\varepsilon_n| \mu_n < \infty \tag{2.6}$$

$$\sum_{n=0}^{\infty} |\Delta \varepsilon_n| \mu_n < \infty \tag{2.7}$$

$$\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} |\Delta^2 \varepsilon_n| \mu_n < \infty \tag{2.8}$$

is that

$$\sum_{n=0}^{\infty}\frac{q_n}{Q_n}|s_n||\varepsilon_n|<\infty \tag{2.9}$$

The object of the present paper is to prove the above theorems for generalized Nörlund summability.

3. We shall prove the following theorems:

THEOREM 1: Suppose hat $\varepsilon_n \in$ M and $\mu_n \neq 0$ $(n \geq 0)$. Then G(N*,p, $\lambda$) has an inverse transformation, whose matrix is given by the transpose of the inverse of G (N,p, $\lambda$), that is, if $b_n$ is given by transformation (1.2), then

$$a_n = r_n \sum_{k=n}^{\infty}\frac{b_k c_{k-n}}{\mu_k} \tag{3.1}$$

THEOREM 2: Suppose $\varepsilon_n \in$ M and $\mu_n \neq 0$ and that $|\mu_n|$ is non-decreasing. If $\sum a_n$ is summable G(N*,p, $\lambda$) to s, then

$$a_n = o\left(\frac{|r_n|}{|\mu_n|}\right)$$

If further $r_n \geq 0$, then

$$t_n = s + o\left(\frac{(\varepsilon * \mu)_n}{|\mu_n|}\right) \tag{3.2}$$

THEOREM 3: Suppose $\varepsilon_n \in \mathrm{M}, \mu_n$ is positive, $\{\mu_n\}$is non-decreasing and $\{\mu_n / r_n\}$ is non-increasing. Then if $\sum a_n$ is summable G $|N^*, p, \lambda|$, then

$$\left(\frac{\mu_n t_n}{r_n}\right) \in \mathrm{BV}$$

4. We need the following lemma for the proof of the theorems:

LEMMA: Let $\varepsilon_n \in$ M. Then

(i) $\sum_{n=0}^{\infty}|c_n| < \infty$ ,

(ii) $c_o > 0, c_n \leq 0 (n \geq 1)$

(iii) $\sum c_n \geq 0$,

(iv) $\sum c_n = 0$, if and only if $(\varepsilon * \mu)_n \to \infty$ as $n \to \infty$

The proof of the lemma appears in HARDY [8], Theorem 22.

5. Proof of the Theorem 1: We know from the identity:

$$\left(\sum c_n x^n\right)\left(\sum \varepsilon_n x^n\right) = 1$$

that

$$\sum_{n=0}^{k} \varepsilon_n c_{k-n} = \begin{cases} 1 & (k=0) \\ 0 & (k>0) \end{cases} \tag{5.1}$$

Hence

$$\sum_{k=n}^{N} c_{k-n} \varepsilon_{v-k} = -\sum_{k=N+1}^{v} c_{k-n} \varepsilon_{v-k} \; (v \rangle n). \tag{5.2}$$

Now for N > n and by (1.2) we have,

$$r_n \sum_{k=n}^{N} \frac{b_k c_{k-n}}{\mu_k} = r_n \sum_{k=n}^{N} \frac{c_{k-n}}{\mu_k} \mu_k \sum_{v=k}^{\infty} \frac{a_v \varepsilon_{k-v}}{r_v}$$

$$= r_n \sum_{k-n}^{N} c_{k-n} \left( \sum_{v=k}^{N} + \sum_{v=N+1}^{\infty} \right) \frac{a_v \varepsilon_{k-v}}{r_v}$$

$$= r_n \sum_{v=n}^{N} \frac{a_v}{r_v} \sum_{k=n}^{v} c_{k-n} \varepsilon_{v-k}$$

$$+ r_n \sum_{v=N+1}^{\infty} \frac{a_v}{r_v} \sum_{k=n}^{N} c_{k-n} \varepsilon_{v-k}$$

$$= a_n + r_n \sum_{v=N+1}^{\infty} \frac{a_v}{r_v} \sum_{k=n}^{N} c_{k-n} \varepsilon_{v-k}$$

by (5.1). Thus the necessary and sufficient condition for the validity of (3.1) is that, for each fixed n,

$$\sum_{\nu=N+1}^{\infty} \frac{a_\nu}{r_\nu} \sum_{k=n}^{N} c_{k-n} \varepsilon_{\nu-k} \to 0, \text{ as } N \to \infty .$$

which is the same thing as, for each fixed n,

$$\Phi_N = \sum_{\nu=N+1}^{\infty} \frac{a_\nu}{r_\nu} \sum_{k=N+1}^{\nu} c_{k-n} \varepsilon_{\nu-k} \to 0, \text{ as } N \to \infty, \tag{5.3}$$

in view of (5.2).

