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SOME APPLICATIONS OF FRACTIONAL CALCULUS OPERATORS TO A NEW CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT

The object of the present paper is to prove various distortion theorems for the fractional calculus of the functions in the class $P(n,\lambda,\alpha,r)$ consisting of analytic functions with negative coefficients in the unit disk.

1. INTRODUCTION

Let A(n) denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} \alpha_k z^k (\alpha_k \geq 0; n \in N = \{1,2,...\})$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. A function $f(z) \in A(n)$ is said to be in the class $P(n,\lambda,\alpha,r)$ if it satisfies

$$\mathrm{Re}\left\{ z\frac{(\lambda r z^{r-1} + 1 - \lambda)f'(z) + \lambda z^r f''(z)}{\lambda z^r f'(z) + (1-\lambda)f(z)} \right\} > \alpha, \ (r = 1,2,...)$$

For some α $(0 \leq \alpha < 1)$, λ $(0 \leq \lambda \leq 1)$ and for all $z \in U$. Some properties of the class $P(n,\lambda,\alpha,r)$ were investigated by Kamali and Kadioglu [3].

Lemma 1. If a function $f(z) \in A(n)$ is in the class $P(n,\lambda,\alpha,r)$ then

$$\sum_{k=n+1}^{\infty} \left[(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)\right] a_k \leq (1-\alpha) + \lambda(r-1). \tag{1}$$

Proof. Let $f(z) \in P(n,\lambda,\alpha,r)$. We can write

$$\operatorname{Re}\left\{z\frac{(\lambda r z^{r-1}+1-\lambda)f'(z)+\lambda z^r f''(z)}{\lambda z^r f'(z)+(1-\lambda)f(z)}\right\}>\alpha.$$

Then

$$\frac{[\lambda r z^{r-1}+(1-\lambda)][z-\sum_{k=n+1}^{\infty}ka_k z^k]+\lambda z^{r+1}[-\sum_{k=n+1}^{\infty}k(k-1)a_k z^{k-2}]}{\lambda z^r[1-\sum_{k=n+1}^{\infty}ka_k z^{k-1}]+(1-\lambda)[z-\sum_{k=n+1}^{\infty}a_k z^k]}$$

$$=\frac{\lambda r z^r+(1-\lambda)z-\sum_{k=n+1}^{\infty}[\lambda rk+\lambda k^2-\lambda k]a_k z^{k+r-1}-\sum_{k=n+1}^{\infty}(1-\lambda)ka_k z^k}{\lambda z^r+(1-\lambda)z-\sum_{k=n+1}^{\infty}[\lambda k a_k z^{k+r-1}-\sum_{k=n+1}^{\infty}(1-\lambda)a_k z^k}$$

If we choose z real and let $z\to 1^-$, we get

$$\sum_{k=n+1}^{\infty}\left[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)\right]a_k\le(1-\alpha)+\lambda(r-1).$$

2 Fractional Calculus

I begin with the statements of the following definitions of fractional calculus (that is, fractional derivatives and fractional integrals) which were defined by Owa ([4],[5]) and were used recently by Srivastava and Owa, Altintas, and Cho and Auof ([1],[2],[6]).

Definition 2. The fractional integral of order δ is defined, for a function f(z), by

$$D_z^{-\delta}f(z)=\frac{1}{\Gamma(\delta)}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{1-\delta}}d\zeta\quad(\delta>0),$$

where f(z) is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 3. The fractional derivative of order δ is defined, for a function f(z), by

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\delta}}d\zeta\quad(0\le\delta<1),$$

where f(z) is constrained, and the multiplicity of $(z-\zeta)^{-\delta}$ is removed, as in Definition 2.

Theorem 4. Let the function f(z) be in the class $P(n,\lambda,\alpha,r)$. Then we have

$$\left|D_z^{-\delta}f(z)\right|\ge\frac{|z|^{\delta}}{\Gamma(2+\delta)}\left\{|z|-\frac{\left[(1-\alpha)+\lambda(r-1)\right]\Gamma(n+2)\Gamma(2+\delta)}{\left[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)\right]\Gamma(n+2+\delta)}|z|^{n+1}\right\}$$

and

$$\left|D_z^{-\delta}f(z)\right| \le \frac{|z|^{\delta}}{\Gamma(2+\delta)}\left\{|z| + \frac{[(1-\alpha)+\lambda(r-1)]\Gamma(n+2)\Gamma(2+\delta)}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]\Gamma(n+2+\delta)}|z|^{n+1}\right\}$$

for $\delta > 0$ and $z \in U$. The result is sharp.

