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A SPECIAL NÖRLUND MEAN and ITS EIGENVALUES

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ABSTRACT

In a series of paper, some authors have previously investigated the spectrum of weighted mean matrices considered as bounded operators on various sequence spaces [1]-[3], [5] , [7]-[10]. As far as we know there is no investigation on the spectrum of Nörlund means. In this paper, we determine the set of eigenvalues of a special Nörlund matrix as a bounded operator over some sequence spaces.

1. INTRODUCTION

Nörlund mean matrix is an infinite triangular matrix $N = (q_{nk})$ with

$$q_{nk} = \begin{cases} \dfrac{q_{n-k}}{Q_n} & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

where $q_0 > 0, q_k \ge 0$ for $k \ge 1$ and $Q_n = \sum_{k=0}^{n} q_k$ ([5], p.9). It is well known that *N* is regular if and only if $q_n / Q_n \to 0$, as $n \to \infty$ ([5], p.10).

In this paper we define $q_k = r^k$,$0 < r < 1$, and consider the operator *N* generated by

$$(Nx)_n = \frac{1-r}{1-r^{n+1}} \sum_{k=0}^{n} r^{n-k} x_k . \tag{1.1}$$

Let $\pi(X)$ denote the point spectrum of N acting on X, where X is one of the following sequence spaces

$$\ell_\infty = \{(x_n) : \sup_n |x_n| < \infty\} ,$$
$$\ell_p = \left\{(x_n) : \sum_{n=0}^{\infty} |x_n|^p < \infty\right\},$$
$$c = \{(x_n) : \lim_n x_n \text{ exists}\},$$
$$c_0 = \{(x_n) : \lim_n x_n = 0\},$$
$$bv = \left\{(x_n) : \sum_{n=0}^{\infty} |x_n - x_{n+1}| < \infty\right\},$$
$$bv_0 = bv \cap c_0 .$$

The spectrum of weighted mean operators have been investigated on c, c_0, bv and bv_0 by several authors [2]-[3], [5], [7]-[10]. In the second section, we show that *N* is bounded on sequence spaces ℓ_∞, ℓ_p, c, c_0, bv and bv_0. In the third section, we determine its set of eigenvalues on these sequence spaces.

1. BOUNDEDNESS

The following theorem shows that N is a bounded linear operator on some sequence spaces.

Theorem 2.1. N is a bounded linear operator on sequence spaces ℓ_∞, ℓ_p, c, c_0, bv and bv_0 .

Proof. From (1.1), N is given by $N = (q_{nk})$ where

$$q_{nk} = \begin{cases} \dfrac{(1-r)r^{n-k}}{1-r^{n+1}} & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases} . \tag{2.1}$$

From (2.1) it is evident that

$$\sum_{k=0}^{\infty} |q_{nk}| = \sum_{k=0}^{n} \frac{(1-r)r^{n-k}}{1-r^{n+1}} = 1 \tag{2.2}$$

for all n. So N is a bounded on ℓ_∞. Since $0 < r < 1$ we get

$$\lim_{n\to\infty} q_{nk} = \lim_{n\to\infty} \frac{(1-r)r^{n-k}}{1-r^{n+1}} = 0 \tag{2.3}$$

for all k . Then (2.2) and (2.2)gives the boundedness of N on c and c_0. (see, [10], p.116).

On the other hand, from (2.1) we have

$$\sum_{n=0}^{\infty}|q_{nk}| = \sum_{n=k}^{n}\frac{(1-r)r^{n-k}}{1-r^{n+1}} = \frac{1-r}{r^k}\sum_{n=k}^{n}\frac{r^n}{1-r^{n+1}}$$

$$\leq \frac{1-r}{r^k}\int_k^{\infty}\frac{r^t}{1-r^{t+1}}dt = \frac{1-r}{r^k}\,\frac{\ln(1-r^{k+1})}{r\ln r}$$

$$\leq \frac{(1-r)\ln(1-r)}{r\ln r} = O(1) \tag{2.4}$$

From (2.4) we see that N is bounded on ℓ_1 and also from (2.2) and (2.4) it is clear that N is bounded on $\ell_p, 1 < p < \infty$, ([4], p.174).

From (2.1) we find

$$Ne = e \tag{2.5}$$

where $e = (1,1,1,\ldots)$ and

$$\sum_{n=0}^{\infty}\left|\sum_{k=0}^{m}(q_{nk}-q_{n-1,k})\right| = \sum_{n=0}^{m}\left|\sum_{k=0}^{m}(q_{nk}-q_{n-1,k})\right| + \sum_{n=m+1}^{\infty}\left|\sum_{k=0}^{m}(q_{nk}-q_{n-1,k})\right|$$

$$= \sum_{n=0}^{m}\left|\sum_{k=o}^{n}q_{nk} - \sum_{k=0}^{n-1}q_{n-1,k}\right| + \sum_{n=m+1}^{\infty}\left|\sum_{k=0}^{m}\left[\frac{(1-r)r^{n-k}}{1-r^{n+1}} - \frac{(1-r)r^{n-k-1}}{1-r^n}\right]\right|$$

