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CONGRUENCE AND GREEN'S EQUIVALENCE RELATION ON TERNARY SEMIGROUP

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ABSTRACT

In this paper we have defined the left, lateral and right congruence on a ternary semigroup. We discuss Green's Equivalence relations **L, M, R, H, D, J,** on T. We give one new relation called **M**-equivalence relation. We also prove that under certain conditions a ternary semigroup reduces to an ordinary semigroup or even to a band. We prove the Green's Lemma - Let a and b be R-equivalent (**M**-equivalent, **L**-equivalent) elements in a ternary semigroup T with an idempotent $e(T^e)$ and x_1, x_2, y_1, y_2 are in T^e such that $[ax_1x_2] = b$ and $[by_1y_2] = a([x_1ax_2] = b$ and $[y_1by_2] = a$, $[x_1x_2a] = b$ and $[by_1y_2] = a)$, then the maps $\rho_{x_1x_2}|L_a$ and $\rho_{y_1y_2}|L_b$ $(\rho_{x_1x_2}|M_a$ and $\rho_{y_1y_2}|M_b$, $\rho_{x_1x_2}|R_a$ and $\rho_{y_1y_2}|R_b)$ are mutually inverse R-class (**M**-class, **L**-class) preserving bijections from L_a to L_b and from L_b to L_a (M_a to M_b and M_b to M_a, R_a to R_b and R_b to R_a). Further we prove Green's theorem -If H is a **H**-class in a ternary semigroup T, then either $[HHH] \cap H = \phi$ or $[HHH] = H$ and H is a ternary subgroup of T.

1. INTRODUCTION

Definition 1.1. A non-empty set T is called a ternary semigroup if a ternary operation [ ] on T is defined and satisfies the associative law

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]] = [x_1x_2x_3x_4x_5]$$

For all x_i in T, $1 \leq i \leq 5$. The idea of such an algebraic structure was given by S. Banach who showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup.

Sioson [9] defined the following definitions of ideals.

Definition 1.2. A non-empty subset L(R,M) of a ternary semigroup T is called a left (right, lateral) ideal of T is $[TTL] \subseteq L([RTT] \subseteq R$, $[TMT] \subseteq M,)$.

A non-empty subset I of T is said to be ideal if it is left, right and lateral ideal of T.

Definition 1.3. A ternary semigroup (G, [ ]) is said to be a ternary group if it has an additional property that for all a, b, c in G there exists unique x, y, z in G such that

$$[xab] = c, \ [ayb] = c, \ [abz] = c$$

Definition 1.4. e is said to be an idempotent of ternary semigroup T if [eee] = e.

Definition 1.5. An idempotent e is said to be the identity of a ternary group G if for all in G, there exists an unique element e in G such that [eaa] = a, [aea] = a and [aae] = a.

Definition 1.6. If for all a in G, there exists an unique element x in G such that

$$[x\ aa] = e, \ [a\ xa] = e \text{ and } [aa\ x] = e.$$

then x is called the inverse of a in G.

Due to associative. Law in T one may write Sioson [9].

$$[x_1x_2 \ \dots \ x_{2n+1}] = [x_1 \ \dots \ x_m \ x_{m+1} \ \dots \ x_{2n+1}]$$
$$= [x_1 \ \dots \ x_{m-1} \ [x_m x_{m+1} x_{m+2}] \ \dots \ x_{2n+1}], \ m \leq 2n+1$$

2. CONGRUENCE RELATION ON A TERNARY SEMIGROUP

Definition 2.1. An equivalence relation ρ on a ternary semigroup T is said to be left (right, lateral) congruence if it is left (right, lateral) compatible i.e., for all a and b in T. aρb implies $[ac_1c_2]$ ρ $[bc_1c_2]$ ($[c_1c_2a]$ ρ $[c_1c_2b]$ and $[c_1ac_2]$ ρ $[c_1bc_2]$) for all c_1 and c_2 in T.

A relation ρ is said to be compatible if it left, right and lateral compatible on T.

Definition 2.2. An equivalence relation ρ is said to be congruence on T if for all a, b, c_1, c_2, c_3 and c_4 in T, aρb, $c_1\rho c_2$ and $c_3\rho c_4$ imply $[ac_1c_3]$ ρ $[bc_2c_4]$

Now we at once get the following result.

Lemma 2.3. An equivalence relation ρ on a ternary semigroup T is congruence if and only if it is right, left and lateral congruence.

