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Dynamic Behavior of Euler-Maclaurin Methods for Differential Equations with Piecewise Constant Arguments of Advanced and Retarded Type

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Abstract

The paper deals with three dynamic properties of the numerical solution for differential equations with piecewise constant arguments of advanced and retarded type: oscillation, stability and convergence. The Euler-Maclaurin methods are used to discretize the equations. According to the characteristic theory of the difference equation, the oscillation and stability conditions of the numerical solution are obtained. It is proved that the convergence order of numerical method is $2n+2$. Furthermore, the relationship between stability and oscillation is discussed for analytic solution and numerical solution, respectively. Finally, several numerical examples confirm the corresponding conclusions.

1. Introduction

As a special type of delay differential equations [1]- [4], differential equations with piecewise constant argument [5]- [9] (abbreviated as EPCA) has some characteristics of continuous and discrete dynamic system, so it has important value in practical application such as population biology [10], neural networks [11, 12], predator-prey model [13], epidemiology [14] and so on. In recent years, the comprehensive exploration of EPCA has become a scientific issue widely concerned by scholars in various fields. Because of the complexity of this kind of equation in structure, it is difficult to solve it accurately. Therefore, it is necessary to study the numerical solution of EPCA, and then clarify the applicability of numerical method in EPCA.
In the study of differential equations with piecewise constant arguments, much research has been focused on the properties of numerical solution of EPCA. Gao [15] considered numerical oscillation of the Runge-Kutta method for EPCA of mixed type. In [16], convergence and stability of stochastic EPCA in split-step theta method was considered. The stability of the Runge-Kutta method for nonlinear neutral EPCA was studied in [17]. Wang and Yao [18] studied the stability and oscillation of a kind of functional differential equation. Liang et al. [19] considered numerical stability of system $u'(t) = Lu(t) + Mu([t])$ with matrix coefficients in the case of 2-norm. Different from previous studies, this paper mainly considers the numerical oscillation, stability and convergence of Euler-Maclaurin methods for forward EPCA with advanced and retarded type, and gives some new conclusions.
Consider the following equation:

$$x'(t) = ax(t) + a_0x([t]) + a_1x([t+1]), x(0) = c_0, \tag{1.1}$$

where $[\cdot]$ designates the greatest-integer function.

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Denote

$$b_0(t)=e^{at}+a^{-1}a_0(e^{at}-1), b_1(t)=a^{-1}a_1(e^{at}-1), \lambda=b_0(1)/(1-b_1).$$

Theorem 1.1. *[20] Eq. (1.1) has on a unique solution*

$$x(t)=(b_0(\{t\})+\lambda b_1(\{t\}))\lambda^{[t]}c_0, \tag{1.2}$$

where $\{t\}$ is the fractional part of t, if $b_1(1)\neq 1$.

In particular, the solution of Eq. (1.1) is

$$x(t)=\left(1+\frac{a_0+a_1}{1-a_1}\{t\}\right)\left(\frac{1+a_0}{1-a_1}\right)^{[t]}c_0,$$

for $a=0$.

Theorem 1.2. *[20] The solution $x=0$ of Eq. (1.1) is stable (asymptotically stable) as $t\to+\infty$, if and only if*

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\geq 0. \tag{1.3}$$

Theorem 1.3. *[20] In each internal $(n,n+1)$, the solution of Eq. (1.1) with the condition $x(0)=c_0\neq 0$ has exact roots*

$$t_n=n+\frac{1}{a}ln\frac{a_0+a_1e^a}{a+a_0+a_1}$$

if

$$\left(a_0+\frac{ae^a}{e^a-1}\right)\left(a_1-\frac{a}{e^a-1}\right)>0. \tag{1.4}$$

If (1.4) is not satisfied and $a_0\neq -ae^a/(e^a-1), c_0\neq 0$, then solution (1.2) has no zero in $[0,+\infty)$.

2. Numerical oscillation and non-oscillation

2.1. Euler-Maclaurin methods and convergence

Firstly, we introduce Bernoulli's numbers and Bernoulli's polynomials as follows:

$$\frac{z}{e^z-1}=\sum_{j=0}^{\infty}\frac{B_j}{j!}z^j, |z|<2\pi,$$

$$\frac{ze^{xz}}{e^z-1}=\sum_{j=0}^{\infty}\frac{B_j(x)}{j!}z^j, |z|<2\pi,$$

where B_j and $B_j(x), j=0,1,2\cdots$ are called Bernoulli's number and the jth-order Bernoulli's polynomial, respectively.

Lemma 2.1. *[21] B_j and $B_j(x)$ have the following several properties:*
(I) $B_0=1, B_1=-\frac{1}{2}, B_{2j}=2(-1)^{j+1}(2j)!\sum_{k=1}^{\infty}(2k\pi)^{-2j}, B_{2j+1}=0, j\geq 1,$
(II) $B_0(x)=1, B_1(x)=x-\frac{1}{2}, B_2(x)=x^2-x+\frac{1}{6}, B_k(x)=\sum_{j=0}^{k}\binom{k}{j}B_jx^{k-j}.$

Lemma 2.2. *[22] Suppose that $f(x)$ has $2n+3$rd continuous derivative on $[t_i,t_{i+1}]$, then we have*

$$\left|\int_{t_i}^{t_{i+1}}f(t)dt-\frac{h}{2}[f(t_{i+1})+f(t_i)]+\sum_{j=1}^{n}\frac{B_{2j}h^{2j}}{(2j)!}\left[f^{(2j-1)}(t_{i+1})-f^{(2j-1)}(t_i)\right]\right|=O\left(h^{2n+3}\right). \tag{2.1}$$

Let $h=\frac{1}{m}$ be a given step-size and t_i be defined by $t_i=ih, i=0,1,2\cdots$, then let $i=km+l, l=0,1,2,\cdots,m-1$. The derivative $x^{(j)}(t)$ exists in every interval $[k,k+1)$. We suppose

$$f(t)=x'(t)=ax(t)+a_0x([t])+a_1x([t+1])$$

for all $j=0,1,2\cdots$, then we have

$$\begin{aligned} f'(t)&=x''(t)=ax'(t)=a^2x(t)+aa_0x([t])+aa_1x([t+1]),\\ f^{(j)}(t)&=x^{(j+1)}(t)=a^{j+1}x(t)+a^ja_0x([t])+a^ja_1x([t+1]). \end{aligned} \tag{2.2}$$

