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ON KENMOTSU MANIFOLD

B. PRASAD AND R.K. VERMA

Dedicated to Prof.Dr. M.C. CHAKI on his 90th birthday

ABSTRACT. The object of this paper is to study a type of Kenmotsu manifold called Kenmotsu $(GR)_n$-manifold and Kenmotsu $G(PRS)_n$ manifold $(n > 2)$. W_4-curvature tensor on Kenmotsu manifold have been also studied.

1. INTRODUCTION

Let $M = M^{2m+1}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the associated Riemannian metric on M. Then by definition [1], we have

$$\phi^2 + I = \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \eta \circ \phi = 0 \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \tag{1.2}$$

for all vector fields X, Y tangent to M and I is the identity tensor field.

If we further have, for any vector fields X, Y, Z tangent to M,

$$(\nabla_X \phi)(Y) = -\eta(Y)\phi X - g(X, \phi Y)\xi \tag{1.3}$$

where ∇ is the Riemannian connection in M, then M is known as Kenmotsu manifold [2]. From (1.3), we get

$$(\nabla_X \xi) = X - \eta(X)\xi. \tag{1.4}$$

Let R be the curvature of the connection ∇. Then a Kenmotsu M is of constant ϕ-holomorphic sectional curvature C (K. Kenmotsu [2])

$$\begin{aligned} R(X,Y)Z \quad &= \frac{C-3}{4}[g(Y,Z)X - g(X,Z)Y] + \frac{C+1}{4}[\eta(X)\eta(Z)Y \\ &-\eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi \\ &+g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z]. \end{aligned} \tag{1.5}$$

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Now,

$$
\begin{aligned}
{}'R(X,Y,Z,W) &= \frac{C-3}{4}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\
&+\frac{C+1}{4}[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\
&+\eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) \\
&+g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\
&+2g(X,\phi Y)g(\phi Z,W)],
\end{aligned}
\tag{1.6}
$$

where $'R$ is the curvature tensor of type $(0,4)$ of M.

From (1.6) we obtain

$$'R(X,Y,Z,\xi) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{1.7}$$

which gives

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X. \tag{1.8}$$

Also on contraction of (1.5), we have Ricci tensor and scalar curvature respectively as follows

$$
\begin{aligned}
&\text{a) } Ric(Y,Z) = \frac{(C+1)(n+1)}{4}g(\phi Y,\phi Z) - (n-1)g(Y,Z). \\
&\text{b) } r = \frac{n+1}{4}[(C+1)(n+1) - 4n].
\end{aligned}
\tag{1.9}
$$

From (1.9), we get

$$Ric(Y,\xi) = -(n-1)\eta(Y). \tag{1.10}$$

The following W_4-curvature tensor is defined [3].

$$W_4(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}[g(X,Z)\,QY - g(X,Y)QZ] \tag{1.11}$$

where Q is the field of symmetric endomorphism corresponding to the Ricci tensors, i.e.,

$$g(QX,Y) = Ric(X,Y).$$

From (1.11), we get

$$W_4(X,Y,Z,U) = {}'R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(X,Y)Ric(Z,U)] \tag{1.12}$$

where

$$'W_4(X,Y,Z,U) = g(W_4(X,Y,Z),U)$$

and

$$'R(X,Y,Z,U) = g(R(X,Y,Z),U).$$

In recent paper De, Guha and Kamilya [4] introduced and studied a type of Riemannian manifold (M^n,g) $(n>2)$ whose Ricci tensor Ric of type $(0,2)$ satisfies the condition

$$(\nabla_X Ric)(Y,Z) = A(X)Ric(Y,Z) + B(X)g(Y,Z) \tag{1.13}$$

where A and B are two 1-forms, B is non-zero, P, Q are two vector fields such that,

$$g(X, P) = A(X) \tag{1.14}$$

$$g(X, Q) = B(X) \tag{1.15}$$

for every vector fields X.

Such a manifold were called by them a generalized Ricci-recurrent manifold and an n-dimensional manifold of this kind were denoted by $(GR)_n$.

On the other hand in 1993 M.C. Chaki and S. Koley [5] introduced another type of non-flat Riemannian manifolds (M^n, g) $(n > 2)$, whose Ricci tensor of the type $(0, 2)$ satisfies the condition

$$(\nabla_X Ric)(Y, Z) = 2A(X)Ric(Y, Z) + B(Y)Ric(X, Z) + C(Z)Ric(Y, X) \tag{1.16}$$

where A, B, C are three non-zero 1-forms and ∇ denotes the operator of covariant differentiation with respect to g. Such a manifold were called by them a generalized pseudo Ricci symmetric manifold and an n-dimensional manifold were denoted by $G(PRS)_n$.

