Adıyaman University Journal of Science
https://dergipark.org.tr/en/pub/adyujsci
DergiPark AKADEMİK
ISSN 2147-1630
e-ISSN 2146-586X

On Surfaces in Pseudo-Galilean Space with Prescribed Mean Curvature

Muhittin Evren AYDIN[1], Alper Osman ÖĞRENMİŞ[2,*]

[1]*Fırat University, Faculty of Science, Department of Mathematics, 23100, Elazığ, Türkiye*
meaydin@firat.edu.tr, ORCID: 0000-0001-9337-8165
[2]*Fırat University, Faculty of Science, Department of Mathematics, 23100, Elazığ, Türkiye*
aogrenmis@firat.edu.tr, ORCID: 0000-0001-5008-2655

Received: 01.04.2021 **Accepted:** 04.08.2021 **Published:** 31.12.2021

Abstract

In this study, we consider certain classes of surfaces in the pseudo-Galilean space, the translation and factorable surfaces. We obtain these surfaces that satisfy the equation $H = v^{\perp}$, where H is the mean curvature and $v^{\perp}$ is the normal component of an isotropic vector v.

Keywords: Translating soliton; Manifolds with density; Mean curvature; Pseudo-Galilean space.

Yarı-Galileo Uzayında Belli Bir Ortalama Eğriliğe Sahip Yüzeyler Üzerine

Öz

Bu çalışmada; yarı-Galileo uzayında, öteleme ve ayrışabilir yüzeyler denilen iki belirgin sınıf ele alınmıştır. $v^{\perp}$, bir v izotropik vektörün normal bileşeni olmak üzere bu yüzeylerden ortalama eğriliği $H = v^{\perp}$ denklemini sağlayanlar elde edilmiştir.

Anahtar Kelimeler: Ötelenen soliton; Yoğunluklu manifoldlar; Ortalama eğrilik; Yarı-Galileo uzayı.

* Corresponding Author DOI: 10.37094/adyujsci.907765

1. Introduction

We are interested in the pseudo-Galilean geometry which is one of the real Cayley-Klein geometries. Let G_3^1 denote the pseudo-Galilean 3-space, $S \subset G_3^1$ an admissible surface, H and N are the mean curvature and unit normal vector field on S, respectively. Moreover, let $(L^2, \langle \cdot,\cdot \rangle_L)$ denote the Lorentzian 2-space. We consider the following:

$$H = \mathrm{v}^{\perp}, \tag{1}$$

where $\mathrm{v}^{\perp}$ is the normal component of a unit isotropic vector $\mathrm{v} \in G_3^1$. Note that $v^{\perp} = \langle N, v \rangle_L$ is the Lorentzian angle function of S between N and v. Up to the abolute figure of G_3^1, since N is completely isotropic and orthogonal to all non-isotropic vectors, some minimal surface obeys to Eqn. (1) if v is non-isotropic. This is the justification why we take v as isotropic in Eqn. (1).

The importance of Eqn. (1) is due to the theories of manifolds with density and mean curvature flow. A surface whose mean curvature holds Eqn. (1) is called *translating soliton* of the mean curvature flow [1-5]. In the Euclidean setting, besides straight lines, one-dimensional solution to Eqn. (1) is the curve $s \mapsto -logcos\, s$, which is called *grim reaper* and known for moving upwards with constant speed under the flow, see [6, 7]. The hyperbolic versions of those functions are the so-called *Lorentzian grim reapers,* $s \mapsto logsinh\, s$ and $s \mapsto logcosh\, s$ [8]. In the Galilean setting, the situation is different. More explicitly, let κ be the curvature of a smooth curve γ in the Galilean plane G_2 and $\langle \cdot,\cdot \rangle_G$ the Galilean scalar product in G_2. Then, Eqn. (1) writes $\kappa = \langle (0,1), v \rangle_G$, admitting solutions as straight lines ($\kappa = 0$) and parabolic circles ($\kappa = 1$).