Definition 2.4. Now we define the ternary operation on the quotient set T/ρ where ρ is a congruence on T in a natural way as follows:

$[a\rho\ b\rho\ c\rho] = [abc]\ \rho$, clearly the quotient is well-defined and this is associative too as $[a\rho\ b\rho\ c\rho\ d\rho\ f\rho] = [[a\rho\ b\rho\ c\rho]\ d\rho\ f\rho] = [a\rho\ [b\rho\ c\rho\ d\rho]\ f\rho] = [a\rho\ b\rho\ [c\rho\ d\rho\ f\rho]]$ Hence $(T/\rho, [\ ])$ is a ternary semigroup.

Proposition 2.5. If ρ is a congruence on a ternary semigroup T, then T/ρ is a ternary semigroup under the operation

$[a\rho\ b\rho\ c\rho] = [a\ bc]\ \rho$

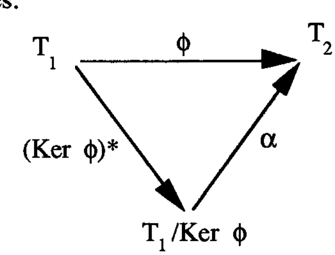
and the mapping $\rho^* : T \to T/\rho$ defined by

$t\rho^* = t\rho$ is a homomorphism.

If $\phi: T_1 \to T_2$ is a homomorphism of ternary semigroup T_1 and T_2, then the relation Ker ϕ: $\phi o \phi^{-1} = \{(a,b) \in T_1 \times T_2 : a\varnothing = b\varnothing\}$ is a congruence on T_1 and there is a monomorphism.

$\alpha : T_1/\mathrm{Ker}\phi \to T_2$ such that range (α) = range (ϕ)

and diagram commutes.



Proof. It is very easy to see that Ker ϕ is a congruence on T_1. Now we define α from the quotient ternary semigroup T_1/Ker ϕ to T_2 as follows:

$\alpha : T_1/\mathrm{Ker}\ \phi \to T_2$ by

$(tKer\phi)\alpha = t\phi,\ t \in T_1$

Clearly α is well-defined and one-one and since.

$([t_1Ker\phi\ t_2Ker\phi\ t_3Ker\phi])\ \alpha = [t_1\phi\, t_2\phi\, t_3\phi]$

$= [(t_1Ker\phi)\alpha\ (t_2Ker\phi)\alpha\ (t_3Ker\phi)\alpha]$

α is a homomorphism map.

range $(\alpha) = \{x/x = (tKer\phi)\alpha \in T_2,\ t \in T_1\}$

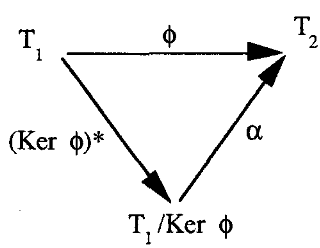
$= \{x/x = t\phi \in T_2\}$

$=$ range (ϕ)

Let a be any element of T_1, then

$a((Ker\phi)^*\ o\alpha) = (a(Ker\phi)^*)\alpha = (a\ Ker\phi)\alpha = a\phi$, where $(Ker\phi)^*$ is a homomorphism map from T_1 to $T_1/Ker\phi$.

Hence the following diagram commutes.



The proof of the following proposition follows easily:

Proposition 2.6. Let ρ, σ be congruences on a ternary semigroup T such that $\rho \subseteq \sigma$, then,

$\sigma/\rho = \{(x\rho,\ y\rho) \in T/\rho \times T/v : (x,y) \in \sigma\}$

is a congruence on T/ρ and

$$\frac{(T/\rho)}{(\sigma/\rho)} = T/\sigma$$

3. REDUCTION OF TERNARY SEMIGROUP INTO ORDINARY SEMIGROUP

In this section we shall prove that a ternary semigroup under certain conditions reduced to an ordinary semigroup. It reduces even to a band under different conditions.

Lemma 3.1. If T is a ternary semigroup which admits an idempotent e satisfying the properties (α) and (β) given below, then (β) implies (α) in general does not imply (β).