Apply (2.2) to (2.1), we get

$$x_{i+1}=x_i+\frac{ha}{2}\left(x_{i+1}+x_i\right)+ha_0x_{km}+ha_1x_{(k+1)m}-\sum_{j=1}^{n}\frac{B_{2j}(ah)^{2j}}{(2j)!}\left(x_{i+1}-x_i\right). \tag{2.3}$$

Since $i=km+l, l=0,1,2,\cdots,m-1$, (2.3) can be expressed as:

$$x_{(k+1)m}=\frac{1+a_0}{1-a_1}x_{km}, \tag{2.4}$$

$$x_{km+l+1}=(1+(l+1)ha_0)x_{km}+(l+1)ha_1x_{(k+1)m}, \tag{2.5}$$

for $a=0$, and

$$x_{(k+1)m}=\frac{R(z)^m+\frac{a_0}{a}\left(R(z)^m-1\right)}{1-\frac{a_1}{a}\left(R(z)^m-1\right)}x_{km}, \tag{2.6}$$

$$x_{km+l+1}=\left(R(z)^{l+1}+\frac{a_0}{a}\left(R(z)^{l+1}-1\right)\right)x_{km}+\frac{a_1}{a}\left(R(z)^{l+1}-1\right)x_{(k+1)m}, \tag{2.7}$$

for $a\neq 0$, where $l=0,1,\cdots,m-2$, $z=ah$, $\phi(z)=1-\frac{z}{2}+\sum_{j=1}^{n}\frac{B_{2j}z^{2j}}{(2j)!}$ and $R(z)=1+\frac{z}{\phi(z)}$ is the stability function of the Euler-Maclaurin methods.

Theorem 2.3. *For every given $n\in N$, the Euler-Maclaurin method is of order $2n+2$.*

Proof. Let $km\le i<(k+1)m-1$, then by Lemma 2.2 and $f(t)=x'(t)$, we get

$$\begin{aligned}x(t_{i+1})-x(t_i)=\int_{t_i}^{t_{i+1}}x'(t)dt=&\frac{ha}{2}\left[x(t_{i+1})+x(t_i)\right]+ha_0x(k)+ha_1x(k+1)\\&-\sum_{j=1}^{n}\frac{B_{2j}(ah)^{2j}}{(2j)!}\left[x(t_{i+1})-x(t_i)\right]+O\left(h^{2n+3}\right).\end{aligned}$$

Let $i=(k+1)m-1$, then for any $0<\varepsilon<h$, we have

$$\begin{aligned}x(t_{i+1}-\varepsilon)-x(t_i)=\int_{t_i}^{t_{i+1}-\varepsilon}x'(t)dt=&\frac{ha}{2}\left[x(t_{i+1}-\varepsilon)+x(t_i)\right]+ha_0x(k)+ha_1x(k+1)\\&-\sum_{j=1}^{n}\frac{B_{2j}(ah)^{2j}}{(2j)!}\left[x(t_{i+1}-\varepsilon)+x(t_i)\right]+O\left(h^{2n+3}\right).\end{aligned} \tag{2.8}$$

Let $\varepsilon\to 0^+$ in (2.8), (2.7) holds true for $i=(k+1)m-1$. Suppose

$$\left(x(t_{i+1})-x_{i+1}\right)\left(1+\frac{ha}{2}+\sum_{j=1}^{n}\frac{B_{2j}(ha)^{2j}}{(2j)!}\right)=O\left(h^{2n+3}\right),$$

then from (2.4)-(2.7) we obtain

$$\left(x(t_{i+1})-x_{i+1}\right)\left(1+\frac{ha}{2}+\sum_{j=1}^{n}\frac{B_{2j}(ha)^{2j}}{(2j)!}\right)=O\left(h^{2n+3}\right),$$

the proof is complete. □

2.2. Oscillation analysis

Theorem 2.4. *If $\{x_n\}$ and $\{x_{km}\}$ are given by (2.5), (2.7) and (2.4), (2.6), respectively, then $\{x_n\}$ is non-oscillatory if and only if $\{x_{km}\}$ is non-oscillatory.*

Proof. The necessity is obvious for $a\neq 0$. Sufficiency: if $\{x_{km}\}$ is non-oscillatory, without loss of generality, we assume that $\{x_{km}\}$ is an eventually negative solution of (2.6), that is, there exists a $k_0\in R$ such that $x_{km}<0$ for $k>k_0$. In order to prove $x_{km+l}<0$ for all $k>k_0+1$ and $l=0,1,\cdots,m-1$, we suppose $a_0<0, a_1<0$. If $a>0$, then $1<R(z)<\infty$ and $R(z)^{-m}\le R(z)^{-l}$, therefore from (2.7) we have

$$\begin{aligned}R(z)^{-l}x_{km+l}&=\left(1+\frac{a_0}{a}\left(1-R(z)^{-l}\right)\right)x_{km}+\frac{a_1}{a}\left(1-R(z)^{-l}\right)x_{(k+1)m}\\&\le\left(1+\frac{a_0}{a}\left(1-R(z)^{-m}\right)\right)x_{km}+\frac{a_1}{a}\left(1-R(z)^{-m}\right)x_{(k+1)m}\\&=R(z)^{-m}x_{(k+1)m}<0.\end{aligned}$$

So $x_{km+l}<0$. The case of $a<0$ and $a=0$ can be studied in the same way. The proof is complete. □

By Theorem 2.4, we can get the following theorem.

Theorem 2.5. *The following propositions are equivalent:*
(I) $\{x_n\}$ *is oscillatory ;*
(II) $\{x_{km}\}$ *is oscillatory ;*
(III) *The two cases hold*
(i) $a_0 < -\frac{aR(z)^m}{R(z)^m-1}$ *and* $a_1 < \frac{a}{R(z)^m-1}$,
(ii) $a_0 > -\frac{aR(z)^m}{R(z)^m-1}$ *and* $a_1 > \frac{a}{R(z)^m-1}$,
for $a \neq 0$, *and*
(i) $a_0 < -1$ *and* $a_1 < 1$,
(ii) $a_0 > -1$ *and* $a_1 > 1$,
for $a = 0$.

