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# THE TANGENT BUNDLE ON $C^1$ FUZZY MANIFOLDS

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## ABSTRACT

Let X be a $C^1$ fuzzy manifold and p be a point in X. At first, it is given that the tangent space at p denoted by $T_p(X)$ is a vector space. In this paper, constructing the tangent bundle $T(X) = \underset{p\in X}{\mathrm{V}} T_p(X)$ on X, it is shown that there is a covariant functor from the category of $C^1$ fuzzy manifolds and fuzzy differentiable functions to the category of the tangent bundles on $C^1$ fuzzy manifolds and fuzzy manifold derivative functions.

## 1.INTRODUCTION

We begin by giving the following definitions.

**Definition 1.1.** Let $(X,\tau_1)$, $(Y,\tau_2)$ be fuzzy topological spaces and $f : X \to Y$ be a mapping. If for each open fuzzy set V in $\tau_2$ the inverse image $f^{-1}(V)$ is open in $\tau_1$, then f is called a fuzzy continuous [3].

**Definition 1.2.** A fuzzy topological vector space is a vector space E over the field K of real or complex numbers, E equipped with a fuzzy topology $\tau$ and K equipped with the usual topology T, such that two mappings

$$\begin{aligned} + :(E,\tau)\times(E,\tau) &\to (E,\tau) \\ (x,y) &\to x+y \end{aligned}$$

and

$$\cdot : (K,T) \times (E,\tau) \quad \rightarrow (E,\tau)$$
$$(\alpha, x) \qquad \rightarrow \alpha x$$

are fuzzy continuous [5].

**Definition 1.3.** Let $E_1$, $E_2$ be fuzzy topological vector spaces and $f : E_1 \rightarrow E_2$ be a bijection. If f and $f^{-1}$ are fuzzy differentiable, and $f'$ and $(f^{-1})'$ are fuzzy continuous, then f is called a $C^1$ fuzzy diffeomorphism [2,4].

**Definition 1.4.** Let X be a set and $\{(X_\alpha, \phi_\alpha)\}_{\alpha\in A}$ be a collection of pairs. If,

Each $X_\alpha$ is a fuzzy set in X and $\sup_\alpha \{\mu_{X_\alpha}(x)\} = 1$ for all $x\in X$,

Each $\phi_\alpha$ is a bijection, defined on the support of $X_\alpha$, $\{x\in X : \mu_{X_\alpha}(x) > 0\}$, which maps $X_\alpha$ onto an open fuzzy set $\phi_\alpha[X_\alpha]$ in some fuzzy topological vector space $E_\alpha$, and, for each β in the index set, $\phi_\alpha[X_\alpha \cap X_\beta]$ is an open fuzzy set in $E_\alpha$,

The mapping $\phi_\beta o \phi_\alpha^{-1}$, which maps $\phi_\alpha[X_\alpha \cap X_\beta]$ onto $\phi_\beta[X_\alpha \cap X_\beta]$ is a $C^1$ fuzzy diffeomorphism for each pair of indices α,β,

then the family $\{(X_\alpha, \phi_\alpha)\}_{\alpha\in A}$ is called a $C^1$ fuzzy atlas.

**Definition 1.5.** Each pair $(X_\alpha, \phi_\alpha)$ is called a fuzzy chart of the fuzzy atlas. If a point $x\in X$ lies in the support of $X_\alpha$, then $(X_\alpha, \phi_\alpha)$ is said to be a fuzzy chart at x.

Let (X,τ) be a fuzzy topological space. Suppose there exist an open fuzzy set χ in X and a fuzzy continuous bijective mapping ϕ defined on the support of χ and mapping χ onto an open fuzzy set V in some fuzzy topological vector space E. Then (χ,ϕ) is said to be compatible with the $C^1$ atlas $\{(X_\alpha, \phi_\alpha)\}_{\alpha\in A}$ if each mapping $\phi_\alpha o \phi^{-1}$ of $\phi[\chi \cap X_\alpha]$ onto $\phi_\alpha[\chi \cap X_\alpha]$ is a fuzzy diffeomorphism of class $C^1$.