Let us write

$$b_0 = \mu_0 \sum_{k=0}^{\infty} \frac{\varepsilon_k a_k}{r_k} \tag{5.4}$$

$$w_\nu = \mu_0 \sum_{k=\nu}^{\infty} \frac{\varepsilon_k a_k}{r_n}$$

since G(N*,p, $\lambda$) method is applicable to $\sum a_n$, $b_0$ is finite and hence, $w_\nu$ is well defined and tends to zero as $\nu \to \infty$. Now from (5.4)

$$\frac{a_\nu}{r_\nu} = \frac{w_\nu - w_{\nu+1}}{\mu_0 \varepsilon_\nu}.$$

Hence

$$\Phi_N = \frac{1}{\mu_0} \sum_{\nu=N+1}^{\infty} \frac{w_\nu - w_{\nu+1}}{\mu_0 \varepsilon_\nu} \sum_{k=N+1}^{\nu} c_{k-n} \varepsilon_{\nu-k}$$

Now for M > N,

$$\frac{1}{\mu_0} \sum_{\nu=N+1}^{M} \frac{w_\nu - w_{\nu+1}}{\varepsilon_\nu} \sum_{k=N+1}^{\nu} c_{k-n} \varepsilon_{\nu-k} = \frac{1}{\mu_0} \sum_{\nu=N+1}^{M} w_\nu \left( \sum_{k=N+1}^{\nu} \frac{\varepsilon_{\nu-k} c_{k-\nu}}{\varepsilon_\nu} - \sum_{k=N+1}^{\nu-1} \frac{\varepsilon_{\nu-k-1} c_{k-n}}{\varepsilon_{\nu-1}} \right)$$
$$- \frac{1}{\mu_0} \frac{w_{M+1}}{\varepsilon_M} \sum_{k=N+1}^{M} \varepsilon_{M-k} c_{k-n}$$

Since $\varepsilon_n \in$ M (by lemma)

$$\left| \sum_{k=N+1}^{M} \varepsilon_{M-k} c_{k-n} \right| = O(1), \text{ as } M \to \infty ,$$

and by definition, $W_M = o\,(1)$, as $M \to \infty$.

We see that

$$\Phi_N = \frac{1}{\mu_0} \sum_{v=N+1}^{\infty} w_v \sum_{k=N+1}^{v} c_{k-n} \left( \frac{\varepsilon_{v-k}}{\varepsilon_v} - \frac{\varepsilon_{v-k-1}}{\varepsilon_{v-1}} \right).$$

Since $\{w_v\}$ is an arbitrary sequence tending to 0, hence (5.3) is valid, that is, $\Phi_N \to 0$ if and only if (See Hardy [8], Theorem 8) for fixed n,

$$J_N = \sum_{v=N+1}^{\infty} \sum_{k=N+1}^{v} \left( \frac{\varepsilon_{v-k}}{\varepsilon_v} - \frac{\varepsilon_{v-k-1}}{\varepsilon_{v-1}} \right) c_{k-n} = O(1)$$

as $N \to \infty$. But by virtue of (5.1)

$$\sum_{k=N+1}^{v} \left( \frac{\varepsilon_{v-k}}{\varepsilon_v} - \frac{\varepsilon_{v-k-1}}{\varepsilon_{v-1}} \right) c_{k-n} = -\sum_{k=n}^{N} \left( \frac{\varepsilon_{v-k}}{\varepsilon_v} - \frac{\varepsilon_{v-k-1}}{\varepsilon_{v-1}} \right) c_{k-n}$$

for $v > n$ and also

$$\frac{\varepsilon_{v-k}}{\varepsilon_v} - \frac{\varepsilon_{v-k-1}}{\varepsilon_{v-1}} \le 1, \text{ for } k \le (v-1).$$

Hence

$$J_N = \sum_{v=N+1}^{\infty} \sum_{k=n}^{N} \left( \frac{\varepsilon_{v-k}}{\varepsilon_v} - \frac{\varepsilon_{v-k-1}}{\varepsilon_{v-1}} \right) c_{k-n}$$

$$\le \sum_{v=N+1}^{\infty} c_o \, \frac{\varepsilon_{v-n}}{\varepsilon_v} - \frac{\varepsilon_{v-k-1}}{\varepsilon_{v-1}}$$

$$+ \sum_{v=N+1}^{\infty} \sum_{k=n+1}^{N} c_{k-n} \left( \frac{\varepsilon_{v-k}}{\varepsilon_v} - \frac{\varepsilon_{v-k-1}}{\varepsilon_{v-1}} \right)$$