Proof. According to Theorem 2.4, the equivalence of (I) and (II) is obvious, then we prove that (II) and (III) are equivalent. $\{x_n\}$ is oscillatory for $a \neq 0$ if and only if the corresponding characteristic equation has no positive roots, which is equivalent to

$$\frac{R(z)^m + \frac{a_0}{a}\left(R(z)^m - 1\right)}{1 - \frac{a_1}{a}\left(R(z)^m - 1\right)} < 0,$$

so we have

$$R(z)^m + \frac{a_0}{a}\left(R(z)^m - 1\right) < 0 \text{ and } 1 - \frac{a_1}{a}\left(R(z)^m - 1\right) > 0$$

or

$$R(z)^m + \frac{a_0}{a}\left(R(z)^m - 1\right) > 0 \text{ and } 1 - \frac{a_1}{a}\left(R(z)^m - 1\right) < 0,$$

that is

$$a_0 < -\frac{aR(z)^m}{R(z)^m-1} \text{ and } a_1 < \frac{a}{R(z)^m-1}$$

or

$$a_0 > -\frac{aR(z)^m}{R(z)^m-1} \text{ and } a_1 > \frac{a}{R(z)^m-1}.$$

In the same way, $\lambda = \frac{1+a_0}{1-a_1}$ for $a = 0$. The proof is complete. □

From Theorem 1.3, we have the following corollary.

Corollary 2.6. *If any of the following conditions holds true:*
(I) *When* $a \neq 0$,
(i) $a_0 < -\frac{ae^a}{e^a-1}$ *and* $a_1 < \frac{a}{e^a-1}$,
(ii) $a_0 > -\frac{ae^a}{e^a-1}$ *and* $a_1 > \frac{a}{e^a-1}$,
(II) *When* $a = 0$,
(i) $a_0 < -1$ *and* $a_1 < 1$,
(ii) $a_0 > -1$ *and* $a_1 > 1$,
then every solution of Eq. (1.1) is oscillatory.

Lemma 2.7. *[21] If* $|z| \leq 1$, *then we have* $\phi(z) \geq \frac{1}{2}\cdot$ *for* $z > 0$ *and* $\phi(z) \geq 1$ *for* $z \leq 0$.

Lemma 2.8. *[21] If* $|z| \leq 1$, *then*
(I) $\phi(z) \leq \frac{z}{e^z-1}$, *n is even ;*
(II) $\phi(z) \geq \frac{z}{e^z-1}$, *n is odd.*

Theorem 2.9. *If* $a \neq 0$, *then the Euler-Maclaurin methods preserve the oscillation of Eq. (1.1) if and only if n is even.*

Proof. According to Theorem 2.5 and Corollary 2.6, we can get the Euler-Maclaurin methods preserve the oscillation of (1.1) if and only if

$$\frac{ae^a}{e^a-1} \leq -\frac{aR(z)^m}{R(z)^m-1} \text{ or } \frac{a}{e^a-1} \geq \frac{a}{R(z)^m-1}$$

holds true. If $a > 0$, we have

$$\frac{e^a}{e^a-1} \geq \frac{R(z)^m}{R(z)^m-1} \text{ or } e^a \leq R(z)^m.$$

Since the function $y = \frac{x}{x-1}$ is decreasing, so

$$e^a \leq R(z)^m.$$

Therefore,

$$\phi(z) \leq \frac{z}{e^z - 1}.$$

From Lemma 2.8, n is even. The case of $a < 0$ can be proved in the same way. □

Theorem 2.10. *If $a \neq 0$, then the Euler-Maclaurin methods preserve the non-oscillation of (1.1) if and only if n is odd.*

From Theorem 2.5 and Corollary 2.6, we can get this proof.

Theorem 2.11. *When $a = 0$, the Euler-Maclaurin methods preserve the oscillation and non-oscillation of (1.1) for any $n \in N$.*

3. Relationship between stability and oscillation

From Theorem 1.2, we have the following corollary.

Corollary 3.1. *The analytic solution of Eq. (1.1) is asymptotically stable as $t \to +\infty$, if and only if*

$$(a + a_0 + a_1)\left(a_1 - a_0 - \frac{a\,(e^a + 1)}{e^a - 1}\right) > 0$$

for $a \neq 0$, and

$$(a_0 + a_1)(a_1 - a_0 - 2) > 0$$

for $a = 0$.

Theorem 3.2. *The numerical solution of Eq. (1.1) is asymptotically stable ($x_n \to 0$ as $n \to \infty$) if and only if*

$$(a + a_0 + a_1)\left(a_1 - a_0 - \frac{a\,(R(z)^m + 1)}{R(z)^m - 1}\right) > 0$$

for $a \neq 0$, and

$$(a_0 + a_1)(a_1 - a_0 - 2) > 0$$

for $a = 0$.

Proof. According to (2.3) and (2.5), it is well known that $x_n \to 0$ as $n \to \infty$ if and only if $|\hat{\lambda}| < 1$, where

$$\hat{\lambda} = \frac{R(z)^m + \frac{a_0}{a}\,(R(z)^m - 1)}{1 - \frac{a_1}{a}\,(R(z)^m - 1)}$$

for $a \neq 0$, and

$$\hat{\lambda} = \frac{1 + a_0}{1 - a_1}$$

for $a = 0$. So we have

$$(a + a_0 + a_1)\left(a_1 - a_0 - \frac{a\,(R(z)^m + 1)}{R(z)^m - 1}\right) > 0$$

for $a \neq 0$, and

$$(a_0 + a_1)(a_1 - a_0 - 2) > 0$$

for $a = 0$. This completes the proof. □

According to Corollary 2.6 and Corollary 3.1, we get the conclusion for the analytic solution.

Theorem 3.3. *When $a \neq 0$, the analytic solution of Eq. (1.1) is*

(A1) non-oscillatory and asymptotically stable if

$$(a + a_0 + a_1)\left(a_1 - a_0 - \tfrac{a(e^a+1)}{e^a-1}\right) > 0,\ a_0 < -\tfrac{ae^a}{e^a-1} \text{ and } a_1 \geq \tfrac{a}{e^a-1}$$

or

$$(a + a_0 + a_1)\left(a_1 - a_0 - \tfrac{a(e^a+1)}{e^a-1}\right) > 0,\ a_0 > -\tfrac{ae^a}{e^a-1} \text{ and } a_1 \leq \tfrac{a}{e^a-1}$$

holds true.

