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STATE-SPACE SOLUTIONS TO STANDARD H_∞ CONTROL PROBLEM

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ABSTRACT

Simple state-space formulas are derived for all controllers solving a standard H_∞ problem. In this paper, we assume that the systems are finite dimension, continuous-time, linear and time-invariant. There are efficient algorithms to solve Linear Matrix Inequalities. It is enough to reduce the H_∞ problem to a linear matrix inequality by using Riccati inequality.

Key Words: State-space, Riccati inequalities, H_∞ control.
AMS Subject Classifications: 93B36, 93B52, 93D25, 93B17, 93C05.

1 . INTRODUCTION

H_∞ control that tried to provide answers to plant uncertainty under problem. This control problem was first formulated by Zames [10] and is developed in Zames and Francis [12] and Kimura [7]. Most of the solution techniques available at that time involved analytic functions, (Nevanlinna-Pick Interpolation) or operator-theoretic methods [1], [8]. The early attempts to solve these problems were based in the frequency domain (see, [4]). Many papers have been published in H_∞ control theory (see, [3], [4], [10], [11]).

Additional progress on the 2×2-block problem came from Ball and Cohen [2], who gave a state space solution involving three Riccati equations. In addition to these, Youla parametrization and 2×2-block problem techniques have played an important role in H_∞ theory. Consider the transfer function G(s) in H_∞ control, and compare transfer functions with their norm. The H_∞ norm of transfer function is defined by

$$\|G(s)\|_\infty = \sup_{\omega\in\mathbf{R}} |G(iw)|$$

The transfer function of a system with state-space matrices [A;B;C;D] is given by.

$$G(s) = C(sI - A)^{-1}B + D\,.$$

We will develop a state-space theory by following an approach based on the work of Gahinet [6] and relevance here Özdemir [9]. Throughout this paper, we assume that the systems are finite dimension, continuous-time, linear and time-invariant.

The following notation will be used; *ker* M is null space, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Also λ (A), σ (A) are eigenvalue and singular value of A, respectively. $E^{\dagger}$ is the *Moore-Pseudo inverse* of E. N^T is the transpose of N. V^* is Complex conjugate transpose of a matrix V. In addition P > 0 and P < 0 denote the matrix P is positive and negative definite, respectively. Moreover, $A \in \mathbb{K}^{n \times n}$ is a $n \times n$ matrix. Finally as usual notation, the *transfer matrix* of the linear system, defined as $G(s) = C(sI - A)^{-1}B + D$.

2. STATEMENT AND MODIFICATION OF THE H_∞ PROBLEM

Consider a linear time-invariant plant G(s) with state-space equations

$$\Sigma : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, & x(0) = x^o \\ y = C_2 x + D_{21} w + D_{22} u \\ z = C_1 x + D_{11} w + D_{12} u \end{cases}$$

where the matrices

$$(A, B_1, B_2, C_1, C_2) \in \mathbf{K}^{n \times n} \times \mathbf{K}^{n \times l} \times \mathbf{K}^{n \times m} \times \mathbf{K}^{q \times n} \times \mathbf{K}^{p \times n},$$
$$(D_{11}, D_{12}, D_{21}, D_{22}) \in \mathbf{K}^{q \times l} \times \mathbf{K}^{q \times m} \times \mathbf{K}^{p \times l} \times \mathbf{K}^{p \times m}.$$

We regard u as a control input, y the measured output, w an unknown disturbance input and z the controlled output. This is very general model since it allows for each of x, u, w to affect both y, z and $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. This can be accommodated by setting

$$B_2 = \begin{bmatrix} B_2^1 & 0 & 0 \end{bmatrix}, \; D_{22} = \begin{bmatrix} 0 & D_{22}^2 & 0 \end{bmatrix}, \; D_{12} = \begin{bmatrix} 0 & 0 & D_{12}^3 \end{bmatrix}$$

Similarly with respect to w,

$$B_1 = [\ B_1^1 \quad 0 \quad 0\ ],\ D_{21} = [\ 0 \quad D_{21}^2 \quad 0\ ],\ D_{11} = [\ 0 \quad 0 \quad D_{11}^3\ ]$$

All the norms of vectors will be Euclidean with the corresponding induced norms for matrices. The transfer function G(.) from $\begin{bmatrix} w \\ u \end{bmatrix}$ *to* $\begin{bmatrix} z \\ y \end{bmatrix}$ *is;*

$$G(s) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (sI - A)^{-1}(B_1, B_2), \quad s \in \mathbb{C} / \sigma(A)$$
$$= \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix}$$

This realization is taken minimal and n denotes Its order $A \in \mathbf{K}^{n \times n}$. We suppose that there is a compensator of the form

$$\sum K : \begin{cases} \dot{x} = A_K \hat{x} + B_1 w + B_K y, & \hat{x}(0) = \hat{x}^0, \\ u = C_K \hat{x} + D_K y \end{cases}$$

where

$(A_K, B_K, C_K, D_K) \in \mathbf{K}^{\hat{n} \times \hat{n}} \times \mathbf{K}^{\hat{n} \times p} \times \mathbf{K}^{m \times \hat{n}} \times \mathbf{K}^{m \times p}, \hat{n}, m, p,$ are integer. So the transfer function of the controller is

$$K(s) = D_K + C_K(I - A_K)^{-1} B_K, \qquad s \in \mathbf{C} \setminus \sigma(A_K)$$

The interconnection system $\Sigma \times \Sigma_K$ is well-posed. It is necessary that $I - D_{22} D_K$ is invertible. Throughout this paper we assume that

(A.1 (A, B_2) is stabilizable,

(A.2) (A, C_2) is detectable,

(A.3) $I - D_{22} D_K$ is invertible.

