Commun.Fac.Sci.Univ.Ank.Series A1
Volume 55, Number 1, Pages 1-6(2006)
ISSN 1303-5991

ORE EXTENSIONS OF ZIP AND REVERSIBLE RINGS

MUHITTIN BAŞER

ABSTRACT. We investigate Ore extensions zip and reversible rings. Let α be an endomorphism and δ an α-derivation of a ring R. Assume that R is an α-rigid ring. Then (1) R is a right zip ring if and only if the Ore extension $R[x;\alpha,\delta]$ is a right zip ring. (2) R is a reversible ring if and only if the Ore extension $R[x;\alpha,\delta]$ is a reversible ring.

Throughout this work all rings R are associative with identity and modules are unital right R-modules. Given a ring R, the polynomial ring over R is denoted by $R[x]$ with x its indeterminate and $r_R(-)$ ($l_R(-)$) is used for the right (left) annihilator over R. Faith [6] called a ring R *right zip* provided that if the right annihilator $r_R(X)$ of a subset X of R is zero, $r_R(Y) = 0$ for a finite subset $Y \subseteq X$; equivalently, for a left ideal L of R with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. R is *zip* if it is right and left zip. The concept of zip rings initiated by Zelmanowitz [11] appeared in [2],[3],[5],[6], and references there in.

Extensions of zip rings were studied by several authors. Beachy and Blair [2] showed that if R is a commutative zip ring, then the polynomial ring $R[x]$ over R is zip. Hong et al. [7] showed that if R is an Armendariz ring, then R is a right zip ring if and only if $R[x]$ is a right zip ring.

According to Chon [4], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Kim and Lee [9] showed that if R is an Armendariz ring, then R is a reversible ring if and only if $R[x]$ is a reversible ring.

In this paper, we study Ore extensions of zip rings and reversible rings. In particular, we show: Let α be an endomorphism and δ an α-derivation of a ring R. Assume that R is an α-rigid ring. Then (1) R is a right zip ring if and only if the

Received by the editors Dec. 16, 2005, Accepted: Jan. 16, 2006.
1991 *Mathematics Subject Classification.* Primary 16D25,16U80; Secondary 16U10,16P60

Key words and phrases. zip rings, reversible rings.

Ore extension $R[x;\alpha,\delta]$ is a right zip ring. (2) R is a reversible ring if and only if the Ore extension $R[x;\alpha,\delta]$ is a reversible ring.

A ring R is called a *reduced ring* if $a^2=0$ in R always implies $a=0$. Recall that for a ring R with a ring endomorphism $\alpha: R\to R$ and an α-derivation $\delta: R\to R$, the *Ore extension* $R[x;\alpha,\delta]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication

$xr=\alpha(r)x+\delta(r)$

for all $r\in R$. If $\delta=0$, we write $R[x;\alpha]$ for $R[x;\alpha,0]$ and is called an *Ore extension* of endomorphism type (also called a *skew polynomial ring*), while $R[[x;\alpha]]$ is called a *skew power series ring.*

Definition 1 (Krempa [10]) Let α be an endomorphism of R. α is called a *rigid endomorphism* if $a\alpha(a)=0$ implies $a=0$ for $a\in R$. A ring R is called to be *α-rigid* if there exists a rigid endomorphism α of R.

Note that α-rigid rings are reduced rings. If R is an α-rigid ring and $r^2=0$ for $r\in R$, then $r\alpha(r)\alpha(r\alpha(r))=r\alpha(r^2)\alpha^2(r)=0$. Thus $r\alpha(r)=0$ and so $r=0$. Therefore, R is reduced.

In this paper, we let α be an endomorphism of R and δ an α-derivation of R, unless especially noted. We need the following lemmas:

Lemma 2 ([8, Lemma 4]) *Let R be an α-rigid ring and $a,b\in R$. Then we have the following:*

(i) *If $ab=0$ then $a\alpha^n(b)=\alpha^n(a)b=0$ for any positive integer n.*

(ii) *If $ab=0$ then $a\delta^m(b)=\delta^m(a)b=0$ for any positive integer m.*

(iii) *If $a\alpha^k(b)=0=\alpha^k(a)b$ for some positive integer k, then $ab=0$.*

Lemma 3 ([8, Proposition 6]) *Suppose that R is an α-rigid ring. Let $p=\sum_{i=0}^{m}a_ix^i$ and $q=\sum_{j=0}^{n}b_jx^j$ in $R[x;\alpha,\delta]$. Then $pq=0$ if and only if $a_ib_j=0$ for all $0\le i\le m$, $0\le j\le n$.*

The following theorem extends [7, Theorem 11].

