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# ON THE CLASSIFICATION OF FUZZY PROJECTIVE LINES OF FUZZY 3-DIMENSIONAL PROJECTIVE SPACE

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ABSTRACT. n this work, the classifications of fuzzy vector planes of fuzzy 4−dimensional vector space and fuzzy projective lines of fuzzy 3−dimensional projective space from fuzzy 4−dimensional vector space are given.

## 1. Introduction and Preliminaries

A general definition of a fuzzy $n-$dimensional projective space $\lambda$ which is obtained from fuzzy $(n+1)-$dimensional vector space $V$ over some field $K$ and a method to find a fuzzy projective line and a fuzzy projective plane are given in [3]. Firstly, the classification of fuzzy vector planes of fuzzy 4−dimensional vector space are introduced. And then we give the classification of the fuzzy projective lines of fuzzy 3−dimensional projective space, from fuzzy 4−dimensional vector space.

The following definitions and theorems concerning the basic concepts of the subject has been taken from [3] with some small modifications.

**Definition 1.1.** Let $\lambda : V \to [0,1]$ be a fuzzy set on $V$. Then we call $\lambda$ a fuzzy vector space on $V$ if and only if $\lambda(a.\overline{u}+b.\overline{v}) \geq \lambda(\overline{u}) \wedge \lambda(\overline{v})$, $\forall \overline{u}, \overline{v} \in V$ and $\forall a, b \in K$.

**Proposition 1.** *Let $V$ be a vector space over some field $K$, $\overline{u}, \overline{v} \in V$ and $a \in K \backslash \{0\}$. If $\lambda : V \to [0,1]$ is a fuzzy vector space, then we have:*
*(i)* $\lambda(a.\overline{u}) = \lambda(\overline{u})$;
*(ii)* $\lambda(\overline{o}) = \sup_{\overline{u} \in V} \lambda(\overline{u})$;
*(iii) if* $\lambda(\overline{u}) \neq \lambda(\overline{v})$, *we have* $\lambda(\overline{u}+\overline{v}) = \lambda(\overline{u}) \wedge \lambda(\overline{v})$.

**Definition 1.2.** Let $\lambda$ is a fuzzy vector space on $V$. The subspace $L$, (linearly) generated by $Supp(\lambda)$ $(\sup p(\lambda) = \{x \in V : \lambda(x) = 0\}$, is called the base vector space of $\lambda$. The dimension $d(\lambda)$ of a fuzzy vector space of $V$ is the dimension of its base subspace.

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**Definition 1.3.** If $U$ is an $i-$dimensional subspace of $V$, and $(\lambda, U)$ is a fuzzy vector space, then it is called a fuzzy $i-$dimensional vector space on $U$. If $i = 1$, i.e. $U$ is a vector line, then $(\lambda, U)$ is a fuzzy vector line on $U$, if $i = 2$, i.e. $U$ is a plane, $(\lambda, U)$ will be called a fuzzy vector plane on $U$. If $i = n - 1$, then $(\lambda, U)$ is called a fuzzy vector hyperplane on $U$.

Let $V$ be an $n-$dimensional vector space over some field $K$, with $n \geq 2$. Let $L$ be a vector line of $V$, so $L$ is uniquely defined by some nonzero vector $\overline{u}$. Let $\alpha$ be a vector plane of the $n-$dimensional vector space $V$ $(n \geq 3)$, then we know that $\alpha$ is uniquely defined by two linearly independent vectors $\overline{u}$ and $\overline{v}$.

**Theorem 1.4.** *If $\lambda : L \longrightarrow [0,1]$ is a fuzzy vector line on $L$, then $\lambda(\overline{u}) = \lambda(\overline{v})$, $\forall \overline{u}, \overline{v} \in L \backslash \{\overline{o}\}$, and $\lambda(\overline{o}) \geq \lambda(\overline{u})$, $\forall \overline{u} \in L$.*

**Theorem 1.5.** *If $\lambda : \alpha \longrightarrow [0,1]$ is a fuzzy vector plane on $\alpha$, then there exists a vector line $L$ of $\alpha$ and real numbers $a_0 \geq a_1 \geq a_2 \in [0,1]$ such that $\lambda$ is of the following form:*

$$\begin{aligned} \lambda : \quad & \alpha \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow a_1 \ for \ \overline{u} \in L \backslash \{\overline{o}\} \\ & \overline{u} \longrightarrow a_2 \ for \ \overline{u} \in \alpha \backslash L, \end{aligned}$$

**Definition 1.6.** Suppose $V$ is an $n-$dimensional vector space. A flag in $V$ is a sequence of distinct, non-trivial subspaces $(U_0, U_1, ..., U_m)$ such that $U_j \subset U_i$ for all $j < i < n - 1$. The rank of a flag is the number of subspaces it contains. A maximal flag in $V$ is a flag of length $n$.