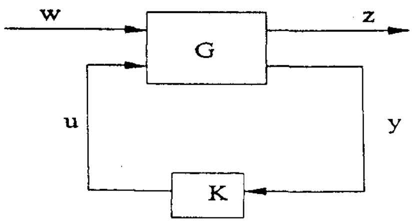

Figure 1: Interconnection system

As a result of the above formulation, the continuous-time basic block diagram H_∞ model problem can be shown as in Figure1. G is the generalized plant and K is controller. Now the closed-loop system is

$$\dot{\bar{x}} = A_{cl}\bar{x} + B_{cl}w,$$
$$z = C_{cl}\bar{x} + D_{cl}w,$$

where $\bar{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \in \mathbf{K}^{n+\hat{n}}$, and

$$A_{cl} = \begin{bmatrix} A + B_2 D_K (I - D_{22}D_K)^{-1} C_2 & B_2 (I - D_K D_{22})^{-1} C_K \\ B_K (I - D_{22}D_K)^{-1} D_{22} C_2 & A_K + B_K (I - D_{22}D_K)^{-1} D_{22} C_K D_{22} \end{bmatrix},$$

$$B_{cl} = \begin{bmatrix} B_1 + B_2 D_K (I - D_{22}D_K)^{-1} C_2 \\ D_K (I - D_{22}D_K)^{-1} C_2 \end{bmatrix},$$

$$C_{cl} = \left[C_1 + D_{12} D_K (I - D_{22}D_K)^{-1} C_2, D_{12}(I - D_K D_{22})^{-1} C_K\right]$$

$$D_{cl} = \left[D_{11} + D_{12} D_K (I - D_{22}D_K)^{-1} D_{21}\right]$$

Let F(G,K)(s) denote the closed-loop transfer function from w to z under dynamic output feedback $u = K(s)y$.

$$F(G,K)(s) = D_{cl} + C_{cl}(sI - A_{cl})^{-1} B_{cl}$$
$$= G_{11}(s) + G_{12}(s)K(s)(I - G_{22}(s)K(s))^{-1} G_{21}(s) \quad s \in \mathbf{C} \setminus \sigma(\mathrm{A_{cl}}).$$

Our aim is to characterize all those dynamic output feedback control K(s), i.e. all quadruple (AK,BK,CK,DK) for the interconnection system $\Sigma\times\Sigma_K$ is well-posed and the resulting closed-loop system, whose transfer matrix is F, internally stable such that for some $\gamma > 0$, the transfer function F(G,K) satisfies

$$\|F(G,K)(.)\|_{H_\infty} = \max_{w\in\mathbf{R}}\|F(G,K)(\imath w)\| < \gamma .$$

3 . PRELIMINARY MATERIAL

We will use the notation $D \in H_n(\mathbf{K}),\ Q \in H_n^+(\mathbf{K}),\ D|_{kerQ} > 0$, to mean $\langle x, Dx\rangle > 0,\ x \in kerQ, x \neq 0$ with no constraint on D in the case where $kerQ = \{0\}$. The following lemma will be needed.

Lemma 3.1. The block matrix

$$\begin{pmatrix} P & N \\ N^T & Q \end{pmatrix} \geq 0$$

is equivalent to

$$\begin{cases} Q < N \\ P - NQ^{-1}N^T < 0. \end{cases}$$

In the sequel, $P - NQ^{-1}N^T$ will be referred to as the Schur complement of Q (see,[6]).

Lemma 3.2. Suppose $D \in H_n(\mathbf{K}),\ Q \in H_n^+(\mathbf{K})$ and $D|_{kerQ} > 0$. Then there exists $\alpha \succ 0$ such that $D + \alpha Q > 0$. Conversely if $D + \alpha Q > 0$ for some $\alpha \succ 0$, then $D|_{kerQ} > 0$.

Proof. With respect to the decomposition

$$\mathbf{K}^n = (kerQ)^\perp \oplus kerQ$$

D and Q have the form

$$D = \begin{bmatrix} D_1 & D_2 \\ D_2^* & D_3 \end{bmatrix},\ Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}, D_3 > 0, Q_1 > 0$$

Hence

$$D + \alpha Q_1 = \begin{bmatrix} D_1 + \alpha Q_1 & D_2 \\ D_2^* & D_3 \end{bmatrix} > 0$$

follows for sufficiently large α from Lemma 3.1, since

$$D_3 > 0, \quad D_1 + \alpha Q_1 > 0, \quad D_1 + \alpha Q_1 - D_2 D_3^{-1} D_2^* > 0, \quad \alpha >> 1$$

The converse is obvious. □

Lemma 3.3 (Gahinet, [5]) Suppose $E \in H_n(\mathbf{K}),\ Q \in H_m(\mathbf{K}),\quad M \in \mathbf{K}^{l\times m}, L \in \mathbf{K}^{n\times m}$. Then there exists a matrix $X \in \mathbf{K}^{n\times l}$ satisfying

$$(L + XM)E^*(L + XM) < Q \tag{3.1}$$

if and only if $L^* EL < Q$ on *ker* M.