2. Generalized Ricci-recurrent Kenmotsu manifold admitting codazzi type Ricci-Tensor

We know that

$$(\nabla_X Ric)(Y, Z) = XRic(Y, Z) - Ric(\nabla_X Y, Z) - Ric(Y, \nabla_X Z). \tag{2.1}$$

Therefore from (1.13) and (2.1), we get

$$A(X)Ric(Y, Z) + B(X)g(Y, Z) = XRic(Y, Z) - Ric(\nabla_X Y, Z) - Ric(Y, \nabla_X Z).$$

Putting $Z = \xi$ in above relation, we get

$$A(X)Ric(Y, \xi) + B(X)g(Y, \xi) = XRic(Y, \xi) - Ric(\nabla_X Y, \xi) - Ric(Y, \nabla_X \xi).$$

Using $(1.2), (1.4), (1.10)$ we get

$$-(n-1)\eta(Y)A(X) + B(X)\eta(Y) = -(n-1)(\nabla_X \eta)(Y) - Ric(Y, X - \eta(X)\xi). \tag{2.2}$$

The above equation can also be written as

$$\begin{aligned} -(n-1)\eta(Y)A(X) + B(X)\eta(Y) &= -(n-1)g(\phi Y, X) - Ric(Y, X - \eta(X)\xi) \\ &= -(n-1)g(\phi Y, X) - Ric(Y, X) - (n-1)\eta(X)\eta(Y). \end{aligned}$$

Putting $Y = \xi$ in above relation we have,

$$-[(n-1)A(X) - B(X)]\eta(\xi) = -(n-1)g(\phi\xi, X) - Ric(\xi, X) - (n-1)\eta(X). \tag{2.3}$$

By virtue of (1.1) and $(1.10), (2.3)$ reduces to

$$(n-1)A(X) - B(X) = 0. \tag{2.4}$$

Here we assume a generalized Ricci-recurrent manifold admits codazzi type Ricci-tensor Ric, that is

$$(\nabla_X Ric)(Y, Z) = (\nabla_X Ric)(X, Z). \tag{2.5}$$

Then in virtue of (1.13) it follow from (2.5) that

$$A(X)Ric(Y, Z) + B(X)g(Y, Z) = A(Y)Ric(X, Z) + B(Y)g(X, Z). \tag{2.6}$$

Putting $X = \xi$ in (2.6) we get, by using (1.2) and (1.10), that

$$A(\xi)Ric(Y, Z) + B(\xi)g(Y, Z) = A(Y)Ric(\xi, Z) + B(Y)g(\xi, Z)$$

or

$$A(\xi)Ric(Y,Z) + B(\xi)g(Y,Z) = -[(n-1)A(Y) - B(Y)]\eta(Z). \tag{2.7}$$

In view of (2.4), (2.7) yields

$$A(\xi)Ric(Y,Z) + B(\xi)g(Y,Z) = 0$$

i.e., $Ric(Y,Z) = \mu g(Y,Z)$ where $\mu = -B(\xi) / A(\xi)$.

Therefore we can state the theorem.

Theorem 2.1. *If a generalized Ricci-recurrent Kenmotsu manifold admits codazzi type Ricci tensor, then it becomes an Einstein manifold.*

3. Kenmotsu $G(PRS)_n$ manifold $(n > 2)$

In this section we assume that a $G(PRS)_n$ is Kenmotsu manifold.

Now we have

$$(\nabla_X Ric)(Y,\xi) = \nabla_X Ric(Y,\xi) - Ric(\nabla_X Y,\xi) - Ric(Y,\nabla_X \xi). \tag{3.1}$$

Using (1.10) in (3.1), we get

$$(\nabla_X Ric)(Y,\xi) = -(n-1)g(\phi X, Y) - Ric(Y, \nabla_X \xi). \tag{3.2}$$

From (1.16), we get

$$(\nabla_X Ric)(Y,\xi) = -(n-1)2A(X)\eta(Y) - (n-1)B(Y)\eta(X) + C(\xi)Ric(Y,X). \tag{3.3}$$

From (3.2) and (3.3), we get

$$-(n-1)2A(X)\eta(Y) - (n-1)B(Y)\eta(X) + C(\xi)Ric(Y,X)$$

$$= -(n-1)g(\phi X, Y) - Ric(Y, \nabla_X \xi). \tag{3.4}$$

Putting ξ for X in (3.4), we get

$$-(n-1)2A(X)\eta(Y) - (n-1)B(Y)\eta(X) - (n-1)C(\xi)\eta(Y) = 0. \tag{3.5}$$

Again putting ξ for Y in (3.5), we get

$$2A(\xi) + B(\xi) + C(\xi) = 0. \tag{3.6}$$

Putting the value of $C(\xi)$ from (3.6) in equation (3.5), we get

$$B(\xi)\eta(Y) + B(Y) = 0. \tag{3.7}$$

Putting ξ for Y in equation (3.7), we get

$$B(\xi) = 0. \tag{3.8}$$

Hence from (3.7) and (3.8), we get

$$B(Y) = 0 \tag{3.9}$$

which is inadmissible by the definition of $G(PRS)_n$. Thus we can state the following theorem.