Let (x, y, z) denote the affine coordinates in G_3^1 and $\varphi(x, y, z) = kx + py + qz, k,\ p, q \in$ R. From theory of manifolds with density, a surface satisfying Eqn. (1) is indeed a *minimal* surface with density e^{φ} [3, 9, 10]. Meanwhile, since $v = grad\, \varphi = ke_1 + pe_2 + qe_3$ is isotropic for standard basis vectors $\{e_1, e_2, e_3\}$ of G_3^1, k must vanish in our case. More generally, a φ *−mean curvature* (or *weighted mean curvature*) H_{φ} with density e^{φ} is given by $2H_{\varphi} = 2H - (d\varphi/dN)$.

One of the basic classes of surfaces in differential geometry is the translation surfaces generated by translating two curves up to isometry of ambient space. Let $S \subset G_3^1$ be a translation surface and $s \mapsto \alpha(s)$ and $t \mapsto \beta(t)$ two parametric curves, $s \in I \subset$ R, $t \in J \subset$ R. Then, S is locally given by

$$x(s, t) = \alpha(s) + \beta(t), \tag{2}$$

where $\alpha\ := \alpha(s)$ and $\beta\ := \beta(t)$ are called *generating curves.* The other class of the surfaces in which we are interested is the one associated with the product of two single variable functions, namely the *factorable* (or *homothetical) surfaces*. Up to the absolute figure a factorable surface is given by one of the explicit forms

$$x = f(s)g(t) \text{ and } z = f(s)g(t), \tag{3}$$

for smooth functions $f(s)$ and $g(t)$. Those surfaces in Galilean and pseudo-Galilean geometries have been considered in several research articles from different geometrical point of views. For example, the results on these surfaces in terms of Gaussian and mean curvatures can be found in [11-19], while the ones in terms of the Laplacian associated with the fundamental forms are in [20-23]. Some surfaces satisfying Eqn. (1) in G_3^1 were already considered from the manifolds with density point of view, [24-27].

In some sense, solving Eqn. (1) is a problem of finding prescribed mean curvature surfaces, which is our main interest. In this paper, we firstly study translation surfaces Eqn. (2) in G_3^1, whose mean curvature satisfies Eqn. (1). When both α and β are planar, the problem was already solved in [26] and for this reason, we deal with the only case that one of α or β is planar and the other spatial. Under this condition, we solve Eqn. (1) completely. In Section 4, we also classify the surfaces given by Eqn. (3) which satisfy Eqn. (1).

2. Preliminaries

In this section, we recall some basics on the curves and surfaces in the pseudo-Galilean geometry from [18, 28-33]. We also refer to [34, 35] for the Lorentzian arguments.

Let $P_3(R)$ denote a real projective 3-space and $(u_0 : u_1 : u_2 : u_3)$ the homogeneous coordinates. The *pseudo-Galilean 3-space* G_3^1 is a Cayley-Klein space $P_3(R)$ with the *absolute figure* $\{\omega, f, I\}$, where ω is the *absolute plane* $u_0 = 0$, f the *absolute line* $u_0 = u_1 = 0$ and I the *fixed hyperbolic involution* of points of f. The hyperbolic involution is $(0 : 0 : u_2 : u_3) \mapsto (0 : 0 : u_3 : u_2)$ and then $u_2^2 - u_3^2 = 0$ is the *absolute conic*.