Proof. Suppose $V_2 \in \mathbf{K}^{m\times k}$ that is a matrix whose columns a basis of kerM. Then (3.3) implies $V_2^* L^* ELV_2 < V_2^* QV_2$, *i.e.* $L^* EL < Q$ on kerM. Replacing L and X, respectively, by $LQ^{-1/2}$ and $XQ^{-1/2}$. We see that we may assume without loss of generality that $Q = I_n$. Now choose $V_1 \in \mathbf{K}^{m\times(m-k)}$ such that V = [V1 V2] is unitary matrix. Write

$$L[V_1 \quad V_2] = [L_1 \quad L_2], \quad M[V_1 \quad V_2] = [M_1 \quad 0]$$

pre and post multiplying by V^* and V, respectively, we see that (3.1) is equivalent to

$$\begin{bmatrix} (L_1 + XM_1)^* E(L_1 + XM_1) & (L_1 + XM_1)^* EL_2 \\ L_2^* E(L_1 + XM_1) & L_2^* EL_2 \end{bmatrix} < Q = I_n$$

But since M1 is full column rank, there exists a matrix $X \in \mathbf{K}^{n\times l}$ such that $(L_1 + XM_1) = 0$. This X proves (3.1). □

Lemma 3.4. (Bounded Real Lemma, [5],[6]) Consider a continuous time transfer matrix *T(s)* of (not necessarily minimal) realization

$$T(s) = D + C(sI - A)^{-1} B.$$

The following statements are equivalent:

(a) A is stable in the continuous-time since $\mathrm{Re}(\lambda(A)) < 0$ and

$$\left\| D + C(sI - A)^{-1} B \right\|_{H_\infty} < \gamma, \quad \gamma > 0.$$

(b) There exists a symmetric positive definite solution X to the LMI:

$$\begin{pmatrix} A^T X & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0 \tag{3.2}$$

Note that LMI *(3.2)* equivalent to

$$\begin{cases} \sigma_{\max}(D) < \gamma \\ A^T X + XA + \gamma^{-1} C^T C + \gamma(\gamma^{-1} C^T D + XB)(\gamma^2 I - D^T D)^{-1}(\gamma^{-1} C^T D + XB)^T < 0. \end{cases}$$

Lemma 3.5. (Projection Lemma) Suppose $N \in \mathbf{K}^{l\times m}, H \in \mathbf{K}^{n\times m}$ and $D \in H_m(\mathbf{K})$. Then the linear matrix inequality

$$D + N^* X^* H + H^* XN < 0 \tag{3.3}$$

has a solution $X \in \mathbf{K}^{n\times l}$ if and only if D is negative definite on *ker*N and on kerH.

Proof. The necessity of the condition is obvious. To prove sufficiency, we assume that D is negative definite on *ker*N and on *ker* H. For every $\gamma > 0$,
(3.3) is equivalent to

$$(\gamma XN + \gamma^{-1} H)^* (\gamma XN + \gamma^{-1} H) < -D + \gamma^2 N^* X^* XN + \gamma^{-2} H^* H .$$

By assumption and Lemma 3.2

$$-D + \gamma^{-2} H^* H > 0 \ \ for\ \gamma \text{ sufficiently small.}$$

We can apply Lemma 3.3 with $E = I_n$, $M = \gamma N$, $L = \gamma^{-1} H$ *and*

$$Q = -D + \gamma^{-2} H^* H ,$$

since $L^* EL = \gamma^{-2} H^* H < -D + \gamma^{-2} H^* H = Q$ *on Ker*N, since -D>0 on ker N by assumption. Hence there exists $X \in \mathbf{K}^{n\times l}$ such that

$$(\gamma XN + \gamma^{-1} H)^* (\gamma XN + \gamma^{-1} H) < -D + \gamma^{-2} H^* H \le -D + \gamma^2 N^* X^* XN + \gamma^{-2} H^* H \ \square$$

All the control information is collected in a single matrix

$$M_K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \in \mathbf{K}^{(n+\hat{n})\times(n+\hat{n})} .$$

4 . Main Results

The following notation will be used.

$$A^0 = \begin{bmatrix} A & 0 \\ 0 & 0_{\hat{n}} \end{bmatrix}, \quad B^0 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad C^0 = [C_1 \quad 0], \quad D_{12}^0 = [0 \quad D_{12}],$$

$$B^I = \begin{bmatrix} 0 & B_2 \\ I_{\hat{n}} & 0 \end{bmatrix}, \quad C^I = \begin{bmatrix} 0 & I_{\hat{n}} \\ C_2 & 0 \end{bmatrix}, \quad D_{21}^0 = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}.$$

Then the closed loop matrices can be written as

$$A_{cl} = A^0 + B^I M_K C^I, \quad B_{cl} = B^0 + B^I M_K D_{21}^0,$$
$$C_{cl} = D^0 + D_{12}^0 M_K C^I, \quad D_{cl} = D_{11} + D_{12}^0 M_K D_{21}^0$$

Theorem 4.1. The following are equivalent:

(i) There exists a stabilizing dynamic output controller K(.) such that

$$\max_{w \in \mathbf{R}} \| F(G,K)(\imath w) \| < \gamma \tag{4.1}$$

(ii) There exists a $X_{cl} \in H_{n+\hat{n}}, X_{cl} > 0$ such that the matrix $\Phi_{X_{cl}}$ is negative definite on *ker*U and $\Psi_{X_{cl}}$ is negative on *ker*V, where

$$\Phi_{X_{cl}} = \begin{bmatrix} A^0 X_{cl}^{-1} + X_{cl}^{-1}(A^0)^* & B^0 & X_{cl}^{-1}(C^0)^* \\ (B^0)^* & -\gamma I & D \\ C^0 X_{cl}^{-1} & D_{11} & -\gamma I \end{bmatrix},$$

$$\Psi_{X_{cl}} = \begin{bmatrix} (A^0)^* X_{cl} + X_{cl} A^0 & X_{cl} B^0 & (C^0)^* \\ (B^0)^* X_{cl} & -\gamma I & D_{11}^* \\ C^0 & D_{11} & -\gamma I \end{bmatrix},$$

and

$$U = [\, (B^I)^*, 0_{(\hat{n}+m)\times l}, (D_{12}^0)^* \,], \quad V = [\, C^I, D_{21}^0, 0_{(\hat{n}+p)\times q} \,] \tag{4.2}$$

Proof. Applying Lemma 3.4 with $A = A_{cl}$ $B = B_{cl}$ we see that $K(s) = D_K + C_K(sI - A)^{-1} B_K$ is a stabilizing controller satisfying (4.1) if and only if there exists $P_{cl} \in H_{n+\hat{n}}, P_{cl} < 0$, such that

$$\begin{bmatrix} P_{cl}A_{cl} + A_{cl}^* P_{cl} - C_{cl}^* C_{cl} & P_{cl} - C_{cl}^* D_{cl} \\ B_{cl}^* P_{cl} - D_{cl}^* C_{cl} & -\gamma^2 I - D_{cl}^* D_{cl} \end{bmatrix} > 0 .$$

We see that this is equivalent to the existence of $X_{cl} \in H_{n+\hat{n}}$, $X_{cl} > 0$ such that

$$\begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -\gamma I & D_{cl}^* \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 . \tag{4.3}$$

Substituting for A_{cl}, B_{cl}, C_{cl}, D_{cl}, (4.3) becomes

$$\begin{bmatrix} (A^0 + B^I M_K C^I)^* X_{cl} + X_{cl}(A^0 + B^I M_K C^I) & X_{cl}(B^0 + B^I M_K D_{21}^0) & (C^0 + D_{12}^0 M_K C^I)^* \\ (B^0 + B^I M_K D_{21}^0)^* X_{cl} & -\gamma I & D_{11} + D_{12}^0 M_K D_{21}^0 \\ C^0 + D_{12}^0 M_K C^I & D_{11} + D_{12}^0 M_K D_{21}^0 & -\gamma I \end{bmatrix} < 0.$$

or

$$\Psi_{X_{cl}} + \begin{bmatrix} X_{cl} B^I \\ 0 \\ D_{12}^0 \end{bmatrix} M_K \left[C^I, D_{21}^0, 0\right] + \left[C^I, D_{21}^0, 0\right]^* M_K^* \begin{bmatrix} X_{cl} B^K \\ 0 \\ D_{12}^0 \end{bmatrix}^* < 0.$$

That is

$$\Psi_{X_{cl}} + U_{X_{cl}}^* M_K V + V^* M_K^* U_{X_{cl}} < 0 , \tag{4.4}$$

where $U_{X_{cl}} = \left[(B^I)^* X_{cl}, 0, (D_{21}^0)^*\right]$. We now use Lemma 3.5 with $D = \Psi_{X_{cl}}$, $N = U_{X_{cl}}$, $H = V$ and $X = M_K$, to conclude that (i) is equivalent to $\Psi_{X_{cl}}$ being negative definite on *ker* V and *ker*UXcl . To complete the proof, note that

$$U_{X_{cl}} = U \begin{bmatrix} X_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \text{and} \quad \Phi_{X_{cl}} = \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \Psi_{X_{cl}} \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

The characterization in the above theorem is awkward since it involves both Xcl and its inverse. However a simpler form can be obtained by partitioning Xcl and X_{cl}^{-1}. In order to show this we need the following lemma

Lemma 4.2. Let $n, \hat{n} \geq 1$. Suppose $X \in H_{n+\hat{n}}(\mathbf{K})$ and its inverse X^{-1} are partitioned as follows

$$X = \begin{bmatrix} S & N \\ N^* & Q \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} P & M \\ M^* & T \end{bmatrix}, P, S \in H_n(\mathbf{K}), \qquad (4.5)$$

and X>0, then

$$S \geq P^{-1} > 0 \text{ and rank} \left[S - P^{-1} \right] \leq \hat{n}. \qquad (4.6)$$

Conversely, if $P, S \in H_n(\mathbf{K})$ are given such that (4.6) is satisfied, then there exists $X \in H_{n+\hat{n}}(\mathbf{K}), X > 0$ such that X and its inverse can be partitioned as in (4.5) (with suitable N, Q, M, T).