## 2. Fuzzy Vector Planes of Fuzzy 4–Dimensional Vector Spaces

In this work, now we classify fuzzy 2-dimensional subspaces of fuzzy 4-dimensional vector spaces to classify fuzzy projective lines of fuzzy 3-dimensional projective space. Since a subspace should not necessarily have the same values (membership degrees different from $a_0$) in its points as the whole space [3], this classification is given in the following theorem.

**Theorem 2.1.** *Let $V$ be a 4-dimensional vector space over some field $K$ and $\lambda : V \longrightarrow [0,1]$ be a fuzzy vector space on $V$. Then the fuzzy 4-dimensional vector space $\lambda$ has exactly six kinds of fuzzy vector planes.*

*Proof.* Let $\lambda : V \longrightarrow [0,1]$ is a fuzzy vector space on $V$ and $(U_0, U_1, U_2, U_3, V)$ is a maximal flag, then there exists a vector plane $\alpha$ of $V$ and a base line $L$ of $\alpha$ and real numbers $a_0 \geq a \geq b \geq c \geq d \in [0,1]$ such that

$$\begin{aligned} \lambda \ : \ & V \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow a \text{ for } \overline{u} \in U_1 \backslash \{U_0\} \\ & \overline{u} \longrightarrow b \text{ for } \overline{u} \in U_2 \backslash U_1 \\ & \overline{u} \longrightarrow c \text{ for } \overline{u} \in U_3 \backslash U_2 \\ & \overline{u} \longrightarrow d \text{ for } \overline{u} \in V \backslash U_3. \end{aligned}$$

The number of points different from zero on a vector line is denoted by $p$, $q$ counts the number of vector lines $L_j$ passing through zero point different from base line in $U_2$, $r$ counts the number of vector lines $L_k$ passing through zero point different from base line in $U_3 \backslash L_j$ and $s$ counts the number of vector lines $L_t$ in $V \backslash U_3$. These fuzzy vector planes of $\lambda$ are of the one of the following forms :

1) Let $\alpha_j$ be 2−dimensional vector spaces,

$$\begin{aligned} \lambda_{ij} \ : \ & \alpha_j \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow a_i \text{ for } \overline{u} \in L \backslash \{\overline{o}\} \\ & \overline{u} \longrightarrow b_{ij} \text{ for } \overline{u} \in \alpha_j \backslash L \end{aligned}$$

such that $a_i \geq b_{ij}$, $i \in \{1,...,p\}$, $j \in \{1,...,q\}$.

2) Let $\beta_k$ be 2−dimensional vector spaces, which

$$\begin{aligned} \mu_{ik} \ : \ & \beta_k \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow a_i \text{ for } \overline{u} \in L \backslash \{\overline{o}\} \\ & \overline{u} \longrightarrow c_{ik} \text{ for } \overline{u} \in \beta_k \backslash L \end{aligned}$$

such that $a_i \geq c_{ik}$, $i \in \{1,...,p\}$, $k \in \{1,...,r\}$.

3) Let $\gamma_t$ be 2−dimensional vector spaces,

$$\begin{aligned} \delta_{it} \ : \ & \gamma_t \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow a_i \text{ for } \overline{u} \in L \backslash \{\overline{o}\} \\ & \overline{u} \longrightarrow d_{it} \text{ for } \overline{u} \in \gamma_t \backslash L \end{aligned}$$

such that $a_i \geq d_{it}$, $i \in \{1,...,p\}$, $t \in \{1,...,s\}$.

4) Let $\alpha_{jk}$ be 2−dimensional vector spaces,

$$\begin{aligned} \psi_{ijk} \ : \ & \alpha_{jk} \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow b_{ij} \text{ for } \overline{u} \in L_j \backslash \{\overline{o}\} \\ & \overline{u} \longrightarrow c_{ik} \text{ for } \overline{u} \in \alpha_{jk} \backslash L_j \end{aligned}$$

such that $b_{ij} \geq c_{ik}$, $i \in \{1,...,p\}$, $j \in \{1,...,q\}$, $k \in \{1,...,r\}$.