Let us introduce the affine coordinates $(u_0 : u_1 : u_2 : u_3) = (1 : x : y : z)$. Up to the absolute figure, the *pseudo-Galilean distance* between the points $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ is

$$d(p,q) = \begin{cases} |q_1 - p_1|, & \text{if } p_1 \neq q_1 \\ \sqrt{|(q_2 - p_2)^2 - (q_3 - p_3)^2|}, & \text{if } p_1 = q_1. \end{cases}$$

The six-parameter group of motions of G_3^1 leaves invariant the absolute figure and the pseudo-Galilean distance, given in terms of affine coordinates as follows:

$$\overline{x} = a_1 + x$$
$$\overline{y} = a_2 + a_3 x + y \cosh \phi + z \sinh \phi$$
$$\overline{z} = a_4 + a_5 x + y \sinh \phi + z \cosh \phi,$$

where $a_1, .., a_5, \phi$ are some constants.

There are two sorts of lines and planes in G_3^1. We call a line *isotropic* when its intersection with the absolute line f is non-empty and *non-isotropic* otherwise. A plane is said to be *isotropic* if it does not involve f, otherwise it is said to be *non-isotropic.* The non-isotropic planes are so-called *Lorentzian* since its induced geometry is Lorentzian. In the affine model of G_3^1, the Lorentzian planes are in the form $x = const$.

A vector $v = (v_1, v_2, v_3)$ is said to be *isotropic* (*non-isotropic)* if $v_1 = 0$ ($\neq 0$). Let $w = (w_1, w_2, w_3)$ and $\langle \cdot,\cdot \rangle_G$ denote the *pseudo-Galilean dot product.* Then, $\langle v, w \rangle_G$ is the Lorentzian scalar product if both v and w are isotropic. Otherwise, $v_1^2 + w_1^2 \neq 0$, it is defined by $\langle v, w \rangle_G = v_1 w_1$. The *pseudo-Galilean angle* between v and w is defined as the Lorentzian angle if v and w are isotropic. Otherwise, it is given by the pseudo-Galilean distance. We call v and w *orthogonal* if $\langle v, w \rangle_G = 0$.

An isotropic vector v is called *spacelike* if $\langle v, v \rangle_L > 0$; *timelike* if $\langle v, v \rangle_L < 0$ and *lightlike* if $\langle v, v \rangle_L = 0$. We call the spacelike and timelike vectors *non-degenerate.* Let $\{e_1, e_2, e_3\}$ be standard basis vectors and v and w no both isotropic vectors. Then, the *pseudo-Galilean cross-product* is

$$v \times_G w = \begin{vmatrix} 0 & -e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Therefore we have $\langle v \times_G w, z \rangle_G = -det(v, w, \tilde{z})$, where $\tilde{z}$ is the projection of z onto the yz −plane. Note that the vector $v \times_G w$ is orthogonal to both v and w.

Let C be a curve given in the parametric form

$$s \mapsto r(s) = (x(s), y(s), z(s)), s \in I \subset R.$$

The curve C is said to be *admissible* if the following conditions hold: for each $s \in I$,

1) $r' = \frac{dr}{ds}$ is non-isotropic;

2) no where C has no inflection points, i.e. r' and $r'' = \frac{d^2 r}{ds^2}$ are linearly independent;

3) $\tilde{r}'$ and $\tilde{r}''$ are non-degenerate.

Then an admissible curve C is said to be *parameterized by arc-length* if the function x is the identity, up to a translation of G_3^1. Let C be such a curve. Then we call $t = r'$ *unit tangent* to C and $\kappa = \sqrt{\langle r'', r'' \rangle_L}$ *curvature* of C. The *normal* and *binormal* to C are defined by

$$n = \frac{1}{\kappa(s)}\left(0, y'', z''\right) \text{ and } b = \frac{1}{\kappa(s)}\left(0, z'', y''\right).$$

The *torsion* of C is introduced by

$$\tau = \frac{\det\left(r', r'', r'''\right)}{\kappa^2}.$$

We call the admissible curve C *spatial* provided $\tau \neq 0$ for each $s \in I$. We call an admissible curve *isotropic planar* if it fully lies in an isotropic plane and in such case τ vanishes identically. We also call a curve *Lorentzian planar* if it fully lies in a Lorentzian plane. For a Lorentzian planar curve the Frenet apparatus are well known.

