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ON A GENERALIZATION OF THE RECIPROCAL LCM MATRIX

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ABSTRACT

Let $S=\{x_1,x_2,...,x_n\}$ be a set of distinct positive integers. The $n\times n$ matrix $1/[S]=(s_{ij})$, where $s_{ij}=1/[x_i,x_j]$, the reciprocal of the least common multiple of x_i and x_j, is called the reciprocal least common multiple (reciprocal LCM) matrix on S. In this paper, we present a generalization of the reciprocal LCM matrix on S, that is the matrix $1/[S^r]$, the ij – entry of which is $1/[x_i,x_j]^r$, where r is a real number. We obtain a structure theorem for $1/[S^r]$ and the value of the determinant of $1/[S^r]$. We also prove that $1/[S^r]$ is positive definite if $r>0$. Then we calculate the inverse of $1/[S^r]$ on a factor closed set. Finally, we show that the matrix $[S^r]=([x_i,x_j]^r)$ defined on S is the product of an integral matrix and the generalized reciprocal LCM matrix $1/[S^r]=(1/[x_i,x_j]^r)$ if S is factor closed and r is a positive integer.

Keywords. The GCD matrix, the LCM matrix, the reciprocal GCD matrix, the reciprocal LCM matrix, Euler's totient function, Jordan's totient function, factor closed set.

1. INTRODUCTION

Let $S=\{x_1,x_2,...,x_n\}$ be a set of distinct positive integers. The $n\times n$ matrix (S), the ij -entry of which is (x_i,x_j) the greatest common divisor of x_i and x_j, is called the greatest common divisor (GDC) matrix on S. Beslin and Ligh [2] initiated the study of GCD matrices in the direction of their structure, determinant and arithmetic in Z_n. They also proved that $\det(S)=\phi(x_1)\phi(x_2)\ldots\phi(x_n)$ if $S=\{x_1,x_2,...,x_n\}$ is factor closed. In [9], Li calculated the value of the determinant of the GCD matrix on S in terms of Euler's totient function ϕ when S is not factor closed. A set S of positive integers is said to be factor closed (FC) if all positive divisors of any element of S belong to S. In [3], Beslin defined the reciprocal GCD matrix on S and investigated the structure of the LCM matrix by using the reciprocal GCD matrix. The $n\times n$ matrix $1/(S)$, the ij-entry of which is $1/(x_i,x_j)$, is called the reciprocal GCD matrix on S. The $n\times n$ matrix $[S]=(s_{ij})$, where $s_{ij}=[x_i,x_j]$, the least common multiple of x_i and x_j, is called the least common multiple (LCM) matrix on S. In [4], Bourque and Ligh calculated the inverses of the GCD matrix and the LCM matrix on S when S is factor closed. They also conjectured that the LCM matrix on S is invertible if S is gcd-closed. A set $S=\{x_1,x_2,\ldots,x_n\}$ is gcd-closed if $(x_i,x_j)\in S$ for all $1\le i,j\le n$. In [7], Haukkanen, Wang, and Sillanpää gave a counterexample for the conjecture of Bouque and Ligh.

Let f be a multiplicative arithmetical function and $(f[x_i,x_j])$ denote the $n\times n$ matrix having f evaluated at the least common multiple $[x_i,x_j]$ of x_i and x_j as its ij-entry. Bourque and Ligh in [5] calculated the determinant of $(f[x_i,x_j])$ if S is factor closed. They calculated its inverse if it is invertible. Furthermore, they determine conditions on f that guarantee the matrix $(f[x_i,x_j])$ is positive definite. They also gave the lower and the upper bounds for $\det(f[x_i,x_j])$, where f be a multiplicative function. In [8], Hong showed for a certain class of semi-multiplicative functions new bounds for $\det(f[x_i,x_j])$, which improved the results obtained by Bourque and Ligh in 1995.