$$= J_N^{(1)} + J_N^{(2)}, \text{ say.}$$

Since $\varepsilon_n \in M$, $\{\varepsilon_n / \varepsilon_{n+1}\}$ is non –increasing and so,

$$J_N^{(1)} = O(1), \text{ as } N \to \infty.$$

Since $\{\varepsilon_n / \varepsilon_{n+1}\} \ge 1$ and $\{\varepsilon_n / \varepsilon_{n+1}\}$ is non-increasing, it follows that, $\lim \{\varepsilon_n / \varepsilon_{n+1}\}$ exists and

$$A=\lim\frac{\varepsilon_n}{\varepsilon_{n+1}}\geq 1.$$

Hence

$$\sum_{v=N+1}^{\infty}\left(\frac{\varepsilon_{v-k}}{\varepsilon_v}-\frac{\varepsilon_{v-k-1}}{\varepsilon_{v-1}}\right)=\lim_{v\to\infty}\ \frac{\varepsilon_{v-k}}{\varepsilon_v}-\frac{\varepsilon_{N-k}}{\varepsilon_N}$$

$$=\lim_{v\to\infty}\left(\frac{\varepsilon_{v-k}}{\varepsilon_{v+1-k}}\frac{\varepsilon_{v+1-k}}{\varepsilon_{v+2-k}}\cdots\frac{\varepsilon_{v-1}}{\varepsilon_v}\right)-\frac{p_{n-k}}{p_N}$$

$$=A^k-\frac{p_{N+k}}{p_N}$$

therefore, by (5.1)

$$J_N^{(2)}=\sum_{k=n+1}^{N}c_{k-n}A^k-\sum_{k=n+1}^{N}c_{k-n}\frac{\varepsilon_{N-K}}{\varepsilon_N}$$

$$=\sum_{k=n+1}^{N}c_{k-n}A^k-\frac{1}{\varepsilon_N}\left(\sum_{k=n}^{N}c_{k-n}\varepsilon_{n-k}-c_0\varepsilon_{N-n}\right)$$

$$=\sum_{k=n+1}^{N}c_{k-n}A^k+c_0\frac{\varepsilon_{N-n}}{\varepsilon_N}$$

Since,

$$\sum_{k=n+1}^{N}c_{k-n}A^k\leq 0,$$

we get,

$$J\frac{(2)}{N}\leq\frac{c_0\varepsilon_{N-n}}{\varepsilon_n}$$

$$=O\,(1),\ \text{as}\ N\to\infty$$

This completes the proof of theorem 1.

6. Proof of the Theorem 2: Since $\sum a_n$ is G (N*,p, $\lambda$) summable, $\sum b_n$ is convergent and hence $b_n=o(1)$. By using the inversion formula as given in Theorem 1, we obtain, by using hypothesis,

$$|a_n|=\left|r_n\sum_{k=n}^{\infty}\frac{b_kc_{k-n}}{\mu_k}\right|$$

$$\leq \frac{|r_n|}{|\mu_n|} \sum_{k=n}^{\infty} |b_k c_{k-n}|$$

$$= \frac{|r_n|}{|\mu_n|} \sum_{k=n}^{\infty} o(1) |c_{k-n}|$$

$$= o\left( \frac{|r_n|}{|\mu_n|} \right)$$

since $\sum |c_n| \langle \infty$ and $b_n = o(1)$,

Next, suppose that $\sum b_n = s$. Since

$$(\sum c_n x^n)(\sum r_n x^n) = \sum \mu_n x^n,$$

$$(\sum c_n^{(1)})(\sum r_n x^n) = \sum (\varepsilon * \mu)_n x^n,$$

it follows that

$$\sum_{v=0}^{n} r_v c_{n-v} = \mu_n, \tag{6.1}$$

$$\sum_{v=0}^{n} r_v c_{n-v}^{(1)} = (\varepsilon * \mu)_n, \tag{6.2}$$

Thus, when $\varepsilon_n \in$ M, we have $c_n^{(1)} \geq 0$ and if $r_n \geq 0$, it follows from (6.2) that $(\varepsilon * \mu)_n \geq 0$, whether or not $\mu_n$ is positive.

$$t_m = \sum_{n=0}^{m} r_n \sum_{k=n}^{\infty} \frac{b_k c_{k-n}}{\mu_k}$$

$$= \sum_{n=0}^{m} r_n \left( \sum_{k=n}^{m} + \sum_{k=m+1}^{\infty} \right) \frac{b_k c_{k-n}}{\mu_k}$$

$$= \sum_{k=0}^{m} \frac{b_k}{\mu_k} \sum_{n=0}^{k} r_n c_{k-n} + \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} \frac{b_k c_{k-n}}{\mu_k}$$

$$= \sum_{k=0}^{m} b_k + \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} \frac{b_k c_{k-n}}{\mu_k}$$