$$= 0 + (1-r)\sum_{n=m+1}^{\infty}\left|\sum_{k=0}^{m}\left[\frac{r^{n-k}}{1-r^{n+1}} - \frac{r^{n-k-1}}{1-r^n}\right]\right|$$

$$= (1-r)\sum_{n=m+1}^{\infty}\frac{1}{(1-r^{n+1})(1-r^n)}\left|\sum_{k=0}^{m}\left[r^{n-k} - r^{n-k-1}\right]\right|$$

$$= (1-r)^2\sum_{n=m+1}^{\infty}\frac{1}{(1-r^{n+1})(1-r^n)}\sum_{k=0}^{m}r^{n-k-1}$$

$$= (1-r)^2\sum_{n=m+1}^{\infty}\frac{1}{(1-r^{n+1})(1-r^n)}\,\frac{r^n(1-r^m)}{r^m(1-r)}$$

$$= \frac{(1-r)(1-r^m)}{r^m}\sum_{n=m+1}^{\infty}\frac{r^n}{(1-r^{n+1})(1-r^n)}$$

$$= \frac{(1-r)(1-r^m)}{r^m}\int_{m+1}^{\infty}\frac{r^t}{(1-r^{t+1})(1-r^t)}dt$$

$$= \frac{(1-r^m)}{r^m \ln r} \ln\left(\frac{1-r^{m+1}}{1-r^{m+2}}\right) = O(1)$$

for all m. Therefore we get

$$\sup_m \sum_{n=0}^{\infty} \left| \sum_{k=0}^{m} (q_{nk} - q_{n-1,k}) \right| = O(1). \tag{2.6}$$

Equations (2.5) and(2.6) are the necessary and sufficient conditions of boundedness of N on bv. Also equations (2.3) and (2.6) give the boundedness of N on bv_0 (see, [10], p.127).

2. EIGENVALUES

Let $Nx = \lambda x$ where $x \neq \theta = (0,0,\ldots)$ and λ is a complex number. By (2.1) we get $x_0 = \lambda x_0$ and for $n \geq 1$

$$\frac{1-r}{1-r^{n+1}} \sum_{k=0}^{n} r^{n-k} x_k = \lambda x_n \ . \tag{3.1}$$

If m is the smallest integer for which $x_m \neq 0$, then from (3.1) we have $\lambda = \dfrac{1-r}{1-r^{m+1}}$ and for $n > m$

$$x_n = r^{n-m} \left(\prod_{k=1}^{n-m} \frac{1-r^{m+k}}{1-r^{m+k+1}} \right) x_m. \tag{3.2}$$

Using the above arguments we get following results about the set of eigenvalues of *N*.

Theorem 3.1. $\pi(c_0) = \pi(\ell_p) = \pi(bv_0) = \phi$.

Proof. If $x = (x_n)$ is a solution of the equation $Nx = \lambda x$ where $x \neq \theta$ and λ is a complex number then x_n is given by (3.2) and

$$\frac{x_n}{x_{n-1}} = \frac{1-r^n}{r^m - r^n} \geq 1 \tag{3.3}$$

is satisfied. The inequality (3.3) implies that $x = (x_n)$ does not belong to c_0. On the other hand, c_0 contains ℓ_p and bv_0 and hence $x = (x_n) \notin \ell_p$ and $x = (x_n) \notin bv_0$. This completes the proof.

Theorem 3.2. $\pi(\ell_\infty) = \pi(c) = \pi(bv) = \{1\}$.

Proof. If the sequence $x=(x_n)$ is not a null sequences which satisfies the equation $Nx=x$ then $x_n=x_0$ for all n. Since $x=(x_0,x_0,\cdots)$ is an element of ℓ_∞, c and bv then $\lambda=1$ is an eigenvalue of N.

Let us assume that the sequence $x=(x_n)$ is a solution of $Nx=\lambda x$ where $x\neq\theta$ and $\lambda\neq 1$ .If we consider the equation (3.2) we have

$$x_n=r^{n-m}\left(\prod_{k=1}^{n-m}\frac{1-r^{m+k}}{r^{m+1}-r^{m+k+1}}\right)x_m=\frac{r^{n-m}}{r^{(n-m)(m+1)}}\left(\prod_{k=1}^{n-m}\frac{1-r^{m+k}}{1-r^k}\right)x_m$$

$$\geq\frac{r^{n-m}}{r^{(n-m)(m+1)}}\left(\prod_{k=1}^{n-m}\frac{1-r^k}{1-r^k}\right)x_m=\frac{1}{r^{m(n-m)}}x_m. \tag{3.4}$$

for $n>m$ .Since $x_m\neq 0$, $x=(x_n)$ is unbounded. So $\lambda\neq 1$ is not eigenvalue of *N*.

Therefore $\lambda=1$ is the only eigenvalue of N acting on ℓ_∞, c and bv.

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