Two $C^1$ fuzzy atlases are compatible if each fuzzy chart of one atlas is compatible with each fuzzy chart of the other atlas. It is shown that the relation of compatibility between $C^1$ fuzzy atlases is an equivalence relation. An equivalence class of $C^1$ fuzzy atlases on X is said to define a $C^1$ fuzzy manifold on X [1].

## 2. THE TANGENT BUNDLE ON $C^1$ FUZZY MANIFOLDS

Let X be a $C^1$ fuzzy manifold and let p be a point in X. Consider triples $(X_\alpha, \phi_\alpha, v)$, where $(X_\alpha, \phi_\alpha)$ is a fuzzy chart at p and v is a fuzzy point of the fuzzy topological vector space in which $\phi_\alpha(X_\alpha)$ lies.

Two such triples $(X_\alpha, \phi_\alpha, v)$, $(X_\beta, \phi_\beta, w)$, are said to be related, written $(X_\alpha, \phi_\alpha, v) \sim (X_\beta, \phi_\beta, w)$, if the fuzzy derivative of $\phi_\beta o \phi_\alpha^{-1}$ at $\phi_\alpha(p)$ maps v into w. That is,

$$(\phi_\beta o \phi_\alpha^{-1})'(\phi_\alpha(p))v = w.$$

**Lemma 2.1.** The relation $(X_\alpha, \phi_\alpha, v) \sim (X_\beta, \phi_\beta, w)$ is an equivalence relation.

**Proof.** Straightforward.

**Definition 2.1.** An equivalence class of triples $(X_\alpha, \phi_\alpha, v)$ is called a tangent vector of the fuzzy manifold X at p and this equivalence class is denoted by $[X_\alpha, \phi_\alpha, v]_p$. The tangent space of p denoted by $T_p(X)$ is defined as the set of all tangent vectors at p.

The set $T_p(X)$ can be given the structure of a vector space. Define the sum of two tangent vectors at $p \in X$ as

$$[X_\alpha, \phi_\alpha, v]_p + [X_\beta, \phi_\beta, w]_p = [X_\beta, \phi_\beta, (\phi_\beta o \phi_\alpha^{-1})'(\phi_\alpha(p))v + w]_p.$$

Define the product of a tangent vector with a scalar c as

$$c\,[X_\alpha, \phi_\alpha, v]_p = [X_\alpha, \phi_\alpha, cv]_p.$$

Now, let X be a $C^1$ fuzzy manifold on E and the tangent bundle of X is defined as the disjoint union of the tangent space $T_p(X)$, p running over X, and will be denoted by T(X). i.e.,

$$T(X) = \bigvee_{p \in X} T_p(X).$$

The next proposition shows that T(X) can always be given, in a natural manner, a fuzzy topology and $C^1$ fuzzy atlas under which it becomes a $C^1$ fuzzy manifold on $E \times E$. In the sequel T(X) will always be assumed to have this extra structure. Define a map

$\pi$: T(X) $\rightarrow$ X,

called the natural projection, by $\pi([X_\alpha, \phi_\alpha, v]_p)$=p. Corresponding to each $\alpha$ in A define

$\tau_\alpha : \pi^{-1}(X_\alpha) \rightarrow U_\alpha \times E$

by $\tau_\alpha\left([X_\alpha, \phi_\alpha, v]_{\phi_\alpha^{-1}(u)}\right) = (u, v)$. Notice that $\tau_\alpha$ is a bijection and that the union of the codomains of $\tau_\alpha^{-1}$ is equal to T(X). Suppose that for $\alpha$, $\beta$ in A the codomains of $\tau_\alpha^{-1}$ and $\tau_\beta^{-1}$ overlap, that is, that $\pi^{-1}(X_\alpha)\cap\pi^{-1}(X_\beta)\neq\varnothing$. Since $\pi^{-1}(X_\alpha)\cap\pi^{-1}(X_\beta)=\pi^{-1}(X_\alpha\cap X_\beta)$, we have $X_\alpha\cap X_\beta\neq\varnothing$. Let (u, v) belong to $\tau_\alpha(\pi^{-1}(X_\alpha)\cap\pi^{-1}(X_\beta))$, then

$$\begin{aligned}
\tau_\beta o \tau_\alpha^{-1}(u, v) &= \tau_\beta\left([X_\alpha, \phi_\alpha, v]_{\phi_\alpha^{-1}(u)}\right) \\
&= \tau_\beta([X_\beta, \phi_\beta, (\phi_\beta \circ \phi_\alpha^{-1})'(u)v]_{\phi_\beta^{-1}\circ(\phi_\beta\circ\phi_\alpha^{-1}(u))}) \\
&=((\phi_\beta o \phi_\alpha^{-1})(u), (\phi_\beta o \phi_\alpha^{-1})'(u)v).
\end{aligned}$$