5) Let $\beta_{jt}$ be 2−dimensional vector spaces,

$$\begin{aligned} \varphi_{ijt} \ : \ & \beta_{jt} \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow b_{ij} \text{ for } \overline{u} \in L_j \backslash \{\overline{o}\} \\ & \overline{u} \longrightarrow d_{it} \text{ for } \overline{u} \in \beta_{jt} \backslash L_j \end{aligned}$$

such that $b_{ij} \geq d_{it}$, $i \in \{1,...,p\}$, $j \in \{1,...,q\}$, $t \in \{1,...,s\}$.

6) Let $\gamma_{kt}$ be 2−dimensional vector spaces,

$$\begin{aligned} \eta_{ikt} : \ & \gamma_{kt} \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow c_{ik} \text{ for } \overline{u} \in L_j \backslash \{\overline{o}\} \\ & \overline{u} \longrightarrow d_{it} \text{ for } \overline{u} \in \gamma_{kt} \backslash L_j \end{aligned}$$

such that $c_{ik} \geq d_{it}$, $i \in \{1,...,p\}$, $k \in \{1,...,r\}$, $t \in \{1,...,s\}$. Any fuzzy vector plane is in the one of the six classes □

Now, we give an example of two subclasses of fuzzy vector planes from $\lambda_{ij}$ and $\eta_{ikt}$.

**Example 2.2.** For $j = 2$, $k = 2$ and $t = 3$, fuzzy subspaces $\lambda_{i2}$ and $\eta_{i23}$ are given as follows:

$$\begin{aligned} \lambda_{i2} \ : \ & \alpha_2 \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow a_i \text{ for } \overline{u} \in L \backslash \{0\} \\ & \overline{u} \longrightarrow b_{i2} \text{ for } \overline{u} \in \alpha_j \backslash L \end{aligned}$$

such that $a_i \geq b_{i2} \geq$, $i \in \{1,...,p\}$ and

$$\begin{aligned} \eta_{i23} : \ & \gamma_{23} \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow c_{i2} \text{ for all } \overline{u} \in \beta_2 \backslash \{0\} \\ & \overline{u} \longrightarrow d_{i3} \text{ for all } \overline{u} \in \gamma_{23} \backslash \{0\}. \end{aligned}$$

such that $c_{i2} \geq d_{i3}$, $i \in \{1,...,p\}$

## 3. Fuzzy Projective Lines of Fuzzy 3−Dimensional Projective Space

A general definition of fuzzy $n$−dimensional projective space $\lambda'$ is well-known [3]. Here, we restrict ourselves to the case a fuzzy 3−dimensional projective space $\lambda'$ from a fuzzy 4−dimensional vector space $(\lambda, V)$, having following form:

$$\begin{aligned} \lambda \ : \ & V \longrightarrow [0,1] \\ & \overline{o} \longrightarrow a_0 \\ & \overline{u} \longrightarrow a \text{ for } \overline{u} \in U_1 \backslash \{U_0\} \\ & \overline{u} \longrightarrow b \text{ for } \overline{u} \in U_2 \backslash U_1 \\ & \overline{u} \longrightarrow c \text{ for } \overline{u} \in U_3 \backslash U_2 \\ & \overline{u} \longrightarrow d \text{ for } \overline{u} \in V \backslash U_3 \end{aligned} \tag{1}$$

with $U_i$ an $i-$dimensional subspace of $V$, containing all $U_j$ for $j < i$, and $a_0 \geq a \geq b \geq c \geq d$ are reals in $[0, 1]$. We define a fuzzy 3−dimensional projective space $\lambda'$ on $V'$ as follows, where it will be denoted $FPG(3, K)$.

$$\begin{aligned} \lambda' : \quad & V' \to [0,1] \\ & q \to a \\ & p \to b \text{ for } p \in U_1' \backslash \{q\} \\ & p \to c \text{ for } p \in U_2' \backslash U_1' \\ & p \to d \text{ for } p \in V' \backslash U_2' \end{aligned} \tag{2}$$

with $q$ the fuzzy projective point corresponding to the fuzzy vector line $U_1$ in (2) and $U_i'$ the $i$-dimensional projective space, corresponding to the vector space $U_{i+1}$. Then, the sequence $(q, U_1', U_2', V')$ is a maximal flag and $a \geq b \geq c \geq d$ are reals in $[0, 1]$.

The following theorem deals with the classification of fuzzy projective lines of fuzzy 3-dimensional projective space from fuzzy 4-dimensional vector space.