(A2) non-oscillatory and unstable if

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0<-\frac{ae^a}{e^a-1}\ \textit{and}\ a_1\geq\frac{a}{e^a-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0>-\frac{ae^a}{e^a-1}\ \textit{and}\ a_1\leq\frac{a}{e^a-1}$$

holds true.

(A3) oscillatory and unstable if

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0<-\frac{ae^a}{e^a-1}\ \textit{and}\ a_1<\frac{a}{e^a-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0>-\frac{ae^a}{e^a-1}\ \textit{and}\ a_1>\frac{a}{e^a-1}$$

holds true.

(A4) oscillatory and asymptotically stable if

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)>0,\ a_0<-\frac{ae^a}{e^a-1}\ \textit{and}\ a_1<\frac{a}{e^a-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)>0,\ a_0>-\frac{ae^a}{e^a-1}\ \textit{and}\ a_1>\frac{a}{e^a-1}$$

holds true.

According to Theorem 2.5 and Theorem 3.2, we get the corresponding conclusion for the numerical solution.

Theorem 3.4. *When $a\neq 0$, the numerical solution of (1.1) is*

(B1) non-oscillatory and asymptoticallystable if

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(R(z)^m+1)}{R(z)^m-1}\right)>0,\ a_0<-\frac{aR(z)^m}{R(z)^m-1}\ \textit{and}\ a_1\geq\frac{a}{R(z)^m-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(R(z)^m+1)}{R(z)^m-1}\right)>0,\ a_0>-\frac{aR(z)^m}{R(z)^m-1}\ \textit{and}\ a_1\leq\frac{a}{R(z)^m-1}$$

holds true.

(B2) non-oscillatory and unstable if

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0<-\frac{aR(z)^m}{R(z)^m-1}\ \textit{and}\ a_1\geq\frac{a}{R(z)^m-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0>-\frac{aR(z)^m}{R(z)^m-1}\ \textit{and}\ a_1\leq\frac{a}{R(z)^m-1}$$

holds true.

(B3) oscillatory and unstable if

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0<-\frac{aR(z)^m}{R(z)^m-1}\ \textit{and}\ a_1<\frac{a}{R(z)^m-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0>-\frac{aR(z)^m}{R(z)^m-1}\ \textit{and}\ a_1>\frac{a}{R(z)^m-1}$$

holds true.

(B4) oscillatory and asymptotically stable if

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(R(z)^m+1)}{R(z)^m-1}\right)>0,\ a_0<-\frac{aR(z)^m}{R(z)^m-1}\ \textit{and}\ a_1<\frac{a}{R(z)^m-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(R(z)^m+1)}{R(z)^m-1}\right)>0,\ a_0>-\frac{aR(z)^m}{R(z)^m-1}\ \textit{and}\ a_1>\frac{a}{R(z)^m-1}$$

holds true.

Theorem 3.5. *When $a=0$, the analytic solution and numerical solution of Eq. (1.1) are both*

(C1) non-oscillatory and asymptotically stable if

$$(a_0+a_1)(a_1-a_0-2)>0,\ a_0<-1\ \textit{and}\ a_1\geq 1$$

or

$$(a_0+a_1)(a_1-a_0-2)>0,\ a_0>-1 \text{ and } a_1\le 1.$$

(C2) non-oscillatory and unstable if

$$(a_0+a_1)(a_1-a_0-2)\le 0,\ a_0<-1 \text{ and } a_1\ge 1$$

or

$$(a_0+a_1)(a_1-a_0-2)\le 0,\ a_0>-1 \text{ and } a_1\le 1.$$

(C3) oscillatory and unstable if

$$(a_0+a_1)(a_1-a_0-2)\le 0,\ a_0<-1 \text{ and } a_1<1$$

or

$$(a_0+a_1)(a_1-a_0-2)\le 0,\ a_0>-1 \text{ and } a_1>1.$$

(C4) oscillatory and asymptotically stable if

$$(a_0+a_1)(a_1-a_0-2)>0,\ a_0<-1 \text{ and } a_1<1$$

or

$$(a_0+a_1)(a_1-a_0-2)>0,\ a_0>-1 \text{ and } a_1>1.$$

4. Numerical examples

Consider the following equations

$$x'(t)=-x(t)-2x([t])+5x([t+1]),\quad x(0)=1, \tag{4.1}$$

$$x'(t)=x(t)+4x([t])-3x([t+1]),\quad x(0)=1, \tag{4.2}$$

$$x'(t)=x(t)+x([t])+2x([t+1]),\quad x(0)=1, \tag{4.3}$$

$$x'(t)=-2x(t)-3x([t])-2x([t+1]),\quad x(0)=1. \tag{4.4}$$

From Theorem 1.1, the analytic solution of Eq. (4.1) is $x(10)\approx 1.51037040806E-4$ at $t=10$. We listed the absolute errors (AE) and the relative errors (RE) at $n=2$ and $t=10$ and the ratio of the errors of the case $m=20$ over that of $m=40$. We can see from Table 1 that the Euler-Maclaurin methods is of order 6 when $n=2$. The Euler-Maclaurin methods have good convergence for this kind of equations.

Further, from (4.1) we know that the coefficients are $a=-1, a_0=-2, a_1=5$, then

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\approx 9.6721>0, a_0<-\frac{ae^a}{e^a-1}\approx -0.5820 \text{ and } a_1\ge\frac{a}{e^a-1}\approx 1.5820,$$

so (A1) in Theorem 3.3 holds true. On the other hand, let $m=50, n=3$, we have

$$z=ha=\frac{a}{m}=-0.02, B_{2j}=2.3404\times 10^{-8}, \phi(z)=1.0100, R(z)=1+\frac{z}{\phi(z)}=0.9802.$$

Table 1: The errors of the Euler-Maclaurin methods $(n=2)$

	AE	RE
$m=2$	$3.0083E-10$	$1.9918E-6$
$m=3$	$2.6198E-11$	$1.7345E-7$
$m=5$	$1.2172E-12$	$8.0591E-9$
$m=10$	$1.8986E-14$	$1.2570E-10$
$m=20$	$2.9751E-16$	$1.9697E-12$
$m=40$	$4.0115E-18$	$2.6560E-14$
ratio	74.16	74.16

Because

$$\phi(z) \geq \frac{z}{e^z - 1} \approx 1.0100,$$

then we obtain

$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a\,(R(z)^m+1)}{R(z)^m-1}\right) \approx 9.6721 > 0, a_0 < -\frac{aR(z)^m}{R(z)^m-1} \approx -0.5820 \text{ and } a_1 \geq \frac{a}{R(z)^m-1} \approx 1.5820,$$

so (B1) in Theorem 3.4 holds true.