Theorem 4. *Let R be an α-rigid ring. Then R is a right zip ring if and only if $R[x;\alpha,\delta]$ is a right zip ring.*

Proof. Suppose that $R[x,\alpha,\delta]$ is right zip. Let $Y\subseteq R$ with $r_R(Y)=0$. If $f(x)=a_0+a_1x+\ldots+a_nx^n\in r_{R[x;\alpha,\delta]}(Y)$, then $bf(x)=ba_0+ba_1x+\ldots+ba_nx^n=0$ for all $b\in Y$. Thus $ba_i=0$, and so $a_i\in r_R(Y)=0$ for all i. Therefore, $f(x)=0$ and hence $r_{R[x;\alpha,\delta]}(Y)=0$. Since $R[x;\alpha,\delta]$ is right zip, there exists a finite subset

$Y_0 \subseteq Y$ such that $r_{R[x;\alpha,\delta]}(Y_0) = 0$. Thus $r_R(Y_0) = r_{R[x;\alpha,\delta]}(Y_0) \cap R = 0 \cap R = 0$. Consequently, R is a right zip ring.

Conversely, assume that R is a right zip ring. Let $X \subseteq R[x;\alpha,\delta]$ with $r_{R[x;\alpha,\delta]}(X) = 0$. Now let Y be the set of all coefficients of elements in X. Then $Y \subseteq R$. If $a \in r_R(Y)$, then $ba = 0$ for all $b \in Y$. By Lemma 2, $f(x)a = 0$ for all $f(x) \in X$, and so $a \in r_{R[x;\alpha,\delta]}(X) = 0$. That is $r_R(Y) = 0$. Since R is a right zip, there exists a finite subset $Y_0 \subseteq Y$ such that $r_R(Y_0) = 0$. For each $a \in Y_0$, there exists $h_a(x) \in X$ such that at least one of the coefficients of $h_a(x)$ is a. Let X_0 be a minimal subset of X such that $h_a(x) \in X_0$ for each $a \in Y_0$. Then X_0 is a nonempty finite subset of X. Let Y' be the set of all coefficients of elements in X_0. Then $Y_0 \subseteq Y'$ and so $r_R(Y') \subseteq r_R(Y_0) = 0$. If $f(x) = a_0 + a_1x + \ldots + a_kx^k \in r_{R[x;\alpha,\delta]}(X_0)$, then $g(x)f(x) = 0$ for all $g(x) = b_0 + b_1x + \ldots + b_tx^t \in X_0$. Since R is α-rigid, then $b_ia_j = 0$ for all i and j, by Lemma 3. Thus $a_j \in r_R(Y') = 0$ for all j, and so $f(x) = 0$. Hence $r_{R[x;\alpha,\delta]}(X_0) = 0$ and therefore $R[x;\alpha,\delta]$ is a right zip ring. □

Corollary 5. *Let R be an α-rigid ring. Then R is a right zip ring if and only if $R[x;\alpha]$ is a right zip ring.*

Corollary 6. *Let R be a reduced ring. Then R is a right zip ring if and only if $R[x]$ is a right zip ring.*

Theorem 7. *Let R be an α-rigid ring. Then R is a right zip ring if and only if $R[[x;\alpha]]$ is a right zip ring.*

Proof. Similar to the proof of Theorem 4 by using Lemma 2 and [8, Proposition 17]. □

Corollary 8. *Let R be a reduced ring. Then R is a right zip ring if and only if $R[[x]]$ is a right zip ring.*