Proof. Suppose that

$$X = \begin{bmatrix} S & N \\ N^* & Q \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} P & M \\ M^* & T \end{bmatrix}, P, S \in H_n(\mathbf{K})$$

Then

$$SP + NM^* = I_n, \quad N^*P + QM^* = 0$$

and since $X>0$, we have

$$S > 0, \quad Q > 0, \quad S - NQ^{-1}N^* > 0,$$

$$P > 0, \quad T > 0, \quad P - MT^{-1}M^* > 0.$$

Now $SP - NQ^{-1}N^*P = I_n$ and hence $S - NQ^{-1}N^* = P^{-1}$. *So*

$$S \geq P^{-1} > 0 \quad \text{and} \quad \text{rank}[S - P^{-1}] \leq \hat{n}.$$

Conversely, assume that $P, S \in H_n(\mathbb{K})$ are given such that the above conditions are satisfied. Let

$$r = rank(S - P^{-1}) \leq \hat{n}.$$

It sufficiency to show that there exists $M, N \in \mathbf{K}^{n \times r}$, $Q, T \in H_r(\mathbf{K}), Q > 0$ such that

$$SP + NM^* = I_n, N^*P + QM^* = 0, \; SM + NT = 0, \quad N^*M + QT = I_{\hat{n}} \quad (4.7)$$

In fact, setting

$$X = \begin{bmatrix} S & 0 & 0 \\ N^* & Q & 0 \\ 0 & 0 & I_{\hat{n}-r} \end{bmatrix} \in H_{n+\hat{n}}(\mathbf{K}),$$

We obtain

$$X^{-1} = \begin{bmatrix} P & 0 & 0 \\ M^* & T & 0 \\ 0 & 0 & I_{\hat{n}-r} \end{bmatrix}$$

X>0 because Q>0 and $S - NQ^{-1}N^* = P^{-1} > 0$. We now construct matrices M, N, Q, T such that (4.7) holds and Q > 0. Let $[U\ V] \in \mathbf{K}^{n \times n}$ be a unitary matrix with $U \in \mathbf{K}^{n \times (n-r)}$ in $\ker(S - P^{-1})$ and define

$$N = \left[rank\left(S - P^{-1}\right)\right]^{\frac{1}{2}} V, \ M = -PN, \ Q = I_r, \ T = I_r - N^* M$$

Since VV^* is the orthogonal projection from $\mathbb{K}^n$ into the linear subspace in $\ker\left(S - P^{-1}\right)$ we have

$$NN^* = \left[rank\left(S - P^{-1}\right)\right]^{\frac{1}{2}} VV^* \left[rank\left(S - P^{-1}\right)\right]^{\frac{1}{2}} = (S - P^{-1}).$$

Using this fact the equations in (4.7) are obtained by direct calculation.

Theorem 4.3. For any $\gamma > 0$, the following are equivalent:

(i) There exists a stabilizing dynamic output feedback controller K(.) of dimension $\hat{n}$ such that

$$\max_{w \in \mathbf{R}} \left\| F(G,K)(\imath w) \right\| < \gamma . \tag{4.8}$$

(ii) There exists $(P,S) \in H_n \times H_n$, $P > 0$, $S > 0$ such that

$$S \ge \gamma^2 P^{-1} > 0 \qquad \text{and } rank[\ S - \gamma^2 P^{-1}\ ] \le \hat{n} \tag{4.9}$$

$$\begin{bmatrix} AP + PA^* + B_1 B_1^* & PC_1^* + B_1 D_{11}^* \\ C_1 P + D_{11} B_1^* & -(\gamma^2 I_q - D_{11} D_{11}^*) \end{bmatrix} < 0, \text{ on } \ker[\ B_2^*\ D_{12}^*\ ] \tag{4.10}$$

$$\begin{bmatrix} SA + A^* S + C_1^* C_1 & SB_1 + C_1^* D_{11} \\ B_1^* S + D_{11}^* C_1 & -(\gamma^2 I_l - D_{11}^* D_{11}) \end{bmatrix} < 0, \text{ on } \ker\ [C_2\ D_{21}]. \tag{4.11}$$

Proof. By the Theorem 4.1 (i) is equivalent to the existence of $X_{cl} \in H_{n+\hat{n}}$, $X_{cl} > 0$, such that the matrix $\Phi_{X_{cl}}$ is negative on kerU and $\Psi_{X_{cl}}$ is negative definite on kerV . Let

$$X_{cl}=\begin{bmatrix} S & N \\ N^* & Q \end{bmatrix}, \quad X_{cl}^{-1}=\begin{bmatrix} P & M \\ M^* & T \end{bmatrix}, \tag{4.12}$$

Then

$$SP+NM^*=I_n, \quad N^*P+QM^*=0.$$

Since $X_{cl}>0$, we have

$$S>0, \quad Q>0, \quad S-NQ^{-1}N^*>0,$$

$$P>0, \quad T>0, \quad R-MT^{-1}M^*>0$$

Now $SP-NQ^{-1}N^*P=I_n$ and hence $S-NQ^{-1}N^*=P^{-1}$. So we obtain from Lemma 4.2

$$S\geq P^{-1} \text{ and } \qquad \operatorname{rank}[S-P^{-1}]\leq \hat{n}. \tag{4.13}$$

Let us consider the condition that $\Phi_{X_{cl}}$ is negative definite on *ker*U. Partitioning $\Phi_{X_{cl}}$ and U, we have

$$\Psi_{X_{cl}}=\begin{bmatrix} AP+PA^* & AM & B_1 & PC_1^* \\ M^*A^* & 0 & 0 & M^*C_1^* \\ B_1^* & 0 & -\gamma I & D_{11}^* \\ C_1P & C_1M & D_{11} & -\gamma I \end{bmatrix},$$

and

$$U=\begin{bmatrix} 0 & I_{\hat{n}} & 0 & 0 \\ B_2^* & 0 & 0 & D_{12}^* \end{bmatrix}.$$