Let S be a surface in G_3^1 locally given by a regular map

$$(u_1, u_2) \mapsto x(u_1, u_2) = \left(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)\right), (u_1, u_2) \in D \subset R^2.$$

Let $x_{,i} = \frac{\partial x}{\partial u_i}$ and $x_{,ij} = \frac{\partial^2 x}{\partial u_i \partial u_j}$ and etc., $1 \leq i, j \leq 2$. Then, S is said to be *admissible* if $x_{,i} \neq 0$, for some $i = 1,2$. For such an admissible surface S, the *first fundamental form* is

$$\langle dx, dx \rangle_G = E du_1^2 + 2F du_1 du_2 + G du_2^2,$$

where $E = \left(x_{,1}\right)^2$, $F = x_{,1} x_{,2}$, $G = \left(x_{,2}\right)^2$. Since nowhere an admissible surface has Lorentzian tangent plane, up to the absolute figure, the isotropic vector $x_{,1} \times_G x_{,2}$ is *normal* to S. Let

$$W = \left\langle x_{,1} \times_G x_{,2}, x_{,1} \times_G x_{,2} \right\rangle_L.$$

Then the surface S is called *spacelike* if $W < 0$; *timelike* if $W > 0$; and *lightlike* if $W = 0$. The spacelike and timelike surfaces are so-called *non-degenerate* and then, the *unit normal vector* to the non-degenerate surface S is

$$N = \frac{x_{,1} \times_G x_{,2}}{\sqrt{|W|}}.$$

Let $\varepsilon = \langle N, N \rangle_L = \pm 1$ and

$$L_{ij} = \varepsilon \frac{1}{x_{,1}} \left\langle x_{,1} \tilde{x}_{,ij} - \left(x_{,i}\right)_{,j} \tilde{x}_{,1}, N \right\rangle_L = \varepsilon \frac{1}{x_{,2}} \left\langle x_{,2} \tilde{x}_{,ij} - \left(x_{,i}\right)_{,j} \tilde{x}_{,2}, N \right\rangle_L,$$

where one of $x_{,1}$ and $x_{,2}$ is always nonzero due to the admissibility. Then the *second fundamental form* of S is

$$II = Ldu_1^2 + 2Mdu_1du_2 + Ndu_2^2,$$

where $L = L_{11}$, $M = L_{12}$, $N = L_{22}$. Thereby, *the Gaussian and mean curvatures* are defined as

$$K = -\varepsilon \frac{LN - M^2}{|W|} \text{ and } H = -\varepsilon \frac{GL - 2FM + EN}{2|W|}.$$

We call a surface *minimal* if H vanishes identically. Throughout this study, we deal with only non-degenerate admissible surfaces.

3. Translation Surfaces

Let $S \subset G_3^1$ a translation surface whose one generating curve lies in a Lorentzian plane and the other admissible. Then, it locally parameterizes

$$x(s,t) = \alpha(s) + \beta(t),$$

in which we may assume $\beta(v)$ fully lies in the Lorentzian yz −plane. Then the unit normal vector field and mean curvature are

$$N = n_\beta \text{ and } H = \frac{1}{2} k_\beta,$$

where n_β and k_β the principal normal and Frenet curvature of β. Then, for the surface S, Eqn. (1) is now

$$k_\beta = 2\varepsilon \langle n_\beta, v \rangle_L,$$

which means that β is one dimensional solution in L^2 to translating soliton Eqn. (1). As can be seen the generating curve α does not play a role. Therefore, we may state that

Proposition 1. Let S be a translation surface in G_3^1 given by $x(s,t) = \alpha(s) + \beta(t)$, where α is some admissible curve and β is Lorentzian planar. Then, S holds Eqn. (1) if and only if β is one dimensional solution in L^2 to Eqn. (1).