From Figure 4.1 we can see that the analytic solution and the numerical solution of (4.1) are asymptotically stable and non-oscillatory, which is agreement with Theorems 3.3 (A1) and 3.4 (B1).

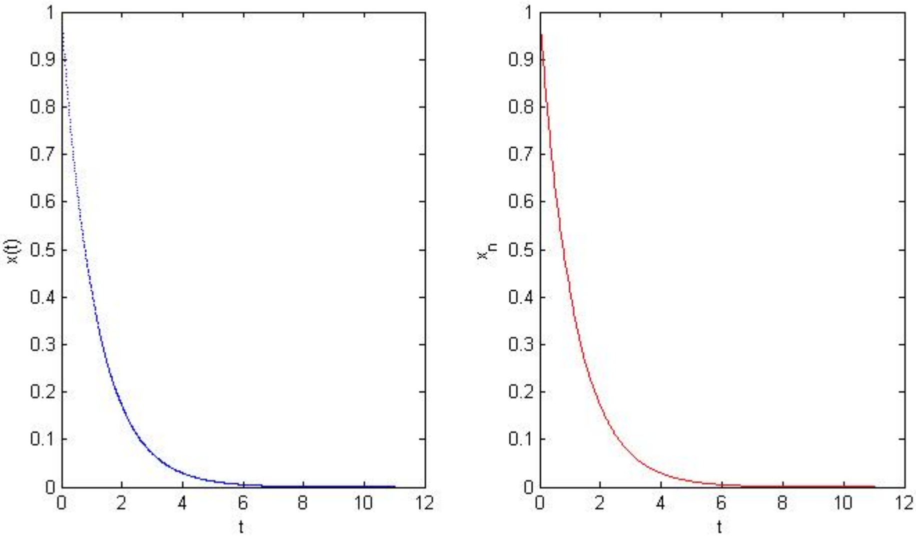

Figure 4.1: The analytic solution (left) and the numerical solution (right, $n = 3$) of (4.1).

From (4.2) we know that the coefficients are $a = 1, a_0 = 4, a_1 = -3$, then

$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a\,(e^a+1)}{e^a-1}\right) \approx -18.3279 \leq 0, a_0 > -\frac{ae^a}{e^a-1} \approx -1.5820 \text{ and } a_1 \leq \frac{a}{e^a-1} \approx 0.5820,$$

so (A2) in Theorem 3.3 holds true. On the other hand, let $m = 50, n = 3$, we have

$$z = ha = \frac{a}{m} = 0.02, B_{2j} = 2.3404 \times 10^{-8}, \phi(z) = 0.9900, R(z) = 1 + \frac{z}{\phi(z)} = 1.0202.$$

Because

$$\phi(z) \geq \frac{z}{e^z - 1} \approx 0.9900,$$

then we obtain

$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a\,(R(z)^m+1)}{R(z)^m-1}\right) \approx -18.3279 \leq 0, a_0 > -\frac{aR(z)^m}{R(z)^m-1} \approx -1.5820 \text{ and } a_1 \leq \frac{a}{R(z)^m-1} \approx 0.5820,$$

so (B2) in Theorem 3.4 holds true.

From Figure 4.2 we can see that the analytic solution and the numerical solution of (4.2) are unstable and non-oscillatory, which is agreement with Theorems 3.3 (A2) and 3.4 (B2).

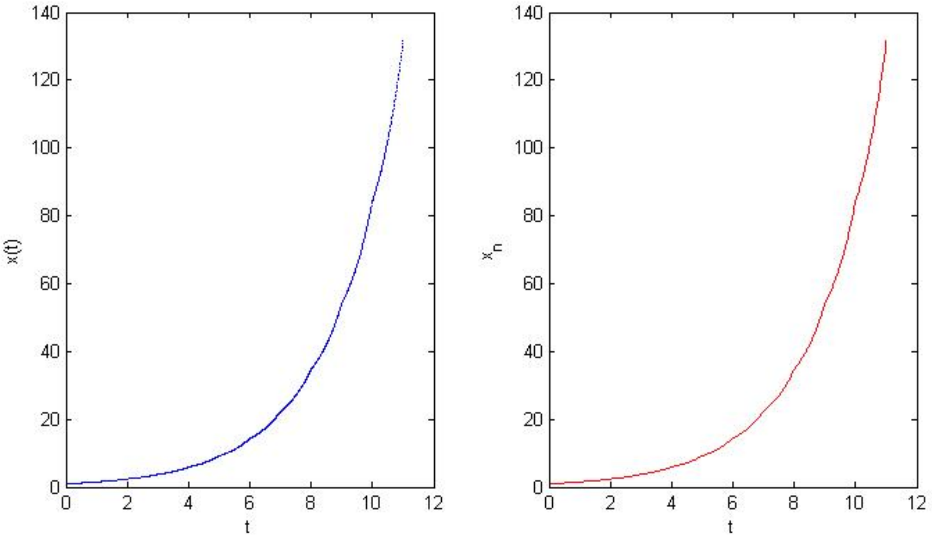

Figure 4.2: The analytic solution (left) and the numerical solution (right, $n = 3$) of (4.2).