It follows that *ker*U has a basis of the form

$$U=\begin{bmatrix} U_1 & 0 \\ 0 & 0 \\ 0 & I_l \\ U_2 & 0 \end{bmatrix},$$

where $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ is basis of ker$[\ B_2^*, \ \ D_{12}^*\ ]$. Now

$$\begin{bmatrix} U_1 & 0 \\ 0 & 0 \\ 0 & I_l \\ U_2 & 0 \end{bmatrix}^* \begin{bmatrix} AP+PA^* & AM & B_1 & PC_1^* \\ M^*A^* & 0 & 0 & M^*C_1^* \\ B_1^* & 0 & -\gamma I & D_{11}^* \\ C_1P & C_1M & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & 0 \\ 0 & I_l \\ U_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} U_1 & 0 \\ 0 & I_l \\ U_2 & 0 \end{bmatrix}^* \begin{bmatrix} AP+PA^* & B_1 & PC_1^* \\ B_1^* & -\gamma I & D_{11}^* \\ C_1P & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & I_l \\ U_2 & 0 \end{bmatrix}$$

Interchanging rows and columns we see that $\Phi_{X_{cl}}$ is negative definite on kerU if and only if

$$\begin{bmatrix} U_1 & 0 \\ U_2 & 0 \\ 0 & I_l \end{bmatrix}^* \begin{bmatrix} AP+PA^* & PC_1^* & B_1 \\ C_1P & -\gamma I & D_{11} \\ B_1^* & D_{11}^* & -\gamma I \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ U_2 & 0 \\ 0 & I_l \end{bmatrix} < 0 .$$

But this is equivalent to

$$\begin{bmatrix} \begin{bmatrix} U_1^* & U_2^* \end{bmatrix} \begin{bmatrix} AP+PA^* & PC_1^* \\ C_1P & -\gamma I \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} & \begin{bmatrix} U_1^* & U_2^* \end{bmatrix} \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} \\ \begin{bmatrix} B_1^* & D_{11}^* \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} & -\gamma I \end{bmatrix} < 0 .$$

By the negativity Lemma 3.4, the above holds if and only if

$$\begin{bmatrix} U_1^* & U_2^* \end{bmatrix} \left\{ \begin{bmatrix} AP+PA^* & PC_1^* \\ C_1P & -\gamma I \end{bmatrix} + \gamma^{-1} \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} \begin{bmatrix} B_1^* & D_{11}^* \end{bmatrix} \right\} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} < 0 .$$

The transform $P \to \gamma^{-1}P$ yields (4.10). (4.11) is proved in similar way and (4.9) obtained form (4.13) after the transformations $P \to \gamma^{-1}P$, $S \to \gamma^{-1}S$.
Conversely, suppose $(P,S) \in H_n \times H_n$, $P > 0, S > 0$ satisfy the conditions in (ii), we first make the transformation $(P,S) \to (\gamma^{-1}P, \gamma^{-1}S)$. Suppose $rank[P - S^{-1}] \le \hat{n}$, then we may choose a basis so that $P - S^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}$, where $H \in H_{\hat{n}}$, $H \ge 0$ and commensurate to this partition

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}, \quad S=\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{bmatrix}.$$

Applying the transform $P \to \gamma P$, $S \to \gamma S$ and using Lemma 4.2 we obtain that there exist $N, M \in \mathbf{K}^{n\times l}$, Q, $T \in H_l(\mathbf{K})$ such that

$$X_{cl} := \begin{bmatrix} S & N \\ N^* & Q \end{bmatrix}, \quad X_{cl}^{-1} = \begin{bmatrix} P & M \\ M^* & T \end{bmatrix}.$$

Now define $\Psi_{X_{cl}}, \Phi_{X_{cl}}$ as in Theorem 4.1. We have just proved that (4.10), (4.11) imply that the matrix $\Phi_{X_{cl}}$ is negative definite on kerU and $\Psi_{X_{cl}}$ is negative definite on kerV. But this is equivalent to (i).

There are now efficient algorithms for solving linear matrix inequalities (LMIs) (see. [6]). Given that P > 0, S > 0 satisfying (4.9), (4.10) and (4.11). $\Psi_{X_{cl}}$ can be constructed as in the above proof, then using (4.4) feasible control matrices M_K are obtained. Nevertheless, we continue the analysis of (4.10) and (4.11). The objectives are:

- to remove the kernel constraint by reducing the dimension in the inequalities,
- to show that the reduced inequalities can be replaced by Riccati inequalities of lower dimension,
- as an alternative to the above we will show that by introducing two scalar parameters, (4.10) and (4.11) can be replaced directly by two Riccati inequalities in H_n.