(α) (i) $[eea] = [eae] = [aee] = a \ \forall \ a \in T$

(ii) $[eab] \in T \ \forall a,b \in t$ such that $a \neq b$, $a, b \neq e$

(β) $[eaa] = [aea] = [aae] = a \ \forall \ a \in T$

Proof (β) implies (α):

(i) $[eea] = [ee[eaa]] = [[eee]aa] = [eaa] = a$

$[eae] = [e[eaa]e] = [[eea]ae] = [aae] = a$

$[aee] = [[aae]ee] = [aa[eee]] = [aae] = a$

(ii) $[eab] = [e[eaa]b] = [[eea]ab] = [aab] \in T$

Conversely, (α) implies $[eaa] = [aea] = [aae]$ only

$$[eaa] = [e[eae][eea]] = [[eea][eee]a] = [aea]$$

$$[aea] = [ae[eae]] = [a[eea]e] = [aae]$$

Therefore $[eaa] = [aea] = [aae]$. But none is equal to a as shown by the example 4.16

$$[eaa] = (e.(aa)) = (e.o) = o \neq a$$

But $[eaa] = [aea] = [aae] = o$

Lemma 3.2. If T is a ternary semigroup which admits an idempotent e satisfying the property (α), then there exists a binary operation "." on T such that (t,.) is a binary semigroup and $[abc] = a.(b.c)$

Proof. We define $a.b = [eab]$ since (ii) of (α) holds, this is a binary operation on T and T is closed under this operation. Now we calculate a.(b.c):

$$a.(b.c) = [eab].c = [e[eab]c] = [[eea]bc] = [abc]$$

Here we use the fact that T is a ternary semigroup and property (i) of α, Now we look at a.(b.c):

$$a.(b.c) = a.[ebc] = [ea[ebc]] = [eae]bc] = [abc]$$

therefore (T,.) is a binary semigroup and [abc] = a.(b.c).

Lemma 3.3. If T is a ternary semigroup which admits an idempotent e satisfying the property (β), then there exists a binary operation "." on T such that (T,.) is a band (a semigroup with $a^2 = a$, for all a in T) and [abc] = a.(b.c).

Proof. By Lemma 3.1, (β) implies (α) therefore a.b = [eab] is an associative binary operation, but we also get a = [eaa] = a.a.

This completes the proof.

4. GREEN'S EQUIVALENCE RELATION ON TERNARY SEMIGROUP

Certain equivalence relations on a semigroup first studied by J.A. Green (1951), have played a fundamental role in the development of semigroup theory. Green's equivalences are especially significant in the study of regular semigroups. Here we have defined these equivalences in ternary semigroup.

We have defined (1.4) idempotent e in a ternary semigroup. If all the elements in T are idempotent, T is said to be an idempotent ternary semigroup. For defining Green's equivalence relation we need in T, a non-zero idempotent e which should satisfy:

(P): [eea] = [eae] = [aee] = [eaa] = a for all a in T

Unfortunately every ternary semigroup does not have such an idempotent. For defining Green's Equivalence Relation we have to enlarge a given ternary group T with an element $e \notin T$ such that e is an idempotent element satisfying the property (P).

We give two examples that show that property (P) must be satisfied by the adjoining idempotent e in T otherwise the adjoining idempotent changes the structure of the given ternary semigroup.

Example 4.1. T = {i, -i} is a ternary semigroup under complex ternary operation, 1 ∉ T. 1 is an idempotent element which does not satisfy property (P) a [1 i -i] = -1 of T.

Example 4.2. S = {0, 1, a, b, c, d, f} is a semigroup whose multiplication table is given below:

( )	0	1	a	b	c	d	f
0	0	0	0	0	0	0	0
1	0	1	a	b	c	d	f
a	0	a	f	0	0	0	0
b	0	b	0	f	0	0	0
c	0	c	0	0	f	0	0
d	0	d	0	0	0	f	d
f	0	f	0	0	0	d	f

T = {0, a, b, c, d} is a ternary semigroup with the same multiplication of elements as defined in S.

[aaa] = 0 ∀a, ≠ 0 in T and ≠ d as [ddd] = d

[abc] = 0 ∀a, b, c ∈ T, a ≠ b, a ≠ c, b ≠ c or b = c.

T is a ternary semigroup which is not a binary semigroup. We see that [111] = 1 is an idempotent element such that 1 ∉ T. If we try to extend the ternary semigroup T to a ternary semigroup T^e = T ∪ {e} where e is an idempotent element under the ternary operation but here e = 1 is not satisfying the property (P). Moreover [1aa] = (1(aa)) = (1f) = f ∉ T and we do not get [1 aa] = a as required in Property (P).

And T ∪ {1} is not a ternary semigroup.

Example 4.3. T = $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}\right\}$ is a ternary semigroup with an idempotent e = $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ satisfying property (P) under matrix multiplication.