From (4.3) we know that the coefficients are $a = 1, a_0 = 1, a_1 = 2$, then

$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a(e^a+1)}{e^a-1}\right) \approx -4.6558 \leq 0, a_0 > -\frac{ae^a}{e^a-1} \approx -1.5820 \text{ and } a_1 > \frac{a}{e^a-1} \approx 0.5820,$$

so (A3) in Theorem 3.3 holds true. On the other hand, let $m = 50, n = 4$, we have

$$z = ha = \frac{a}{m} = 0.02, B_{2j} = 2.3404 \times 10^{-8}, \phi(z) = 0.9900, R(z) = 1 + \frac{z}{\phi(z)} = 1.0202.$$

Because

$$\phi(z) \geq \frac{z}{e^z-1} \approx 0.9900,$$

then we obtain

$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a(R(z)^m+1)}{R(z)^m-1}\right) \approx -4.6558 \leq 0, a_0 > -\frac{aR(z)^m}{R(z)^m-1} \approx -1.5820 \text{ and } a_1 > \frac{a}{R(z)^m-1} \approx 0.5820,$$

so (B3) in Theorem 3.4 holds true.
From Figure 4.3 we can see that the analytic solution and the numerical solution of (4.3) are unstable and oscillatory, which is agreement with Theorems 3.3 (A3) and 3.4 (B3).

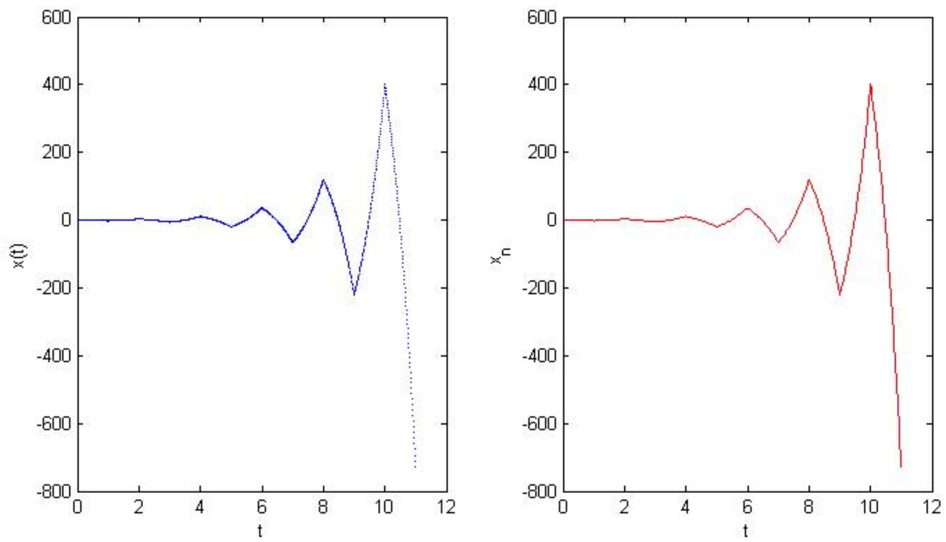

Figure 4.3: The analytic solution (left) and the numerical solution (right, $n = 4$) of (4.3).

From (4.4) we know that the coefficients are $a=-2, a_0=-3, a_1=-2$, then

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\approx 11.3825>0, a_0<-\frac{ae^a}{e^a-1}\approx -0.3130 \text{ and } a_1<\frac{a}{e^a-1}\approx 2.3130,$$

so (A4) in Theorem 3.3 holds true. On the other hand, let $m=50, n=4$, we have

$$z=ha=\frac{a}{m}=-0.04, B_{2j}=2.3404\times 10^{-8}, \phi(z)=1.0201, R(z)=1+\frac{z}{\phi(z)}=0.9608.$$

Because

$$\phi(z)\geq \frac{z}{e^z-1}\approx 1.0201,$$

then we obtain

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(R(z)^m+1)}{R(z)^m-1}\right)\approx 11.3825>0, a_0<-\frac{aR(z)^m}{R(z)^m-1}\approx -0.3130 \text{ and } a_1<\frac{a}{R(z)^m-1}\approx 2.3130,$$

so (B4) in Theorem 3.4 holds true.

From Figure 4.4 we can see that the analytic solution and the numerical solution of (4.4) are asymptotically stable and oscillatory, which is agreement with Theorems 3.3 (A4) and 3.4 (B4).

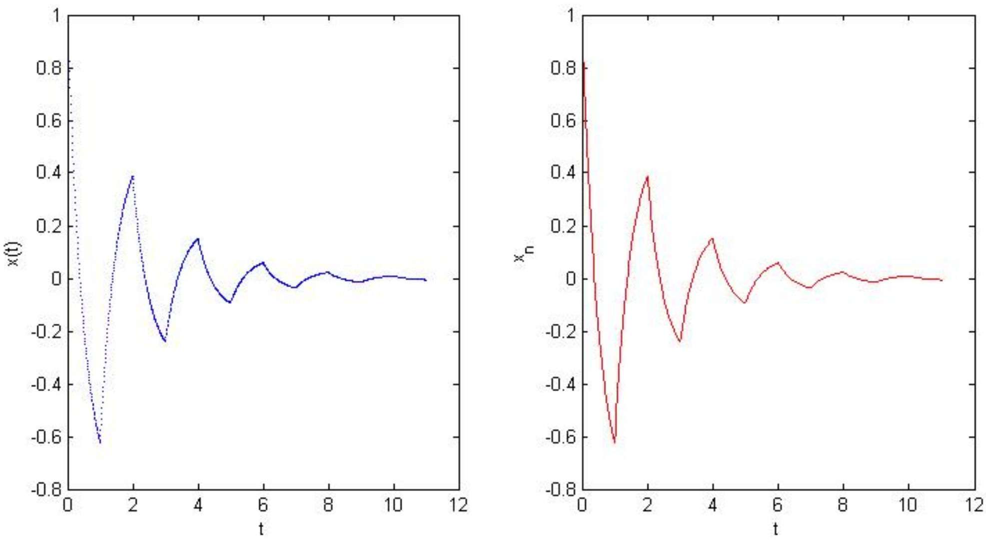

Figure 4.4: The analytic solution (left) and the numerical solution (right, $n=4$) of (4.4).

In particular, when $a=0$, Eq. (4.1) becomes

$$x'(t)=-2x([t])+5x([t+1]), \quad x(0)=1, \tag{4.5}$$

that is, $a_0=-2, a_1=5$, so we have

$$(a_0+a_1)(a_1-a_0-2)=15>0, \quad a_0<-1 \text{ and } a_1\geq 1,$$

so (C1) in Theorem 3.5 holds true.