We next consider the translation surface S whose one generating curve is isotropic planar, say β, and the other spatial. Let

$$s \mapsto \alpha(s) = \big(s, f(s), h(s)\big) \text{ and } t \mapsto \beta(t) = \big(t, 0, g(t)\big),$$

where $(s,t) \in I \times J \subset R^2$. Then, S is locally given by

$$x(s,t) = (s + t, f(s), h(s) + g(t)). \tag{4}$$

Denote a prime the derivative with respect to the related variable. Since α is assumed to be spatial, the following holds

$$f''h''' - f'''h'' \neq 0, \tag{5}$$

implying that f'' and g'' must be linearly independent. The unit normal vector field and mean curvature are

$$N = \frac{1}{\sqrt{\left|(g'-h')^2 - f'^2\right|}}(0, g' - h', -f')$$

and

$$H = -\frac{f''(g'-h') + f'(h''+g'')}{2\left|(g'-h')^2 - f'^2\right|^{3/2}}.$$

Let $v = (0, p, q)$. Then, Eqn. (1) is now

$$f''(g' - h') + f'(h'' + g'') = 2\varepsilon(p(h' - g') - qf')\left(f'^2 - (g' - h')^2\right). \tag{6}$$

The successive derivatives of Eqn. (6) with respect to s and t yield

$$f'''g'' + f''g''' = 4\varepsilon g''\left(-pf'f'' - 3(g' - h')h'' + q\left(f''g' - (f'h')'\right)\right). \tag{7}$$

Assume that $g'' \neq 0$ in Eqn. (7), for each $t \in J$. Dividing Eqn. (7) with g'' and then taking derivative with respect to t gives

$$f''\left(\frac{g'''}{g''}\right)' = 4\varepsilon g''(-3h'' + qf''), \tag{8}$$

in which the right-hand side of Eqn. (8) is non-vanishing due to Eqn. (5). Then, there exists a nonzero constant c such that

$$\left(\frac{g'''}{g''}\right)' = 4\varepsilon c g''. \tag{9}$$

Hence, Eqn. (8) reduces to $(q - c)f'' = 3h''$, which contradicts with Eqn. (5). This discussion allows our assumption to be false, namely there exists $t_0 \in J$ such that $g'' = 0$ in a neighborhood of t_0 in J. In such a case the generating curve $\beta(t)$ is a non-isotropic straight line parallel to $(1,0,g_0)$, where $g' = g_0 \in R$, namely the surface given by Eqn. (4) is a cylinder with non-isotropic rulings.

Therefore, we have proved

Theorem 1. The cylinders with non-isotropic rulings are the only translating solitons of translation type whose one generating curve is isotropic planar and the other is spatial.

There is a class of translation solitons of translation type that we do not consider: both α and β are spatial, which remains an open problem.

4. Factorable Surfaces

Let $s \mapsto f(s)$ and $t \mapsto g(t)$, $s \in I, t \in J$, be smooth functions and $S \subset G_3^1$ locally the graph of the product of $f \ := f(s)$ and $g \ := g(t)$. Assume that f and g are non-vanishing on $I \times J$. Up to the absolute figure, the geometric properties of S depend on if it is the graph on an isotropic or Lorentzian planes. Thereby, we consider the surfaces $z = f(s)g(t)$ (or equivalently $y = f(s)g(t)$) and $x = f(s)g(t)$, separately.

Let S be locally the surface $z = f(s)g(t)$ which is parameterized by

$$(s,t) \mapsto x(s,t) = \big(s, t, f(s)g(t)\big).$$

The unit normal and mean curvature are given by

$$N = \frac{1}{\sqrt{|(fg')^2 - 1|}}\big(0, fg', 1\big)$$

and

$$H = \frac{fg''}{2|(fg')^2 - 1|^{3/2}}.$$

Let $v = (0, p, q)$ and then Eqn. (1) is

$$fg'' = 2\varepsilon(pfg' - q)\left((fg')^2 - 1\right). \tag{10}$$

Assume that f is a non-constant function. Then, (10) turns to a polynomial of degree 2 in f