Let $S=\{x_1,x_2,\ldots,x_n\}$ be a set of distinct positive integers and r be a real number. In this paper, we present a generalization of the reciprocal LCM matrix defined on S, that is the $n\times n$ matrix $1/[S^r]$, the ij-entry of which is the r-th power of the ij-entry of the $n\times n$ reciprocal LCM matrix. We investigate the structure and the determinant of $1/[S^r]$. We also prove that $1/[S^r]$ is positive definite if $r>0$. We calculate the inverse of $1/[S^r]$ if S is factor closed. Furthermore, we showed that $[S^r]=\left([x_i,x_j]^r\right)_{n\times n}$ is the product of an $n\times n$ integral matrix and $1/[S^r]$, if S is factor closed and r is a positive integer.

2. THE RECIPROCAL LCM MATRIX AND ITS GENERALIZATION

Let $S=\{x_1,x_2,\ldots,x_n\}$ be a set of distinct positive integers. In this section, we present a generalization of the reciprocal LCM matrix defined on S.

Definition 1. Let $S=\{x_1,x_2,\ldots,x_n\}$ be a set of distinct positive integers and $[x_i,x_j]$ denote the least common multiple of x_i and x_j. The $n\times n$ matrix $1/[S]=(s_{ij})$, where $s_{ij} = 1/[x_i, x_j]$, is called the reciprocal least common multiple (reciprocal LCM) matrix on S.

It is obvious that the reciprocal LCM matrix is symmetric and rearrangements of the elements of S yield similar matrices. Hence we may assume $x_1 < x_2 < \ldots < x_n$. Throughout this paper, $S = \{x_1, x_2, \ldots, x_n\}$ denotes an ordered set of distinct positive integers such that $x_1 < x_2 < \ldots < x_n$.

A set S of positive integers is said to be factor closed (FC) if all positive divisors of any element of S belong to S. A set $\overline{S}$, the factor closure of S is a minimal set of positive integers such that all positive divisors of any element of S belong to $\overline{S}$.

Let r be a real number. The $n \times n$ matrix $1/[S^r]$, the ij – entry of which is the rth power of the $n \times n$ reciprocal LCM matrix on S is a generalization of reciprocal LCM matrix on S.

Theorem 1. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and $\overline{S} = \{d_1, d_2, \ldots, d_m\}$ be the factor closure of S. If $1/[S^r]$ is the $n \times n$ generalized LCM matrix on S then the matrix $1/[S^r]$ is the product of an $n \times m$ matrix A and A^T, the transpose of the matrix A.

Proof. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and $\overline{S} = \{d_1, d_2, \ldots, d_m\}$ be the factor closure of S. Then we define the $n \times m$ $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} \dfrac{\sqrt{J_r(d_j)}}{x_i^r} & \text{if } d_j | x_i, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where J_r is Jordan's totient function [1] such that

$$J_r(d) = \sum_{e|d} e^r \mu(d/e) \tag{2}$$

for every positive integer d. Then the ij – entry of AA^T is:

$$\left(AA^T\right)_{ij} = \sum_{k=1}^{m} a_{ik} a_{jk} = \sum_{\substack{d_k | x_i \\ d_k | x_j}} \frac{J_r(d_k)}{x_i^r x_j^r} = \frac{(x_i, x_j)}{x_i^r x_j^r} = \frac{1}{[x_i, x_j]^r}.$$

The proof is completed.

Corollary 1. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and $\overline{S} = \{d_1, d_2, \ldots, d_m\}$ be the factor closure of S. The matrix $1/[S^r]$ can be written as $1/[S^r] = F_r \Lambda_r F_r^T$, where $\Lambda_r = diag(J_r(d_1), J_r(d_2), \ldots, J_r(d_m))$ and $F_r = (f_{ij})$ is an $n \times m$ matrix given by

$$f_{ij}=\begin{cases}\frac{1}{x_i^r} & if\ d_j|x_i,\\ 0 & otherwise.\end{cases}$$

Proof. The ij – entry of the product $F_r\Lambda_r F_r^T$ is:

$$\left(F_r\Lambda_r F_r^T\right)_{ij}=\sum_{k=1}^{m} f_{ik}J_r(d_k)f_{jk}=\sum_{d_k|(x_i,x_j)}\frac{J_r(d_k)}{x_i^r x_j^r}=\left(1/[S]\right)_{ij}.$$