From Figure 4.5 we also see that the analytic solution and the numerical solution of (4.5) are asymptotically stable and non-oscillatory, which is agreement with Theorem 3.5 (C1).

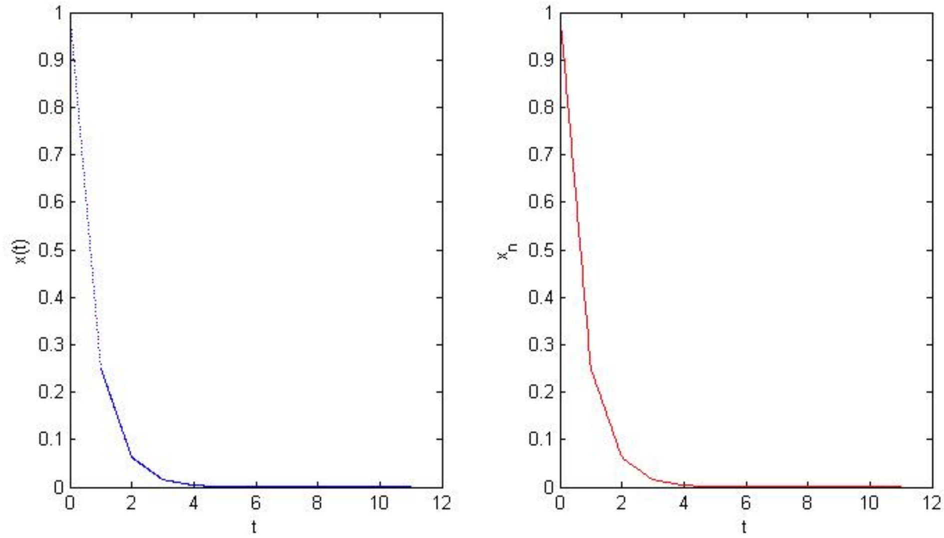

Figure 4.5: The analytic solution (left) and the numerical solution (right) of (4.5).

When $a = 0$, Eq. (4.2) becomes

$$x'(t) = 4x([t]) - 3x([t+1]), \quad x(0) = 1, \tag{4.6}$$

that is, $a_0 = 4, a_1 = -3$, so we have

$$(a_0 + a_1)(a_1 - a_0 - 2) = -9 \leq 0, \quad a_0 > -1 \text{ and } a_1 \leq 1,$$

so (C2) in Theorem 3.5 holds true.
From Figure 4.6 we also see that the analytic solution and the numerical solution of (4.6) are unstable and non-oscillatory, which is agreement with Theorem 3.5 (C2).

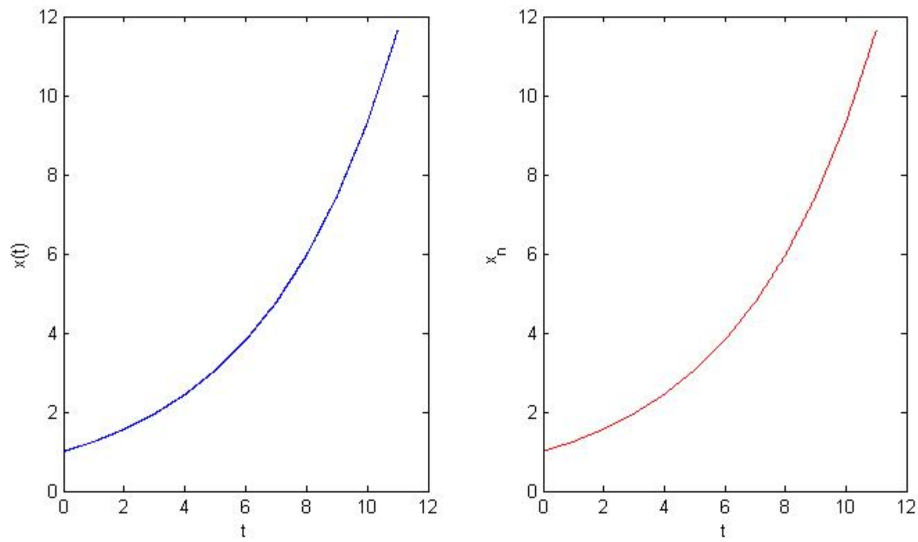

Figure 4.6: The analytic solution (left) and the numerical solution (right) of (4.6).

When $a = 0$, Eq. (4.3) becomes

$$x'(t) = x([t]) + 2x([t+1]), \quad x(0) = 1, \tag{4.7}$$

that is, $a_0 = 1, a_1 = 2$, so we have

$$(a_0 + a_1)(a_1 - a_0 - 2) = -3 \leq 0, \quad a_0 > -1 \text{ and } a_1 > 1,$$

so (C3) in Theorem 3.5 holds true.
From Figure 4.7 we also see that the analytic solution and the numerical solution of (4.7) are unstable and oscillatory, which is agreement with Theorem 3.5 (C3).

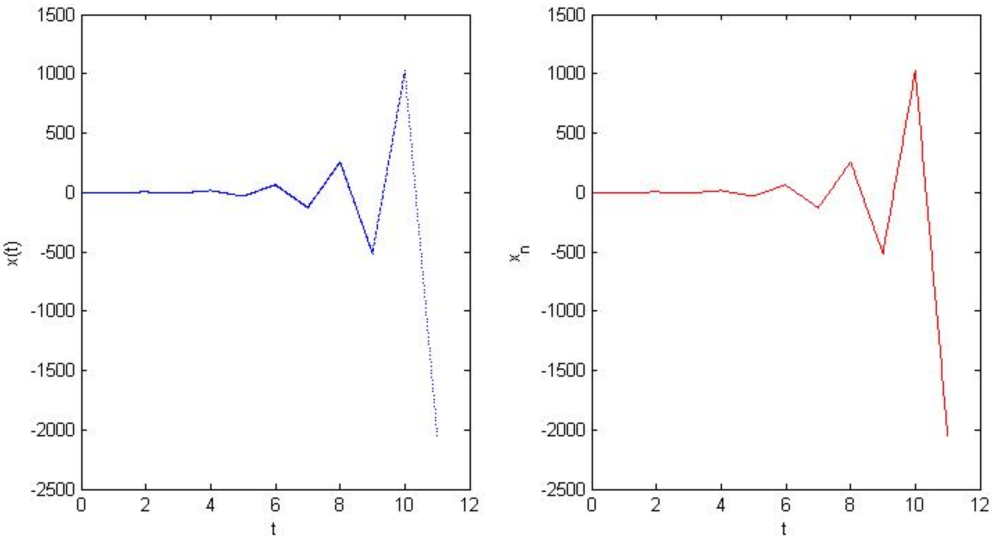

Figure 4.7: The analytic solution (left) and the numerical solution (right) of (4.7).