And T ∪ {e} = T is a ternary semigroup.

Now define four types of principal ideals in ternary semigroup.

Definition 4.4. [9] Let t be any element in a ternary semigroup T.

(a) The principal left deal generated by t is $(t)_L = \{t\} \cup [TTt]$

(b) the principal right deal generated by t is $(t)_R = \{t\} \cup [tTT]$

(c) the principal lateral ideal generated by t is
$(t)_M = \{t\} \cup [TtT] \cup [TTtTT]$

(d) the principal three-sided ideal generated by t is
$t = \{t\} \cup [TTt] \cup [TtT] \cup [tTT] \cup [TTtTT]$

Proposition 4.5. Let T be a ternary semigroup which can be extended to a ternary semigroup $T^e = T \cup \{e\}$ satisfying the property (P). Let t be any element of T, then.

(a) $[TT^et] = (t)_L$

(b) $[tT^eT] = (t)_R$

(c) $[T^etT] = (t)_M$

(d) $[TT^etT^eT] = (t)$

Proof. Clearly $(t)_L \subseteq [TT^et]$

Conversely, let $a \in T$, $b \in T^e$, then either $b = e$ or $b \neq e$

If $b = e$, $[aet] = [aaeet] = [aat] \in [TTt]$

If $b \neq e$, $[abt] \subseteq [TTt]$

Hence $[Tt^eT] \subseteq \{t\} \cup [TTt]$

implies $[TT^et] \subseteq (t)_L$

Similar proofs go to other equalities.

Definition 4.6. Let a, b be any two elements of T, then

(a) aLb if a and b generate the same principal left ideal of T i.e.,

$[TT^ea] = [TT^eb]$

(b) a **R** b if a and b generate the same principal right ideal of T i.e.,

$[aT^eT] = [bT^eT]$

(c) a **M** b if a and b generate the same principal lateral ideal of T i.e.

$[T^eaT] = [T^ebT]$

The proof of the following are trivial.

Proposition 4.7. Let a and b be any two element of T, then

(a) a **L** b iff there exist t_1, t_2, t_3, t_4 in T^e such that $[t_1\ t_2\ a] = b$, $[t_3\ t_4\ b] = a$

(b) a **M** b iff there exist t_1, t_2, t_3, t_4 in T^e such that $[t_1\ a\ t_2] = b$, $[t_3\ b\ t_4] = a$

(c) a **R** b iff there exist t_1, t_2, t_3, t_4 in T^e such that $[ta_1\ t_2] = b$, $[b\ t_3\ t_4] = a$.

Proposition 4.8. L, M and **R** are respectively right, lateral and left congruences on T.

Proposition 4.9. LoMoR is an equivalence relation if the following equalities hold:

(a) **LoMoR = RoMoL**

(b) **LoM = MoL**

(c) **MoR = RoM**

(d) **LoR = RoL**

Now we define two relations **H**-relation and **D**-relation in a ternary semigroup which are different from those defined in a binary semigroup. Here we can not ignore **M**-equivalence (lateral equivalence), justified by examples 4.13, 4.14 given below.

$$\mathbf{H} = \mathbf{L} \cap \mathbf{M} \cap \mathbf{R} = \mathbf{R} \cap \mathbf{M} \cap \mathbf{L} \tag{4.10}$$

$$\mathbf{D} = \mathbf{L} \circ \mathbf{M} \circ \mathbf{R} = \mathbf{R} \circ \mathbf{M} \circ \mathbf{L} \tag{4.11}$$

Definition 4.12. a **J** b if a and b generate the same principal ideal of T i.e,

$[TT^e\ a\ T^eT] = [TT^e\ b\ T^eT],\ \forall a,b \in T.$

Example 4.13. In example 4.3. if we take

$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then

$[eea] = [eae] = [aee] = [eaa] = a\ \forall\ a \in T$, i.e. the property (P) is true,

$T^e = T$ and $\mathbf{L} = \mathbf{R} = \mathbf{M} = \mathbf{H} = \mathbf{D} = \mathbf{J} = \mathbf{I} = \{(0,0), (e,e), (a,a), (b,b)\}$.