Corollary 2. The matrix $1/[S^r]$ can be written as $1/[S^r]=(D_rE)\Lambda_r(D_rE)^T$, where $D_r=diag\left(1/x_1^r,1/x_2^r,\ldots,1/x_n^r\right)$, $\Lambda_r=diag\left(J_r(d_1),J_r(d_2),\ldots,J_r(d_m)\right)$, and $E=(e_{ij})$ is an $n\times m$ matrix defined by

$$e_{ij}=\begin{cases}1 & if\ d_j|x_i,\\ 0 & otherwise.\end{cases}\tag{3}$$

Proof. Let $A=(a_{ij})$ is an $n\times m$ matrix defined by (1). Then the ij – entry of A is:

$$a_{ij}=\frac{1}{x_i^r}e_{ij}\sqrt{J_r(d_j)}$$

while $d_j\,|\,x_i$. Hence, $A=D_rE\Lambda_r^{1/2}$. Thus $1/[S^r]=AA^T=(D_rE)\Lambda_r(D_rE)^T$.

Theorem 2. Let $S=\{x_1,x_2,...,x_n\}$ be a set of distinct positive integers with $x_1<x_2<\cdots<x_n$ and $\overline{S}=\{x_1,x_2,\ldots,x_n,x_{n+1},\ldots,x_m\}$ be the factor closure of S. Then

$$\det\left(1/[S^r]\right)=\frac{1}{x_1^{2r}x_2^{2r}\ldots x_n^{2r}}\sum_{1\le k_1<k_2<\ldots<k_n\le m}\left(\det E_{(k_1,k_2,\ldots,k_n)}\right)^2 J_r(x_{k_1})J_r(x_{k_2})\ldots J_r(x_{k_n}),\tag{4}$$

where $E_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of E consisting of the k_1th, k_2th, ..., k_nth columns of E.

Proof. From Theorem 1, $1/[S^r]=AA^T$. By Cauchy-Binet formula [6], we have

$$\det\left(1/[S^r]\right)=\sum_{1\le k_1<k_2<\ldots<k_n\le m}\left(\det A_{(k_1,k_2,\ldots,k_n)}\right)^2.$$

Therefore, since $\det A_{(k_1,k_2,\ldots,k_n)}=\frac{1}{x_1^r x_2^r\ldots x_n^r}\det E_{(k_1,k_2,\ldots,k_n)}\left(J_r(x_{k_1})J_r(x_{k_2})\ldots J_r(x_{k_n})\right)^{1/2}$,

the proof is completed.

Corollary 3. If $S=\{x_1,x_2,...,x_n\}$ is factor closed, then

$$\det\left(1/[S^r]\right)=\prod_{i=1}^{n}\frac{J_r(x_i)}{x_i^{2r}}.$$

Proof. Since $S=\{x_1,x_2,...,x_n\}$ is factor closed, $\overline{S}=S$ and $E=(e_{ij})$ given by (3) is an $n\times n$ $(0-1)$ matrix having diagonal $(1,1,\ldots,1)$. Then the result is immediate.

Corollary 4. If $S=\{x_1,x_2,...,x_n\}$ is factor closed, then $1/[S^r]$ is invertible.

Proof. From Corollary 3,

$$\det(1/[S^r]) = \prod_{i=1}^{n} \frac{J_r(x_i)}{x_i^{2r}}.$$

Therefore, since $J_r(x_i) \neq 0$ for every $x_i \in S$, $1/[S^r]$ is invertible.

Definition 2. ([5], DEFINITION 1) Given any set S of positive integers, define the class of arithmetical functions

$$C_S = \{f : (f * \mu)(d) > 0 \text{ whenever } d \mid x \text{ for any } x \in S\}.$$

Lemma 1. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and J_r be Jordan's totient function. If r is a positive real number, then $J_r \in C_S$.