Example 9. Let $R = \mathbb{Z}_2[y]/(y^2)$, where (y^2) is a principal ideal generated by y^2 of the polynomial ring $\mathbb{Z}_2[y]$. Since R is finite and commutative, R is a zip ring. Now, let α be the identity map on R and we define an α-derivation δ on R by $\delta(y + (y^2)) = 1 + (y^2)$. Then R is not α-rigid since R is not reduced. However, by [1, Example 11] we get

$$R[x;\alpha,\delta] = R[x;\delta] \cong \mathrm{Mat}_2(\mathbb{Z}_2[y^2]) \cong \mathrm{Mat}_2(\mathbb{Z}_2[t]).$$

Since $\mathbb{Z}_2$ is Armendariz and zip rings, $\mathbb{Z}_2[t]$ is a zip ring by [7, Theorem 11] and so $\mathrm{Mat}_2(\mathbb{Z}_2[t])$ is a zip ring by [3, Proposition 1]. Therefore $R[x;\alpha,\delta]$ is a zip ring.

In the following we obtain more examples of zip rings. Let R be an algebra over a commutative ring S. Recall that the *Dorroh extension* of R by S is the ring $R \times S = D(R, S)$ with operations

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$$
$$(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2)$$

where $r_1, r_2 \in R$ and $s_1, s_2 \in S$. Let R be a commutative ring, M be an R-module and σ be an endomorphism of R. Give $R \oplus M = N(R, M)$ a (possibly noncommutative) ring structure with multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1) m_2 + r_2 m_1)$$

where $r_1, r_2 \in R$ and $m_1, m_2 \in M$. We shall call this extension the *Nagata extension* of R by M and σ.

Proposition 10. *Let R be a commutative zip ring. Then the Nagata extension of R by R is a left zip ring.*

Proof. Assume that R is a zip ring and $X \subseteq N(R, R)$ with $l_{N(R,R)}(X) = 0$. Let $Y = \{x \in R \mid (x, y) \in X\} \subseteq R$. If $b \in r_R(Y)$ then $(0, b)(x, y) = (0x, \sigma(0)y + xb) = (0, 0)$ for any $(x, y) \in X$. Thus $(0, b) \in l_{N(R,R)}(X) = 0$ and so $b = 0$. Therefore $r_R(Y) = 0$. Since R is a right zip, there exists a finite subset $Y_0 = \{x_1, x_2, \ldots, x_m\} \subseteq Y$ such that $r_R(Y_0) = 0$. Let $X_0 = \{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m) \mid (x_i, y_i) \in X,\ 1 \leq i \leq m\} \subseteq X$. If $(a, b) \in l_{N(R,R)}(X_0)$ then $(a, b)(x_i, y_i) = (0, 0)$ for all $(x_i, y_i) \in X_0$. Thus $(0, 0) = (a, b)(x_i, y_i) = (ax_i, \sigma(a)y_i + x_i b)$. So $ax_i = 0$ and $\sigma(a)y_i + x_i b = 0$. Then $a \in l_R(Y_0) = r_R(Y_0) = 0$ since R is commutative. Hence $\sigma(a)y_i + x_i b = x_i b = 0$ and so $b \in r_R(Y_0) = 0$. Consequently $l_{N(R,R)}(X_0) = 0$ and therefore $N(R, R)$ is a left zip ring. □

Proposition 11. *Let R be a commutative ring with $2^{-1} \in R$. If the Dorroh extension of R by R is a right zip ring then R is also right zip ring.*

Proof. Assume $D(R, R)$ is a right zip ring and $X \subseteq R$ with $r_R(X) = 0$. Let $Y = \{(x, x) \mid x \in X\} \subseteq D(R, R)$. If $(a, b) \in r_{D(R,R)}(Y)$, then $(x, x)(a, b) = (0, 0)$ for all $x \in X$. Thus $(xa + xa + bx, xb) = (0, 0)$. So $2xa + bx = 0$ and $xb = 0$. Thus $b \in r_R(X) = 0$ and hence $b = 0$. Therefore $2xa = 0$. By hypothesis $xa = 0$ and so $a \in r_R(X) = 0$ and hence $a = 0$. Consequently, $r_{D(R,R)}(Y) = 0$. Since $D(R, R)$ is a right zip ring, there exists a finite subset $Y_0 = \{(x_1, x_1), (x_2, x_2), \ldots, (x_m, x_m)\} \subseteq Y$ such that $r_{D(R,R)}(Y_0) = 0$. Let $X_0 = \{x_1, x_2, \ldots, x_m\} \subseteq X$. If $c \in r_R(X_0)$, then $(x_i, x_i)(c, 0) = (x_i c + x_i c + 0x_i, x_i 0) = (0, 0)$ for $1 \leq i \leq m$. Thus $(c, 0) \in$