Example 4.14. T = {0,e,a,b,c} is an idempotent ternary semigroup with property (P) defined as:

( )	0	e	a	b	c
0	0	0	0	0	0
e	0	e	a	b	c
a	0	a	a	b	0
b	0	b	a	b	0
c	0	c	0	0	c

$[a\ b\ c] = a.(b.c)\ \forall\ a, b, c \in T.$

$\mathbf{L} = \{(0,0), (e,e), (a,a), (b,b), (c,c)\}$

$\mathbf{R} = \mathbf{M} = \{(0,0), (e,e), (a,a), (a,b), (b,b), (b,b)\ (c,c)\}$

$\mathbf{H} = \mathbf{L} \cap \mathbf{M} \cap \mathbf{R} = \{(0,0)\ (e,e), (a,a), (a,b)\ (b,b), (c,c)\} = \mathbf{R} = \mathbf{M}$

$\mathbf{J} = \{(0,0), (e,e), (a,a), (a,b), (b,b), (c,c)\} = \mathbf{D} = \mathbf{R} = \mathbf{M}$

Remarks 4.15. (i) There exists a ternary semigroup T such that

$[aTT] \cup [TTa] \not\subset [TaT] \cup [TTaTT]$ for some $a \in T$.

(ii) $\mathbf{M} \neq \mathbf{J}$

We show this by the following example:

Example 4.16. T = {0,e,a,b} is a ternary semigroup. The multiplication of elements are shown in the table:

( )	0	e	a	b
0	0	0	0	0
e	0	e	a	b
a	0	b	0	0
b	0	b	0	0

[abc] = (a.(b.c)) = ((a.b).c) $\forall$ a, b, c, $\in$ T

(i) It can be easily shown that [aTT] $\cup$ [TTa] $\not\subset$ [TaT] $\cup$ [TTaTT]

As, [aTT] $\cup$ [TTa] = {0, a, b}

and, [TaT] u [TTaTT] = {0, b}

(ii) Let **M = J**, Then we have

{a}$\cup$[TaT]$\cup$[TTaTT] = {a}$\cup$[aTT]$\cup$[TTa]$\cup$[TaT]$\cup$[TTaTT].

Hence [aTT] $\cup$ [TTa] $\subseteq$ [TaT] $\cup$ [TTaTT]

which is contradiction by above example.

We denote the **L**-class (**R**-class **M**-class, **H**-class, **D**-class, **J**-class) containing the element a bay L_a, ($\mathbf{R}_a$, $\mathbf{M}_a$, $\mathbf{H}_a$, $\mathbf{D}_a$, $\mathbf{J}_a$ where

L_a = {x $\in$ T / x **L** a}

In a ternary semigroup with zero it is easy to see that

$L_o = M_o = R_o = D_o = H_o = J_o = \{0\}$.

Since **L, M, R** and **J** are defined in terms of principal ideals, the inclusion order among the principal ideals induces corresponding order among the equivalence classes as follow:

Remark 4.17. $L_a \leq L_b$ if $[TT^ea] \subseteq [TT^eb]$,
$M_a \leq M_b$ if $[T^eaT] \subseteq [T^ebT]$,
$R_a \leq R_b$ if $[aT^eT] \subseteq [bT^eT]$,
$J_a \leq J_b$ if $[TT^eaT^eT] \subseteq [TT^ebT^eT]$,

Remark 4.18. Clearly

$L_{[x_1x_2a]} \leq L_a$, $R_{[ax_1x_2]} \leq R_a$,

$J_{[x_1x_2ax_3x_4]} \leq J_a$, $M_{[x_1ax_2]} \leq M_a$,

for all $x_1, x_2, x_3, x_4, \in T^e$.

Remark 4.19. Either of $L_a \leq L_b$, $M_a \leq M_b$ or $R_a \leq R_b$ implies $J_a \leq J_b$.

Remark 4.20. D-class is non-empty if H-class is non-empty.

Definition 4.21. T is said to be a left (lateral, right) simple ternary semigroup if the left (lateral, right) ideal of T is T itself.

Definition 4.22. T is said to be a simple ternary semigroup if T has no ideal other than trivial ideal and T itself.

The proofs of the following propositions are trivial:

Proposition 4.23. A ternary semigroup T is left (lateral, right) simple if it consists of a single **L**-class (**M**-class, **R**-class).

Proposition 4.25. A ternary semigroup T is said to be bi-simple if it consists of a single **D**-class.

Proposition 4.26. [LMR] of any **L**-class **L**, **M**-class **M** and any **R**-class **R** of a ternary semigroup T is always contained in a single **D**-class of T.

Proof. It is clear by the following:

The above statement is equivalent to the following statement if a, a′, b, b′, c, c′ are elements of a ternary semigroup T such that a**L**a′, b**M**b′ and c**R**c′ then [abc] **D** [a′b′c′].