Proof. Since J_r and μ, Möbius function are multiplicative functions, $J_r * \mu$, Dirichlet convolution of J_r and μ is multiplicative. Let m be a positive integer such that $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ for distinct primes $p_1, p_2, \ldots, p_k$. Then

$$(J_r * \mu)(m) = (J_r * \mu)(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}) = (J_r * \mu)(p_1^{\alpha_1})(J_r * \mu)(p_2^{\alpha_2}) \ldots (J_r * \mu)(p_k^{\alpha_k}).$$

Since $r > 0$, it is clear that $(J_r * \mu)(p_i^{\alpha_i}) = J_r(p_i^{\alpha_i}) - J_r(p_i^{\alpha_i - 1}) > 0$ for $i = 1, 2, \ldots, k$. Thus $(J_r * \mu)(m) > 0$ for every positive integer m. From the Definition 2, $J_r \in C_S$.

Theorem 3. ([5], Theorem 3) Let f be a multiplicative function. If

$$(\frac{1}{f} * \mu) \in C_S,$$

then each of the following is true:

(i) $(f[x_i, x_j])$ is positive definite,

(ii) $\prod_{i=1}^{n} [f(x_i)]^2 (\frac{1}{f} * \mu)(x_i) \leq \det(f[x_i, x_j]) \leq \prod_{i=1}^{n} f(x_i)$,

(iii) $\det(f[x_i, x_j]) = \prod_{i=1}^{n} [f(x_i)]^2 (\frac{1}{f} * \mu)(x_i)$ if and only if S is factor closed.

Theorem 4. Let $1/[S^r]$ be the generalized LCM matrix on $S = \{x_1, x_2, \ldots, x_n\}$. If $r > 0$, then each of the following is true:

(i) $1/[S^r]$ is positive definite,

(ii) $\prod_{i=1}^{n} \frac{J_r(x_i)}{x_i^{2r}} \leq \det(1/[S^r]) \leq \prod_{i=1}^{n} \frac{1}{x_i^r}$.

Proof. Let r be a positive real number and f be an arithmetical function such that

$$f(n) = \frac{1}{n^r}$$

for every positive integer n. Then

$$\frac{1}{f}*\mu = J_r,$$

where J_r is Jordan's totient function and $\frac{1}{f}$ is an arithmetical function defined as

$$\frac{1}{f}(n) = \begin{cases} 0 & \text{if } f(n)=0, \\ \frac{1}{f(n)} & \text{otherwise.} \end{cases}$$

Then by Lemma 1 and Theorem 3, (i) and (ii) are obtained.

Theorem 5. Let $S=\{x_1,x_2,\ldots,x_n\}$ be a set of distinct positive integers. If S is factor closed, then the inverse of $1/[S^r]$ on S is the matrix $B=(b_{ij})$, where

$$b_{ij} = x_i^r x_j^r \sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{1}{J_r(x_k)} \mu(x_k/x_i)\mu(x_k/x_j).$$

Proof. From Corollary 1 $1/[S^r] = F_r \Lambda_r F_r^T$. Since S is factor closed, the matrix $F_r = (f_{ij})$ is an $n\times n$ matrix given by

$$f_{ij} = \begin{cases} \frac{1}{x_i^r} & \text{if } d_j \mid x_i, \\ 0 & \text{otherwise.} \end{cases}$$

and $\Lambda_r = diag(J_r(d_1), J_r(d_2), \ldots, J_r(d_m))$. Let the $n\times n$ matrix $W_r = (w_{ij})$ be defined as follows:

$$w_{ij} = \begin{cases} x_j^r \mu(x_i/x_j) & \text{if } x_j \mid x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Calculating the ij – entry of the product $F_r W_r$ gives

$$(F_r W_r)_{ij} = \sum_{k=1}^{n} f_{ik} w_{kj} = \sum_{\substack{x_k \mid x_i \\ x_j \mid x_k}} \frac{1}{x_i^r} x_j^r \mu(x_i/x_j) = \frac{x_j^r}{x_i^r} \sum_{x_k \mid \frac{x_i}{x_j}} \mu(x_k) = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$$

Hence $W_r = F_r^{-1}$. Therefore, since $1/[S^r] = F_r \Lambda_r F_r^T$ we have

$$b_{ij} = \left(W_r^T \Lambda_r^{-1} W_r\right)_{ij} = \sum_{k=1}^{n} \frac{1}{J_r(x_k)} w_{ki} w_{kj} = x_i^r x_j^r \sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{1}{J_r(x_k)} \mu(x_k/x_i)\mu(x_k/x_j).$$

Thus, the theorem is proved.