These results will be used to obtain a Riccati equation based characterisation. First we assume that D_{12} and D_{21}^* have full column rank. Then later we will show how this assumption can be removed. Since we want to state a result which covers both (4.10) and (4.11), we will use the following notations.

$$(A,B,C,Q,E,V,W) \in \mathbf{K}^{n\times n} \times \mathbf{K}^{n\times m} \times \mathbf{K}^{q\times n} \times \mathbf{K}^{q\times q} \times \mathbf{K}^{q\times m} \times \mathbf{K}^{n\times q} \times \mathbf{K}^{n\times n} \tag{4.14}$$

$$A_0 = A - BE^{\dagger}C, \quad V_q = V + BE^{\dagger}(\gamma^2 I - Q) \tag{4.15}$$

$$W_0 = W - BE^{\dagger}V^* - V(BE^{\dagger})^* - BE^{\dagger}(\gamma^2 I - Q)(BE^{\dagger})^*, \tag{4.16}$$

$$V_0 = W_Q(I - EE^{\dagger}), \; Q_0 = (I - EE^{\dagger})Q(I - EE^{\dagger}), \; C_0 = (I - EE^{\dagger})C \tag{4.17}$$

where $E^\dagger$ is the *pseudo inverse* of E and since we will assume that E is of full column rank, we have $E^\dagger = (E^* E)^{-1} E^*$.

Lemma 4.4. Suppose (4.14)-(4.17) hold with E being full column rank. Then there exists $P \in H_n, P > 0$, such that

$$\begin{bmatrix} AP + PA^* + W & PC^* + V \\ CP + V^* & -\gamma^2 I + Q \end{bmatrix} < 0 \text{ on } \ker\begin{bmatrix} B^* & E^* \end{bmatrix} \quad (4.18)$$

if and only if $P > 0$ satisfies

$$\gamma^2 I > Q_0 \quad (4.19a)$$

$$A_0 P + PA_0^* + W_0 + (PC_0^* + V_0)(\gamma^2 I - Q_0)^{-1}(PC_0^* + V_0)^* < 0. \quad (4.19b)$$

Proof. Let U_{12} be a basis for $E^* = Range(I - EE^\dagger)$ with $U_{12}^* U_{12} = I$. Then we may choose

$$\begin{bmatrix} I & 0 \\ -(BE^\dagger)^* & U_{12} \end{bmatrix}$$

as a basis for $\ker\begin{bmatrix} B^* & E^* \end{bmatrix}$. So (4.18) is equivalent to

$$\begin{bmatrix} I & 0 \\ -(BE^\dagger)^* & U_{12} \end{bmatrix}^* \begin{bmatrix} AP + PA^* + W & PC^* + V \\ CP + V^* & -\gamma^2 I + Q \end{bmatrix} \begin{bmatrix} I & 0 \\ -(BE^\dagger)^* & U_{12} \end{bmatrix} < 0.$$

The $\{11\}$ component of the LHS of the above inequality is

$$AP + PA^* + W - BE^\dagger(CP + V^*) - (PC^* + V)(BE^\dagger)^* + BE^\dagger(-\gamma^2 I + Q)(BE^\dagger)^*.$$

The $\{12\}$ component is

$$(PC^* + V + BE^\dagger(\gamma^2 I - Q))U_{12},$$

and the $\{22\}$ is

$$-U_{12}^*(\gamma^2 I - Q)U_{12}.$$

So (4.18) is equivalent to

$$\begin{bmatrix} A_0 P + PA_0^* + W_0 & (PC^* + V_Q)U_{12} \\ U_{12}^*(CP + V_Q^*) & -\gamma^2 I + U_{12}^* Q U_{12} \end{bmatrix} < 0.$$

Then using Lemma 3.4, both of the above are equivalent to $\gamma^2 I > Q\Big|_{\ker E^*}$ and the Riccati inequality

$$A_0 P + PA_0^* + W_0 + (PC^* + V_Q)U_{12}(\gamma^2 I - U_{12}^* Q U_{12})^{-1} U_{12}^* (PC^* + V_Q)^* < 0 \tag{4.20}$$

But $\qquad U_{12}(\gamma^2 I - U_{12}^* Q U_{12} = (\gamma^2 I - U_{12}U_{12}^* Q)^{-1} U_{12} U_{12}^*$.

Now $\qquad U_{12}U_{12}^* = (I - EE^\dagger) = (I - EE^\dagger)^2$, so

$$(\gamma^2 I - U_{12}U_{12}^* Q)^{-1} U_{12}U_{12}^* = (\gamma^2 I (I - EE^\dagger)^2 Q)^{-1} (I - EE^\dagger)^2$$

Substitution in (4.20) yields (4.15).

In order to apply this result to (4.9) and (4.10), we have to introduce even more notations.

$$\widetilde{B}_2 = B_2 D_{12}^\dagger, \qquad \widetilde{A} = A - \widetilde{B}_2 C_1, \qquad \widetilde{B}_1 = B_1 - \widetilde{B}_2 D_{11} \tag{4.21a}$$

$$\widetilde{C}_1 = (I - D_{12}D_{12}^\dagger)C_1, \qquad \widetilde{D}_{11} = (I - D_{12}D_{12}^\dagger)D_{11}, \ \widetilde{\Pi}_\gamma = \gamma^2 I - \widetilde{D}_{11}\widetilde{D}_{11}^* \tag{4.21b}$$

and

$$\overline{C}_2 = D_{21}^\dagger C_2, \overline{A} = A - B_1\overline{C}_2, \ \overline{C}_1 = C_1 - D_{11}\overline{C}_2 \tag{4.22a}$$

$$\overline{B}_1 = B_1(I - D_{21}^\dagger D_{21}), \quad \overline{D}_{11} = D_{11}(I - D_{21}^\dagger D_{21}), \ \overline{\Pi}_\gamma = \gamma^2 I - \overline{D}_{11}^* \overline{D}_{11} \tag{4.22b}$$