Green's Lemma 4.27. Let a and b be **R**-equivalent elements in a ternary semigroup T and x_1, x_2, y_1, y_2 be in T^e such that $[ax_1x_2] = b$ and $[by_1y_2] = a$. Then the map $\rho_{x_1x_2}|L_a$ and $\rho_{x_1x_2}|L_b$ are mutually inverse **R**-class preserving bijections from L_a to L_b and from L_b to L_a.

Proof. Define $\rho_{x_1x_2} : T \to T$ by

$\rho_{x_1x_2}(a) = [ax_1x_2],\ x_1x_2 \in T^e$

The map is well-defined because of a **Rb**. In fact it maps L_a into L_b, as $\mu \in L_a$, then by 4.8 we get

$[\mu\ x_1x_2] \in L_{[ax_1x_2]} = L_b$

Similarly, we define $\rho_{y_1y_2} : T \to T$ by

$\rho_{y_1y_2}(b) = [by_1y_2],\ y_1y_2 \in T^e$

as above $\rho_{y_1y_2}$ maps L_b into L_a,

Now $(\rho_{y_1y_2}\ \rho_{x_1x_2})\ (a) = \rho_{y_1y_2}[ax_1x_2] = \rho_{y_1y_2}(b) = [by_1y_2] = a$

Similarly, $(\rho_{x_1x_2}\ \rho_{y_1y_2})\ (b) = b$

Thus $\rho_{x_1x_2} | L_a$ and $\rho_{y_1y_2} | L_b$ are mutually inverse bijections from L_a to L_b and from L_b to L_a respectively. Moreover these bijections are R-class preserving.

The proofs of the following lemmas are similar.

Green's Lemma 4.28. Let a and b be **M**-equivalent element in ternary semigroup T and x_1, x_2, y_1, y_2 be in T^e such that $[x_1ax_2] = b$ and $[y_1by_2] = a$, then maps $\rho_{x_1x_2}|M_a$ and $\rho_{x_1x_2}|M_b$ are mutually inverse M-class preserving bijections from M_a to M_b and from M_b to M_a.

Green's Lemma 4.29. Let a and b be **L**-equivalent elements in a ternary semigroup T and x_1, x_2, y_1, y_2 be in T^e such that $[x_1x_2a] = b$ and $[y_1y_2b] = a$, then the maps $\rho_{x_1x_2}|R_a$ and $\rho_{x_1x_2}|R_b$ are mutually inverse **L**-class preserving bijections from R_a to R_b and R_b to R_a.

Green's Lemmas 4.27, 4.28 and 4.29 at once give the following:

Lemma 4.30. Let a and b be **D**-equivalent elements in a ternary semigroup T. Then

$|H_a| = |H_b|$

This lemma soon gives the following lemma:

Lemma 4.31. If x, y, z are in T such that $[xyz] \in H_x$, then $\rho_{yz}|H_x$ is a bijection, if $[xyz] \in H_y$, then $\rho_{xy}|H_y$ is a bijection and if $[xyz] \in H_z$, then $\rho_{xy}|H_z$ is a bijection.

Proposition 4.32. No **H**-class contains more than one idempotent.

Proof. Let e and f be two idempotents such that $H_e = H_f$. Then each is three sided identity of the other, by Lemma 4.30. Hence e = f.

Now we prove Green's theorem in a ternary semigroup.

Green's Theorem 4.33. If H is a **H**-class in a ternary semigroup T, then either $[HHH] \cap H = \emptyset$ or [HHH] = H and H is a ternary subgroup of T.

Proof. If $[HHH] \cap H = \emptyset$, then there is nothing to prove. Otherwise, there exist a, b, c in H such that $[abc] \in H$. Now by Lemma 4.31, ρ_{bc}, σ_{ac} and λ_{ab} are bijections from H onto [HHH]. Hence [HHH] = H,

Let $\mu \in H$. Then $[\mu bc]$, $[a\mu c]$, $[ab\mu]$ are in H, again by Lemma 4.31 we have ρ_{bc}, σ_{ac} and λ_{ab} are bijections from H onto $[\mu HH]$, H onto $[H\mu H]$ and H onto $[HH\mu]$ respectively. Therefore $[\mu HH] = [H\mu H] = [HH\mu] = H$ and hence H is a ternary subgroup of T.

Corollary 4.34. If e is idempotent in a ternary semigroup T, then H_e is a ternary subgroup of T.

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