$$-2\varepsilon q + (2\varepsilon p g' + g'')f + 2\varepsilon g'^2 f^2 - 2\varepsilon p g'^2 f^3 = 0,$$

in which the coefficients must vanish, giving $q = 0$ and $g' = 0$. We then deduce that $v = (0,1,0)$ and

$$x(s,t) = \big(s, 0, g_0 f(s)\big) + t(0,1,0), \tag{11}$$

where $g' = g_0 \in R - \{0\}$. Eqn. (11) is a parameterization of a cylinder with isotropic rulings. If $f = f_0 \in R - \{0\}$, then

$$x(s,t) = s(1,0,0) + \big(0, t, f_0 g(t)\big),$$

which is a parameterization of a cylinder with non-isotropic rulings.

Therefore, we have proved

Theorem 2. The cylinders are the only translating solitons of the form $z = f(s)g(t)$.

We next take the surface $x = f(s)g(t)$, parameterized by

$$(s,t) \mapsto x(s,t) = (f(s)g(t), s, t). \tag{12}$$

The unit normal and mean curvature are given by

$$N = \frac{1}{\sqrt{|(f'g)^2 - (fg')^2|}}(0, f'g, -fg')$$

and

$$H = \frac{(f'g)^2 fg'' - 2fg(f'g')^2 + (fg')^2 f''g}{2|(f'g)^2 - (fg')^2|^{3/2}}.$$

Then Eqn. (1) with $v = (0, p, q)$ writes

$$fg\{f'^2(gg'' - g'^2) + g'^2(ff'' - f'^2)\} = 2\varepsilon(pf'g + qfg')\left((f'g)^2 - (fg')^2\right). \tag{13}$$

The functions f and g play symmetric roles in Eqn. (13) and we only concentrate for f.

Case (a). $f = f_0 \in R - \{0\}$. Then Eqn. (13) implies $v = (0,1,0)$ and

$$x(s,t) = s(0,1,0) + (f_0 g(t), 0, t),$$

which is a cylinder with isotropic rulings.

Case (b). f and g are both non-constant functions. We divide Eqn. (13) with $fg(f'g')^2$ and write

$$\left(\frac{f'}{f}\right)' + \left(\frac{g'}{g}\right)' = 2\varepsilon\left(p\frac{f'}{f} + q\frac{g'}{g}\right)\left(\left(\frac{g}{g'}\right)^2 - \left(\frac{f}{f'}\right)^2\right). \tag{14}$$

After successive derivatives of Eqn. (14) with respect to s and t we may deduce

$$p\left(\frac{f'}{f}\right)'\left(\frac{g}{g'}\right)\left(\frac{g}{g'}\right)' - q\left(\frac{g'}{g}\right)'\left(\frac{f}{f'}\right)\left(\frac{f}{f'}\right)' = 0. \tag{15}$$

Assume that f and f' are linearly independent and, by symmetry, so are g and g'. Then Eqn. (15) reduces to

$$p\left(\frac{f'}{f}\right)^3 - q\left(\frac{g'}{g}\right)^3 = 0,$$

which implies $p = q = 0$. This is a contradiction. Therefore, f and f' must be linearly dependent and put $f' = b_1 f$, namely $f(s) = b_2 e^{b_1 s}$ for nonzero constants b_1 and b_2. Therefore, the surface becomes $x = b_2 e^{b_1 s} g(t)$ or, up to a translation, Eqn. (12) turns to

$$(x,t) \mapsto \left(x, \frac{1}{b_1} log\left|\frac{x}{g(t)}\right|, t\right), \tag{16}$$

which is a translation surface. Its generating curves are

$x \mapsto \alpha(x) \ := \left(x, \frac{1}{b_1} log|x|, 0\right)$, and $t \mapsto \beta(t) \ := \left(0, -\frac{1}{b_1} log|g(t)|, t\right)$.