Proposition 4.5. Suppose D_{12} and D_{21}^* have full column rank. Then the followings are equivalent:

(i) There exists a stabilizing dynamic output feedback controller K(.) of dimension $\hat{n}$ such that

$$\max_{w \in \mathbf{R}} \| \mathrm{F}(G,K)(\imath w) \| < \gamma .$$

(ii) There exists $(P,S) \in H_n \times H_n, P > 0, S > 0$ such that

$$\gamma > \max \left\{ \|\widetilde{D}_{11}\|, \|\overline{D}_{11}\| \right\}, \tag{4.23a}$$

$$S \ge \gamma^2 P^{-1} \text{ and } \operatorname{rank}\left|S - \gamma^2 P^{-1}\right| \le \tilde{n}\,, \tag{4.23b}$$

$$\widetilde{A}P + P\widetilde{A}^* - \gamma^2 \widetilde{B}_2 \widetilde{B}_2^* + \widetilde{B}_1 \widetilde{B}_1^* + (P\widetilde{C}_1^* + \widetilde{B}_1 \widetilde{D}_{11}^*)\widetilde{\Pi}_\gamma^{-1}(P\widetilde{C}_1^* + \widetilde{B}_1 \widetilde{D}_{11})^* < 0 \tag{4.24}$$

$$\overline{A}^* S + S\overline{A} - \gamma^2 \overline{C}_2^* \overline{C}_2 + \overline{C}_1^* \overline{C}_1 + (S\overline{B}_1 + \overline{C}_1^* \overline{D}_{11})^* \widetilde{\Pi}_\gamma^{-1}(S\overline{B}_1 + \overline{C}_1^* \overline{D}_{11})^* < 0\,. \tag{4.25}$$

Proof. Let

$B = B_2, \quad E = D_{12}, \quad W = B_1 B_1^*, \quad C = C_1, \quad V = B_1 D_{11}^*, \quad Q = D_{11} D_{11}^*$
and applying Lemma 4.4, we see that (4.18) is equivalent to the first inequality in (4.23a), together with the Riccati inequality

$$A_0 P + P A_0^* + W_0 + (P C_0^* + V_0)(\gamma^2 I - Q_0)^{-1}(P C_0^* + V_0)^* < 0\,,$$

where

$$\begin{aligned} A_0 &= A - B_2 D_{12}^\dagger C_1 = A, \quad C_0 = (I - D_{12} D_{12}^\dagger) C_1 = \widetilde{C}_1 \\ V_0 &= [\ B_1 D_{11}^* + B_2 D_{12}^\dagger (\gamma^2 I - D_{11} D_{11}^*\ ](I - D_{12} D_{12}^\dagger) \\ &= \widetilde{B}_1 \widetilde{D}_{11}^* + \gamma^2 B_2 D_{12}^\dagger (I - D_{12} D_{12}^\dagger) \end{aligned}$$

But

$$D_{12}^\dagger D_{12} D_{12}^\dagger = (D_{12}^* D_{12})^{-1} D_{12}^* D_{12} D_{12}^\dagger = D_{12}^\dagger$$

and hence $V_0 = \widetilde{B}_1 \widetilde{D}_{11}^*$.

$$Q_0 = (I - D_{12} D_{12}^\dagger) D_{11} D_{11}^* (I - D_{12} D_{12}^\dagger) = \widetilde{D}_{11} \widetilde{D}_{11}^*$$

$$\begin{aligned} W_0 &= B_1 B_1^* - B_2 D_{12}^\dagger D_{11} B_1^* - B_1 D_{11}^* (B_2 D_{12}^\dagger)^* - B_2 D_{12}^\dagger (\gamma^2 I - D_{11} D_{11}^*)(B_2 D_{12}^\dagger)^* \\ &= -\gamma^2 \widetilde{B}_2 \widetilde{B}_2^* + \widetilde{B}_1 \widetilde{B}_1^* \end{aligned}$$

Thus, the above Riccati inequality is the same as

$$\widetilde{A}P + P\widetilde{A}^* - \gamma^2 \widetilde{B}_2 \widetilde{B}_2^* + \widetilde{B}_1 \widetilde{B}_1^* + (P\widetilde{C}_1^* + \widetilde{B}_1 \widetilde{D}_{11}^*)\widetilde{\Pi}_\gamma^{-1}(P\widetilde{C}_1^* + \widetilde{B}_1 \widetilde{D}_{11}^*)^* < 0. \qquad \square$$

Equation (4.25) and the second inequality in (4.23a) are proved in a similar way. Then the equivalence follows from Theorem 4.3.

REMARKS

Since there are efficient algorithms to solve Linear Matrix Inequality (LMIs), it is enough to reduce the problem (H_∞ control to a linear matrix inequality by using Riccati inequality. However, this requires E to be full column rank which implies D_{12} and D_{21}^* have full column rank. In the case when D_{12} and D_{21}^* do not have full column rank, the reduction algorithm can be used to reduce D_{12} and D_{21}^* to the case where the equivalent reduced version of the D_{12} and D_{21}^* have full column rank. This can be done by using the Lemma 4.4, and it may require multiple steps, but eventually a reduced form of D_{12} and D_{21}^* would be found after which the Riccati inequalities follow. Then the problem can be solved by one of the efficient algorithms which are developed to solve the Linear Matrix Inequality (LMI).

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