Now,

$$\begin{aligned}
\tau_\alpha(\pi^{-1}(X_\alpha)\cap\pi^{-1}(X_\beta)) &= \tau_\alpha\pi^{-1}(X_\alpha\cap X_\beta) \\
&= \phi_\alpha(X_\alpha\cap X_\beta)\times E
\end{aligned}$$

and so the $C^1$ fuzzy diffeomorphism of $\tau_\beta o \tau_\alpha^{-1}$ on $\tau_\alpha(\pi^{-1}(X_\alpha)\cap\pi^{-1}(X_\beta))$ follows from the $C^1$ fuzzy diffeomorphism of $\phi_\beta o \phi_\alpha^{-1}$ on $\phi_\alpha(X_\alpha\cap X_\beta)$.

The collection of fuzzy open sets $\tau_\alpha^{-1}(U\times V)$, $\alpha$ ranging over A, U ranging over fuzzy open subsets of $U_\alpha$, and V ranging over fuzzy open subsets of E, may be seen to from the basis of a fuzzy topology on T(X) and under this topology we have just shown that:

**Proposition 2.2.** (T(X), ($\tau_\alpha : \alpha\in A$)) is a $C^1$ fuzzy manifold on E×E.

**Definition 2.2.** Let X, Y be $C^1$ fuzzy manifolds. If f : X $\rightarrow$ Y is a fuzzy differentiable and p$\in$X, then we define a map

$f_{*,p} : T_p(X) \rightarrow T_{f(p)}(Y)$, called the $C^1$ fuzzy manifold derivative function of f at p, by

$f_{*,p} : [X_\alpha, \phi_\alpha, v]_p \rightarrow [Y_\beta, \psi_\beta, (\psi_\beta o f o \phi_\alpha^{-1})'(\phi_\alpha(p))v]_{f(p)}$.

*Since $(Y_\beta, \psi_\beta, (\psi_\beta o f o \phi_\alpha^{-1})'(\phi_\alpha(p))v) \sim (Y_\delta, \psi_\delta, (\psi_\delta o f o \phi_\gamma^{-1})'(\phi_\gamma(p))w)$ whenever $(X_\alpha, \phi_\alpha, v) \sim (X_\beta, \phi_\beta, w)$, is well-defined.*

By letting p range over X we define a tangent bundle map

$$f_* : T(X) \rightarrow T(Y)$$

by $f_*=f_{*,p}$ on $T_p(X)$.

Suppose that both $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are fuzzy differentiable, where X, Y and Z are $C^1$ fuzzy manifolds. Then

$(gof)_{*,p}=g_{*,f(p)} o f_{*,p}$

and hence

$(gof)_*=g_* o f_*$ .

Now, let $Q_1$ be the category of $C^1$ fuzzy manifolds and fuzzy differentiable functions and $Q_2$ be the category of the tangent bundles on the $C^1$ fuzzy manifolds and fuzzy manifold derivative functions. Then we can define a mapping $F : Q_1 \rightarrow Q_2$ as follows:

For any sheaf $C^1$ fuzzy manifold X and every fuzzy differentiable function $f : X \rightarrow Y$, let $F(X)=T(X)$ and $F(f)=f_* : T(X) \rightarrow T(Y)$. Then,

$f=1_X$, then $F(1_X)=1_{T(X)}$

If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are fuzzy differentiable functions, then

$F(gof)=(gof)_*=g_* o f_*=F(g)oF(f)$,

Thus, the mapping $F : Q_1 \rightarrow Q_2$ is a covariant functor.

Therefore, we can state the following theorem.

**Theorem 2.3.** There is a covariant functor from the category of the $C^1$ fuzzy manifolds and fuzzy differentiable functions to the category of the tangent bundles on the $C^1$ fuzzy manifolds and fuzzy manifold derivative functions.

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