Then, a surface $x = f(s)g(t)$ satisfying Eqn.(1) has to be of form Eqn. (16) if it is non-cylindrical. Since $\beta(t)$ is fully in the Lorentzian yz −plane, it has to be one dimensional solution to Eqn. (1) in L^2 due to Proposition 1.

To sum up, we have proved

Theorem 3. A surface $x = f(s)g(t)$ satisfying Eqn. (1) is either a cylinder with isotropic rulings or a translation surface of the form Eqn. (16), where one generating curve is one dimensional solution to Eqn. (1) in L^2.

Example 1. Let $a^2 - b^2 = \pm 1, a, b \in R - \{0\}$. Consider the surface $x = e^{as+bt}$ and take $v = (0, b, -a)$. This surface is indeed minimal and its normal is $N = (0, a, -b)$ such that Eqn. (1) holds obviously. Notice that it is also given by

$(x, t) \mapsto \left(x, \frac{1}{a} log|x| - \frac{b}{a} t, t\right)$.

Acknowledgement

The authors are thankful to the referees for their valuable comments and suggestions helping improvement of the paper.

References

[1] Bueno, A., *Translating solitons of the mean curvature flow in the space* $H^2 \times R$, Journal of Geometry, 109, Article number 42, 2018. https://doi.org/10.1007/s00022-018-0447-x, 2018.

[2] López, R., *Separation of variables in equations of mean-curvature type*, Proceedings of the Royal Society of Edinburgh Section A, 146(5), 1017-1035, 2016.

[3] López, R., *Some geometric properties of translating solitons in Euclidean space,* Journal of Geometry, 109, Article number 40, 2018. https://doi.org/10.1007/s00022-018-0444-0

[4] Aydin, M.E., López, R., *Ruled translating solitons in Minkowski 3-space*, arXiv:2109.05254v1, 2021.

[5] Aydin, M.E., López, R., *Translating solitons by separation of variables*, arXiv:2109.05253v1 , 2021.

[6] Angenent, S., *On the formation of singularities in the curve shortening flow*, Journal of Differential Geometry, 33, 601-633, 1991.

[7] Halldorsson, H., *Self-similar solutions to the curve shortening flow*, Transactions of the *American Mathematical Society*, 364(10), 5285-5309, 2012.

[8] Castro, I., Castro-Infantes, I., Castro-Infantes, J., *Curves in the Lorentz-Minkowski plane: elasticae, catenaries and grim-reapers,* Open Mathematics, 16,747-766, 2018.

[9] Altin, M., Kazan, A., Karadag, H.B., *Ruled surfaces in* E^3 *with density*, Honam

Mathematical Journal, 41(4), 683-695, 2019.

[10] Hieu, D.T., Hoang, N.M., *Ruled minimal surfaces in R^3 with density e^z* , Pacific Journal of Mathematics, 243(2), 277-285, 2009.

[11] Abdel-Aziz, H.S., Saad, M.K., Ali, H.A., *Affine factorable surfaces in pseudo-Galilean space*, arXiv:1812.00765v1 [math.GM].

[12] Aydin, M.E., Ogrenmis, A.O., Ergut, M., *Classification of factorable surfaces in the pseudo-Galilean space*, Glasnik Matematicki Series III, 50(70), 441-451, 2015.

[13] Aydin, M.E., Kulahci, M.A., Ogrenmis, A.O., *Non-zero constant curvature factorable surfaces in pseudo-Galilean space*, Communications of the Korean Mathematical Society, 33(1), 247-259, 2018.

[14] Aydin, M.E., Kulahci, M.A., Ogrenmis, A.O., *Constant curvature translation surfaces in Galilean 3-space*, International Electronic Journal of Geometry, 12(1), 9-19, 2019.