When $a = 0$, Eq. (4.4) becomes

$$x'(t) = -3x([t]) - 2x([t+1]), \quad x(0) = 1, \tag{4.8}$$

that is, $a_0 = -3, a_1 = -2$, so we have

$$(a_0 + a_1)(a_1 - a_0 - 2) = 5 > 0, \quad a_0 < -1 \text{ and } a_1 < 1,$$

so (C4) in Theorem 3.5 holds true.
From Figure 4.8 we also see that the analytic solution and the numerical solution of (4.8) are asymptotically stable and oscillatory, which is agreement with Theorem 3.5 (C4).

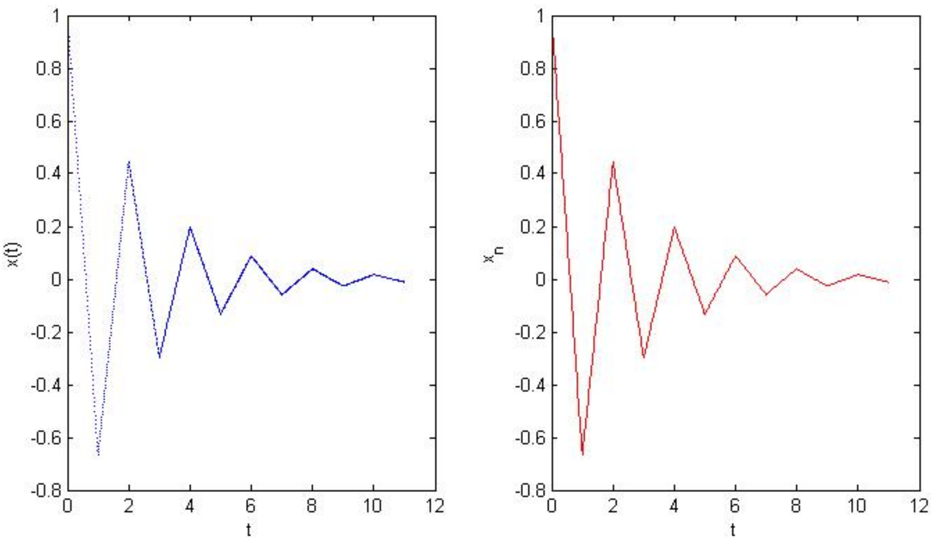

Figure 4.8: The analytic solution (left) and the numerical solution (right) of (4.8).

5. Conclusion

In this paper, the Euler-Maclaurin methods are applied to discrete differential equations with piecewise constant arguments of advanced and retarded type. We obtained the stability, oscillation conditions and convergence order of numerical methods. The type of Euler-Maclaurin methods for solving differential equations with piecewise constant arguments is extended and the results of corresponding literature are generalized. In the future, we will consider the application of the numerical method to the multi-dimensional and fractional cases.

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Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Appendix A

The following code is the Matlab detail of Table 1.

```
%n = 2
syms d;
a = -1;
a0 = -2;
a1 = 5;
x0 = 1;
t = 10;
m = 2;
h = 1/m;
z = h*a;
R1 = symsum(1/d^2,1,inf);R1 = double(R1);
R2 = symsum(1/d^4,1,inf);R2 = double(R2);
A = 1-z/2+R1*z^2/(2*pi^2)-R2*z^4/(8*pi^4);
R = 1+z/A;
k1 = (R^m+(a0/a)*(R^m-1))/(1-(a1/a)*(R^m-1));
x = zeros(1,11);
x(1) = x0;
for k = 1:10
    x(k+1) = k1*x(k)
end
b0 = (exp(a)+(exp(a)-1)*(a0/a))/(1-(a1/a)*(exp(a)-1));
X = b0^10;
AE = abs(x(11)-X)
RE = abs(AE/X)
```

Appendix B

The following code is the Matlab detail of Figure 4.1.

```
a = -1;
a0 = -2;
a1 = 5;
x0 = 1;
%t = 10;
m = 50;
h = 1/m;
z = h*a;
for j = 1:3
    for k = 1:10
        B = 2*(-1)^(j+1)*factorial(2*j)*sum((2*k*pi)^(-2*j));
        A = 1-z/2+sum((B*z^(2*j))/factorial(2*j));
    end
end
R = 1+z/A;
k1 = (R^m+(a0/a)*(R^m-1))/(1-(a1/a)*(R^m-1));
x = zeros(1,12*m);
%x(0) = x0;
x(m) = x0;
t = zeros(1,11*m+1);
for k = 1:11
    x(m*(k+1)) = k1*x(m*k);
    for l = 0:m-2
        k2 = R^(l+1)+(a0/a)*(R^(l+1)-1);
        k3 = (a1/a)*(R^(l+1)-1);
        x(k*m+l+1) = k2*x(k*m)+k3*x((k+1)*m);
    end
end
y = x(m:end);
for i = 0:11*m
```

```
    t(i+1) = i/m;
end
subplot(1,2,2)
plot(t,y,'r–')
xlabel('t');
ylabel('x_n');
hold on;
for   n = 0 : 10
    for   t = n : 0.01 : n+1
        z = ((exp(a*(t-n)) + (a0/a)*(exp(a*(t-n)) - 1)) + (exp(a) + (a0/a)*(exp(a) - 1))/(1 - (a1/a)*(exp(a) -
1))*(a1/a)*(exp(a*(t-n)) - 1))*((exp(a) + (a0/a)*(exp(a) - 1))/(1 - (a1/a)*(exp(a) - 1)))^n;
        subplot(1,2,1)
        plot(t,z,'b-.')
        hold on
    end
end
hold off
xlabel('t');
ylabel('x(t)');
```