**Theorem 3.1.** *Fuzzy 3-dimensional projective space $\lambda'$ from fuzzy 4-dimensional vector space $\lambda$ over some field $K$ has exactly six kinds of fuzzy projective lines.*

*Proof.* Let $\lambda'$ be fuzzy 3-dimensional projective space on $V'$. Then it is form as follows

$$\begin{aligned} \lambda' : \quad & V' \to [0,1] \\ & q \to a \\ & p \to b \text{ for } p \in U_1' \backslash \{q\} \\ & p \to c \text{ for } p \in U_2' \backslash U_1' \\ & p \to d \text{ for } p \in V' \backslash U_2'. \end{aligned}$$

The fuzzy projective lines of $\lambda'$ are one of the following forms:

1) Let $L_j$ be projective lines corresponding to the vector planes $\alpha_j$, and $q$ be the projective point corresponding to the vector line $L \subseteq \alpha$.

$$\begin{aligned} \lambda'_{ij} : \quad & L_j \to [0,1] \\ & q \to a_i \\ & p \to b_{ij}, \text{ for } p \in L_j \backslash \{q\} \end{aligned}$$

such that $a_i \geq b_{ij}$.

2) Let $M_k$ be projective lines corresponding to the vector planes $\beta_k$, and $q$ be a projective point corresponding to the vector line $L \subseteq \beta_k$.

$$\begin{aligned} \mu'_{ik} : \quad & M_k \to [0,1] \\ & q \to a_i \\ & p \to c_{ik} \text{ for } p \in M_k \backslash \{q\} \end{aligned}$$

such that $a_i \geq c_{ik}$.

3) Let $N_t$ be projective lines corresponding to the vector planes $\gamma_t$, and $q$ be a projective point corresponding to the vector line $L \subseteq \gamma_t$.

$$\begin{aligned} \eta'_{it} \ : \ & N_t \to [0,1] \\ & q \to a_i \\ & p \to d_{it} \text{ for } p \in N_t \backslash \{q\}. \end{aligned}$$

such that $a_i \geq d_{it}$.

4) Let $L_{jk}$ be projective lines corresponding to the vector planes $\alpha_{jk}$.

$$\begin{aligned} \psi'_{ijk} \ : \ & L_{jk} \to [0,1] \\ & q_j \to b_{ij}, \text{ for } q_j \in L \\ & p \to c_{ik} \text{ , for } p \in L_{jk} \backslash \{q_j\}. \end{aligned}$$

such that $b_{ij} \geq c_{ik}$.

5) Let $M_{jt}$ be projective lines corresponding to the vector planes $\beta_{jt}$.

$$\begin{aligned} \varphi'_{ijt} \ : \ & M_{jt} \to [0,1] \\ & q_j \to b_{ij}, \text{ for } q_j \in L \\ & p \to d_{it} \text{ , for } p \in M_{jt} \backslash \{q_j\}. \end{aligned}$$

such that $b_{ij} \geq d_{it}$.

6) Let $N_{kt}$ be projective lines corresponding to the vector planes $\gamma_{kt}$.

$$\begin{aligned} \eta'_{ikt} \ : \ & N_{kt} \to [0,1] \\ & p \to c_{ik}, \text{ for } p \in L_j \\ & p \to d_{it} \text{ , for } p \in N_{kt} \backslash L_j. \end{aligned}$$

such that $c_{ik} \geq d_{it}$.

One can easly see that any fuzzy projective line is in one of above the six classes □

**Example 3.2.** If we consider the subclasses $\lambda_{i2}$ and $\eta_{i23}$ in the example 2.1, then the subclasses of fuzzy projective lines $\lambda'_{i2}$ and $\eta'_{i23}$ from fuzzy vector planes $\lambda_{i2}$ and $\eta_{i23}$ are as fallows:

$$\begin{aligned} \lambda'_{i2} \ : \ & L_2 \to [0,1] \\ & q \to a_i \\ & p \to b_{i2} \text{ for } p \in L_2 \backslash \{q\} \end{aligned}$$

and

$$\begin{aligned} \eta'_{i23} : \ & N_{23} \to [0,1] \\ & p \to c_{i2} \text{ for all } p \in L_2 \\ & p \to d_{i3} \text{ for all } p \in N_{23} \backslash L_2 \end{aligned}$$

**ÖZET** Bu çalışmada, fuzzy 4−boyutlu vektör uzayının fuzzy vektör düzlemlerinin sınıflaması ve fuzzy 4−boyutlu vektör uzayından elde edilen fuzzy 3−boyutlu projektif uzayın fuzzy projektif doğrularının sınıflaması veriliyor.

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