Proof. It is easy to see that

$$\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z) = z - \sum_{k=n+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2+\delta)}{\Gamma(k+\delta+1)}a_k z^k = z - \sum_{k=n+1}^{\infty}\psi(k)a_k z^k$$

where

$$\psi(k) = \frac{\Gamma(k+1)\Gamma(2+\delta)}{\Gamma(k+\delta+1)} \quad (k \ge n+1).$$

Noting that $\psi(k)$ is a decreasing function of k, we have

$$0 < \psi(k) \le \psi(n+1) = \frac{\Gamma(n+2)\Gamma(2+\delta)}{\Gamma(n+2+\delta)}.$$

Using Theorem 1, we have

$$[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]\sum_{k=n+1}^{\infty}a_k \le \sum_{k=n+1}^{\infty}[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]a_k$$
$$\le (1-\alpha)+\lambda(r-1),$$

or

$$\sum_{k=n+1}^{\infty}a_k \le \frac{(1-\alpha)+\lambda(r-1)}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-)]}.$$

We can see that

$$\left|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)\right| \ge |z| - \psi(n+1)|z|^{n+1}\sum_{k=n+1}^{\infty}a_k$$
$$\ge |z| - \frac{[(1-\alpha)+\lambda(r-1)]\Gamma(n+2)\Gamma(2+\delta)}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]\Gamma(n+2+\delta)}|z|^{n+1}$$

and

$$\left|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)\right| \le |z| + \psi(n+1)|z|^{n+1}\sum_{k=n+1}^{\infty}a_k$$
$$\le |z| + \frac{[(1-\alpha)+\lambda(r-1)]\Gamma(n+2)\Gamma(2+\delta)}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]\Gamma(n+2+\delta)}|z|^{n+1}$$

which prove the inequalities of the theorem. Further, equalities are attained for the function f(z) defined by

$$D_z^{-\delta}f(z) = \frac{z^{\delta}}{\Gamma(2+\delta)}\left\{z - \frac{[(1-\alpha)+\lambda(r-1)]\Gamma(n+2)\Gamma(2+\delta)}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]\Gamma(n+2+\delta)}z^{n+1}\right\}$$

or

$$f(z) = z - \frac{[(1-\alpha)+\lambda(r-1)]}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]}z^{n+1}.$$

Theorem 5. Let the function f(z) be in the class $P(n,\lambda,\alpha,r)$. Then we have

$$\left|D_z^\delta f(z)\right| \geq \frac{|z|^{-\delta}}{\Gamma(2-\delta)}\left\{|z| - \frac{[(1-\alpha)+\lambda(r-1)]\Gamma(n+2)\Gamma(2-\delta)}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]\Gamma(n+2-\delta)}|z|^{n+1}\right.$$

and

$$\left|D_z^\delta f(z)\right| \leq \frac{|z|^{-\delta}}{\Gamma(2-\delta)}\left\{|z| + \frac{[(1-\alpha)+\lambda(r-1)]\Gamma(n+2)\Gamma(2-\delta)}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]\Gamma(n+2-\delta)}|z|^{n+1}\right\}$$

for $0 \leq \delta < 1$ and $z \in U$. The result is sharp.

Proof. It is easy to see that

$$\Gamma(2-\delta)z^\delta D_z^\delta f(z) = z - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k-\delta+1)} a_k z^k = z - \sum_{k=n+1}^{\infty} \psi(k) k a_k z^k$$

where

$$\psi(k) = \frac{\Gamma(k)\Gamma(2-\delta)}{\Gamma(k-\delta+1)} \quad (k \geq n+1).$$

Noting that $\psi(k)$ is a decreasing function of k, we have

$$0 < \psi(k) \leq \psi(n+1) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+2-\delta)}.$$

Using Theorem 1, we have

$$\frac{(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)}{n+1} \sum_{k=n+1}^{\infty} k a_k \leq (1-\alpha)+\lambda(r-1)$$

or

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)[(1-\alpha)+\lambda(r-1)]}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1]}.$$

We can see that

$$\left|\Gamma(2-\delta)z^\delta D_z^\delta f(z)\right| \geq |z| - \psi(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} k a_k$$

$$\geq |z| - \frac{[(1-\alpha)+\lambda(r-1)]\Gamma(n+2)\Gamma(2-\delta)}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]\Gamma(n+2-\delta)}|z|^{n+1}$$

and

$$\left|\Gamma(2-\delta)z^\delta D_z^\delta f(z)\right| \leq |z| + \psi(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} k a_k$$

$$\leq |z| + \frac{[(1-\alpha)+\lambda(r-1)]\Gamma(n+2)\Gamma(2-\delta)}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]\Gamma(n+2-\delta)}|z|^{n+1}$$

which prove the inequalities of the theorem. Further, equalities are attained for the function f(z) defined by

$$D_z^\delta f(z) = \frac{z^{-\delta}}{\Gamma(2-\delta)}\left\{z - \frac{[(1-\alpha)+\lambda(r-1)]\Gamma(n+2)\Gamma(2-\delta)}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]\Gamma(n+2-\delta)} z^{n+1}\right.$$

or

$$f(z) = z - \frac{[(1-\alpha)+\lambda(r-1)]}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]} z^{n+1}.$$

Remark: If we take r=1 in this paper, then we have the result given by Altıntaş [1]

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