Theorem 3.1. *A $G(PRS)_n$ $(n > 2)$ cannot be Kenmotsu manifold.*

4. W_4-CURVATURE TENSOR IN A KENMOTSU MANIFOLD OF CONSTANT ϕ-HOLOMORPHIC SECTIONAL CURVATURE

Theorem 4.1. *On Kenmotsu manifold of constant ϕ-holomorphic sectional curvature, we have*

$$'W_4(X,Y,Z,\xi) = \eta(Z)g(X,Y) - \eta(X)g(Y,Z) \tag{4.1}$$

$$\begin{aligned} 'W_4(\xi,Y,Z,U) - {}'W_4(\xi,U,Y,Z) &= \eta(U)[-g(Y,Z) + \frac{1}{n-1}Ric(Z,Y) \\ &+\eta(Y)[g(U,Z) - \frac{1}{n-1}Ric(Z,U)] \end{aligned} \tag{4.2}$$

$$'W_4(\xi,Y,Z,U) + {}'W_4(Y,Z,\xi,U) + {}'W_4(Z,\xi,Y,U) = 0. \tag{4.3}$$

Proof. From (1.7), (1.10), (1.12), we get (4.1). Similarly the other results can also proved.

Theorem 4.2. *A W_4-flat Kenmotsu manifold is a manifold of constant Riemannian curvature.*

Proof. From (1.12), we get

$$'W_4(\xi,Y,Z,U) = {}'R(\xi,Y,Z,U) + \frac{1}{n-1}[g(\xi,Z)Ric(Y,U) - g(\xi,Y)Ric(Z,U)]. \tag{4.4}$$

For such a space $W_4(\xi,Y,Z,U) = 0$. Consequently from (4.4), we have

$$(n-1)\,'R(\xi,Y,Z,U) = g(\xi,Y)Ric(Z,U)g(\xi,Z)Ric(Y,U) \tag{4.5}$$

or

$$(n-1)[\eta(Z)g(Y,U) - \eta(U)g(Y,Z)] = \eta(Y)Ric(Z,U) - \eta(Z)Ric(Y,U).$$

Putting ξ for Z in above equation, we get

$$Ric(Y,U) = -(n-1)g(Y,U), \tag{4.6}$$

which shows that Kenmotsu manifold is an Einstein manifold. Now using (4.6) in (4.5), we find

$$'R(\xi,Y,Z,U) = \eta(Y)g(Z,U) - \eta(Z)g(Y,U)$$

or

$$R(Y,Z,U) = \eta(Y)Z - \eta(Z)Y.$$

Hence the manifold is of constant Riemannian curvature -1.

Theorem 4.3. In a W_4-flat η-Einstein Kenmotsu manifold the scalar curvature is given by $-n(n-1)$.

Proof. Since manifold is η-Einstein, there exist functions a and b such that

$$Ric(X,Y) = a\,g(X,Y) + b\,\eta(X)\eta(Y). \tag{4.7}$$

In consequence of (4.7) and (1.10), we have

$$a + b = -2m \text{ and } r = (2m+1)a + b,$$

which yields

$$a = \frac{r+2m}{2m}, \quad b = \frac{-r-2m(2m+1)}{2m}. \tag{4.8}$$

From (1.12) and (4.7), we have

$$\begin{aligned} 2m' R(X,Y,Z,U) &= a[g(X,Y)g(Z,U) - g(X,Z)g(Y,U)] \\ &+b[g(X,Y)\eta(Z)\eta(U) - g(X,Z)\eta(Y)\eta(U)]. \end{aligned} \tag{4.9}$$

Putting ξ for X and Z and using (1.7) , (1.2), we find

$$(a + 2m)\{g(Y,U) - \eta(Y)\eta(U)\} = 0 \tag{4.10}$$

which implies $a = -2m$. This yields $r = -n(n-1)$, and hence the theorem.

ÖZET:Bu çalışmanın amacı, Kenmotsu $(GR)_n$-manifoldu olarak adlandırılan Kenmotsu tipi bir manifold ile, Kenmotsu $G(PRS)_n$ $(n > 2)$ manifoldunu incelenmektir. Ayrıca, Kenmotsu manifoldu üzerindeki W_4-eğrilik tensörü de incelenmiştir.

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Current address: Department of Mathematics, S.M.M. Town (P.G.) College, Ballia-277001, (U.P.), INDIA

E-mail address: bp_kushawa@sify.com