[15] Dede, M., Ekici, C., *On minimal surfaces in Galilean space*, Conference Proceedings of Science and Technology, 2(2), 142-147, 2019.

[16] Kelleci, A., *Translation-factorable surfaces with vanishing curvatures in Galilean 3-spaces,* International Journal of Maps in Mathematics, 4(1), 14-26, 2021.

[17] Milin-Sipus, Z., Divjak, B., *Translation surface in the Galilean space*, Glasnik Matematicki Series III, 46(66), 455-469, 2011.

[18] Milin-Sipus, Z., *On a certain class of translation surfaces in a pseudo-Galilean space*, International Mathematical Forum, 6(23), 1113-1125, 2012.

[19] Yoon, D.W., *Some classification of translation surfaces in Galilean 3-space*, *International Journal* of *Mathematical* Analysis, 28(6), 1355-1361, 2012.

[20] Abdel-Baky, R.A., Unluturk, Y., *A study on classification of translation surfaces in pseudo-Galilean 3-space*, Journal of Coupled Systems and Multiscale Dynamics, (6)3, 233-240, 2018.

[21] Bansal, P., Shahid, M.H., *On classification of factorable surfaces in Galilean space G^3*, Jordan Journal of Mathematics and Statistics, 12(3), 289-306, 2019.

[22] Cakmak, A., Karacan, M.K., Kiziltug, S., Yoon, D.W., *Corrigendum to translation surfaces in the 3-dimensional Galilean space satisfying $\Delta^{II} x_i = \lambda_i x_i$*, *Bulletin* of the *Korean Mathematical Society,* 56(2), 549-554, 2019.

[23] Lone, M.S., *Homothetical surfaces in three dimensional pseudo-Galilean spaces satisfying $\Delta^{II} x_i = \lambda_i x_i$*, Advances in Applied Clifford Algebras, 29(5), 92, 2019. https://doi.org/10.1007/s00006-019-1007-7

[24] Kazan, A., Karadag, H.B., *Weighted minimal and weighted flat surfaces of revolution in Galilean 3-space with density*, International Journal of Analysis and Applications, 16(3), 414-426, 2018.

[25] Mosa, S., Elzawy, M., *Helicoidal surfaces in Galilean space with density*, Frontiers in Physics, 8, Article 81, 2020. https://doi.org/10.3389/fphy.2020.00081.

[26] Yoon, D.W., *Weighted minimal translation surfaces in the Galilean space with density*, Open Mathematics, 15, 459-466, 2017.

[27] Yoon, D.W., Lee, J.W., Lee, C.W., *φ −minimal rotational surfaces in pseudo-Galilean space with density*, *Annales Polonici* Mathematici, 120, 183-196, 2017.

[28] Altin, M., Unal, I., *Surface family with common line of curvature in 3-dimensional*

Galilean space, Facta Universitatis: Series Mathematics and Informatics, 35(5), 1315-1325, 2020.

[29] Divjak, B., *Curves in pseudo-Galilean geometry*, Annales Universitas Scientiarium Budapestinesis, 41,117-128,1998.

[30] Milin-Sipus, Z., Divjak, B., *Surfaces of constant curvature in the pseudo-Galilean space*, International Journal of Mathematics and Mathematical Sciences, Art ID375264, 28pp., 2012.

[31] Mólnar, E., *The projective interpretation of the eight 3-dimensional homogeneous geometries*, Beiträge zur *Algebra* und Geometrie, 38(2), 261-288, 1997.

[32] Onishchick, A., Sulanke, R., *Projective and Cayley-Klein Geometries*, Springer, 2006.

[33] Röschel, O., *Die Geometrie des Galileischen Raumes*, Habilitationsschrift, Leoben, 1984.

[34] López, R., *Differential geometry of curves and surfaces in Lorentz-Minkowski space*, International Electronic Journal of Geometry, 7(1), 44-107, 2014.

[35] O'Neill, B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, 1983.