Example 1. Let $S=\{1,2,3\}$ and $r=1/2$. Then the generalized reciprocal LCM matrix is:

$$1/[S^{1/2}]=\begin{bmatrix} 1 & \sqrt{2}/2 & \sqrt{3}/3 \\ \sqrt{2}/2 & \sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{6}/6 & \sqrt{3}/3 \end{bmatrix}.$$

By Theorem 5, the inverse of $1/[S^{1/2}]$ is the matrix $B=(b_{ij})$, where

$b_{11}=(5+2\sqrt{2}+\sqrt{3})/2$, $b_{12}=-2-\sqrt{2}$, $b_{13}=-(3+\sqrt{3})/2$,

$b_{22}=2+2\sqrt{2}$, $b_{23}=0$, $b_{33}=(3+3\sqrt{3})/2$.

Therefore, since $B=(b_{ij})$ is symmetric, the inverse of $1/[S^{1/2}]$ is

$$\left(1/[S^{1/2}]\right)^{-1}=\begin{bmatrix} (5+2\sqrt{2}+\sqrt{3})/2 & -2-\sqrt{2} & -(3+\sqrt{3})/2 \\ -2-\sqrt{2} & 2+2\sqrt{2} & 0 \\ -(3+\sqrt{3})/2 & 0 & (3+3\sqrt{3})/2 \end{bmatrix}.$$

3. A FACTORIZATION OF THE LCM MATRIX

In this section, we show that the reciprocal LCM matrix is the product of an integral matrix and the matrix $1/[S^r]=\left(1/[x_i,x_j]^r\right)_{n\times n}$. Also we present a relation between the determinants of the reciprocal LCM matrix and $1/[S^r]=\left(1/[x_i,x_j]^r\right)_{n\times n}$.

Lemma 2. Let m, u, and r be positive integers and $t=u^r/(m,u)^r$. If $m=p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ is a product of distinct positive primes p_i, then we have the following

$$\psi(m,u)=\sum_{d|m}d^r[u,d]^r\mu(m/d)=\begin{cases} J_r(m) & \textit{if } p_i^{\alpha_i}|u \textit{ for some } 1\le i\le k, \\ tJ_{2r}(m) & \textit{otherwise.}\end{cases}$$

Proof. Let $\psi(m,u)=u^r f(m)$, where

$$f(m)=\sum_{d|m}\frac{d^{2r}}{(u,d)^r}\mu(m/d).$$

For a prime power of p^α $(\alpha\ge 1)$ we have

$$f(p^\alpha)=\frac{p^{2r\alpha}}{(u,p^\alpha)^r}-\frac{p^{2r(\alpha-1)}}{(u,p^{\alpha-1})^r}=\begin{cases} J_r(p^\alpha) & \text{if } p^\alpha|u, \\ \dfrac{J_{2r}(p^\alpha)}{(u,p^\alpha)^r} & \text{otherwise.}\end{cases}$$

Since f is a multiplicative arithmetical function, we have

$$\psi(m,u)=\sum_{d|m}d^r[u,d]^r\mu(m/d)=\begin{cases} J_r(m) & \text{if } p_i^{\alpha_i}|u \text{ for some } 1\le i\le k, \\ tJ_{2r}(m) & \text{otherwise.}\end{cases}$$

The proof is completed.

Theorem 6. Let $S=\{x_1,x_2,...,x_n\}$ be a set of distinct positive integers and r be a positive integer. If S is factor closed, then the matrix $[S^r]=\left([x_i,x_j]^r\right)_{n\times n}$ defined on S is the product of an integral matrix and the matrix $1/[S^r]=\left(1/[x_i,x_j]^r\right)_{n\times n}$.