Hence, as $b_k = o(1)$,

$$\left| t_m - \sum_{k=0}^{m} b_k \right| \leq \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} o(1) \frac{|c_{k-n}|}{\mu_k}$$

$$= o(1)\frac{1}{|\mu_m|}\sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} |c_{k-n}|$$

But when $p_n \in$ M, we have

$$\sum_{k=m+1}^{\infty} |c_{k-n}| \le c_{m-n}^{(1)} \tag{6.3}$$

and hence, by identity (6.2)

$$\left| t_m - \sum_{k=0}^{m} b_k \right| = o(1)\frac{1}{\mu_m}\sum_{n=0}^{m} r_n c_{m-n}^{(1)}$$

$$= o(1)\frac{(\varepsilon * \mu)_m}{|\mu_m|}$$

This completes the proof of Theorem 2.

7. Proof of Theorem 3: We have

$$\sum_{n=0}^{\infty} \frac{t_n\mu_n}{r_n} - \frac{t_{n+1}\mu_{n+1}}{r_{n+1}} = \sum_{n=0}^{\infty} \Delta\left(\frac{t_n\mu_n}{r_n}\right)$$

$$\le \sum_{n=0}^{\infty} |a_{n+1}|\frac{\mu_{n+1}}{r_{n+1}} + \sum_{n=0}^{\infty} |t_n| \Delta\left|\frac{\mu_n}{r_n}\right|$$

$$= L_n + M_n \text{, (say).}$$

By using (3.1), we get (as $\mu_n$ is non-decreasing)

$$L_n \le \sum_{n=0}^{\infty} \frac{\mu_{n+1}}{r_{n+1}} r_{n+1} \sum_{k=n+1}^{\infty} \frac{|b_k||c_{k-n-1}|}{\mu_k}$$

$$\le \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} |b_k||c_{k-n-1}|$$

$$= \sum_{k=0}^{\infty} |b_k| \sum_{n=0}^{k-1} |c_{k-n-1}|$$

$$= O(1),$$

since $\sum |b_k| \langle \infty$ and $\sum |c_n| \langle \infty$ as $\varepsilon_n \in$ M.

Since $\{\mu_n / r_n\}$ is decreasing we have.

$$\sum_{n=v}^{\infty} \left|\Delta \frac{\mu_n}{r_n}\right| = \sum_{n=v}^{\infty} \left(\frac{\mu_n}{r_n} - \frac{\mu_{n+1}}{r_{n+1}}\right) \le \frac{\mu_v}{r_v}$$

Hence,

$$\begin{aligned}
M_n &= \sum_{n=0}^{\infty} \left|\Delta \frac{\mu_n}{r_n}\right| \left|\sum_{v=0}^{n} r_v \sum_{k=v}^{\infty} \frac{b_k c_{k-v}}{\mu_k}\right| \\
&\le \sum_{n=0}^{\infty} \left|\Delta \frac{\mu_n}{r_n}\right| \sum_{v=0}^{n} r_v \sum_{k=v}^{\infty} \frac{|b_k||c_{k-v}|}{\mu_k} \\
&= \sum_{v=0}^{\infty} r_v \sum_{n=v}^{\infty} \left|\Delta \frac{\mu_n}{r_n}\right| \sum_{k=0}^{\infty} \frac{|b_k||c_{k-v}|}{\mu_k} \\
&= \sum_{v=0}^{\infty} r_v \sum_{k=v}^{\infty} \frac{|b_k||c_{k-v}|}{\mu_k} \sum_{n=v}^{\infty} \left|\Delta \frac{\mu_n}{r_n}\right| \\
&\le \sum_{v=0}^{\infty} \frac{r_v}{\mu_v} \sum_{k=v}^{\infty} |b_k||c_{k-v}| \frac{\mu_v}{r_v} \\
&= \sum_{v=0}^{\infty} \sum_{k=v}^{\infty} |b_k||c_{k-v}| \\
&= \sum_{k=0}^{\infty} |b_k| \sum_{v=0}^{\infty} |c_{k-v}| \\
&< \infty, \quad \text{by hypothesis.}
\end{aligned}$$

Hence

$$\sum \left|\Delta\left(\frac{t_n \mu_n}{r_n}\right)\right| \le L_n + M_n = O(1) \qquad \text{as} \qquad n \to \infty,$$

and therefore

$$\left\{\frac{t_n \mu_n}{r_n}\right\} \in BV.$$

This completes the proof of Theorem 3.

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