$r_{D(R,R)}(Y_0) = 0$ and so $c = 0$. Therefore $r_R(X_0) = 0$ and so R is a right zip ring. □

Lemma 12. *Let R be a reversible ring. Then R is a right zip ring if and only if R is a left zip ring.*

Proof. Clear. □

Lemma 13. *If R is reduced and $R[x;\alpha]$ is reversible then R is an α-rigid ring.*

Proof. If $a\alpha(a) = 0$ for $a \in R$, then $(ax)(a + ax) = a\alpha(a)x + a\alpha(a)x^2 = 0$. Since $R[x;\alpha]$ is reversible $0 = (a + ax)(ax) = a^2x + a\alpha(a)x^2$ and so $a^2 = 0$. Thus $a = 0$ since R is reduced. Consequently R is an α-rigid ring. □

For any ring R, if $R[x;\alpha,\delta]$ is a reversible ring then R is also reversible. One may conjecture that if a ring R is reversible then $R[x;\alpha,\delta]$ is also reversible. However there may be a counterexample for this as follows.

Example 14. Let $R = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and define $\alpha : R \to R$ by $\alpha((a,b)) = (b,a)$ for $a, b \in \mathbb{Z}_3$, where $\mathbb{Z}_3$ is the ring of integers modulo 3. Then α is an automorphism of R. Thus R is reversible and R is not α-rigid. Take $f(x) = (1,0) + (0,1)x$ and $g(x) = (0,1) + (0,1)x$ in $R[x;\alpha]$. Then

$$f(x)g(x) = (1,0)(0,1) + ((1,0)(0,1) + (0,1)(1,0))x + (0,1)(1,0)x^2 = (0,0)$$

but

$$g(x)f(x) = (0,1)(1,0) + ((0,1)(0,1) + (0,1)(0,1))x + (0,1)(1,0)x^2 = (0,1)x \neq 0$$

Hence $R[x;\alpha]$ is not reversible.

Moreover, this example shows that the condition "R is α-rigid" in the following theorem is not superflous.

Theorem 15. *Let R be an α-rigid ring. Then R is a reversible ring if and only if $R[x;\alpha,\delta]$ is a reversible ring.*

Proof. Assume $R[x;\alpha,\delta]$ is a reversible ring. Since class of reversible rings is closed under subrings, R is a reversible ring.

Conversely, assume that R is a reversible ring. Since R is α-rigid, $R[x;\alpha,\delta]$ is a reduced ring by [8, Proposition 5]. Therefore $R[x;\alpha,\delta]$ is a reversible ring. □

Theorem 16. *Let R be an α-rigid ring. Then R is a reversible ring if and only if $R[[x;\alpha]]$ is a reversible ring.*

Proof. Proof is clear by [8, Corollary 18]. □

ÖZET: Bu makalede zip ve reversible halkaların Ore genişlemeleri çalışılmıştır. R bir halka, α; R nin bir endomorfizması ve δ bir α-türev olmak üzere; aşağıdakiler ispatlanmıştır. (1) R nin bir sağ zip halka olması için gerek ve yeter koşul $R[x;\alpha,\delta]$ nın bir sağ zip halka olmasıdır. (2) R nin bir reversible halka olması için gerek ve yeter koşul $R[x;\alpha,\delta]$ nın bir reversible halka olmasıdır.

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Current address: A*Department of Mathematics, Faculty of Science and Arts, Kocatepe University, ANS Campus TR-03200, Afyon - Turkey*
E-mail address: mbaser@aku.edu.tr
URL: http://math.science.ankara.edu.tr