Proof. By Theorem 5, the inverse of $1/[S^r]$ is the matrix $B=(b_{ij})$, where

$$b_{ij}=x_i^r x_j^r \sum_{\substack{x_i|x_k\\ x_j|x_k}} \frac{1}{J_r(x_k)} \mu(x_k/x_i)\mu(x_k/x_j).$$

The ij – entry of the product $[S^r]B$ is:

$$\begin{aligned} v_{ij}=\left([S^r]B\right)_{ij} &= \sum_{m=1}^{n}[x_i,x_m]^r \sum_{\substack{x_m|x_k\\ x_j|x_k}} \frac{x_m^r x_j^r}{J_r(x_k)}\mu(x_k/x_m)\mu(x_k/x_j) \\ &= \sum_{x_j|x_k}\frac{x_j^r}{J_r(x_k)}\mu(x_k/x_j)\sum_{x_m|x_k}x_m^r[x_i,x_m]^r\mu(x_k/x_m) \\ &= \sum_{x_j|x_k}\frac{x_j^r}{J_r(x_k)}\mu(x_k/x_j)\psi(x_k,x_i), \end{aligned}$$

where

$$\psi(x_k,x_i)=\sum_{x_m|x_k}x_m^r[x_i,x_m]^r\mu(x_k/x_m).$$

Thus $[S^r]=V\left(1/[S^r]\right)$, where $V=(v_{ij})$ is the $n\times n$ matrix. By Lemma 2 and the fact that $J_r(m)\,|\,J_{2r}(m)$ for every positive integer m, v_{ij} is an integer. The proof is completed.

Corollary 5. Let $S=\{x_1,x_2,...,x_n\}$ be a set of distinct positive integers and r be a positive integer. If S is factor closed, then the determinant of $V=(v_{ij})$ given in Theorem 6 is:

$$\det V=\prod_{i=1}^{n}\pi_r(x_i)\,x_i^{2r},$$

where

$$\pi_r(m)=\prod_{p|m}(-p^r)$$

for every positive integer m.

Proof. Since S is factor closed, by Corollary 3

$$\det\left(1/[S^r]\right)=\prod_{i=1}^{n}\frac{J_r(x_i)}{x_i^{2r}}.$$

From the fact that $J_r(m)=m^{2r}\pi_{-r}(m)\ J_{-r}(m)$ for every positive integer m, it is clear that

$$\det[S^r] = \left(\prod_{i=1}^{n} \pi_r(x_i)\, x_i^{2r}\right) \det(1/[S^r]).$$

By Theorem 6, the result is immediate.

Example 2. Let $S = \{1, 2, 3, 4\}$ and $r = 5$. Then we have

$$[S^5] = \begin{bmatrix} 1 & 32 & 243 & 1024 \\ 32 & 32 & 7776 & 1024 \\ 243 & 7776 & 243 & 248832 \\ 1024 & 1024 & 248832 & 1024 \end{bmatrix}$$

and

$$1/[S^5] = \begin{bmatrix} 1 & 1/32 & 1/243 & 1/1024 \\ 1/32 & 1/32 & 1/7776 & 1/1024 \\ 1/243 & 1/7776 & 1/243 & 1/248832 \\ 1/1024 & 1/1024 & 1/248832 & 1/1024 \end{bmatrix}.$$

Then by Theorem 6, the matrix $V = (v_{ij})$ is

$$V = \begin{bmatrix} -276 & -32736 & 59292 & 1081344 \\ -7808 & -32768 & 1897344 & 1081344 \\ -8019 & -7954848 & 59049 & 262766592 \\ -249856 & 0 & 60715008 & 1048576 \end{bmatrix}$$

such that $[S^5] = V\ 1/[S^5]$. Therefore, by Corollary 5, we have

$$\det V = \pi_5(1)\,\pi_5(2)\,\pi_5(3)\,\pi_5(4)\ 1^{10}\, 2^{10}\, 3^{10}\, 4^{10} = -15776790092376440832.$$

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