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Hadamard Type Integral Inequalities for Differentiable (h,m) −Convex Functions

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Abstract

In this article, firstly, basic information on convex function types and classical inequalities is given. Then, some Hadamard type inequalities are proved by using (h,m) −convex functions with the help of an integral identity that has the potential to generate Hadamard type inequalities.

Keywords: Hadamard type inequalities, Hölder inequality, (h,m) −convex functions.

1. *Introduction*

Convex function concept is a structure that has come to the forefront among the known function classes in mathematics with its features, geometric interpretation, wide usage areas and aesthetic structure. Convex functions whose definition is given by an inequality have a structure expressed with the help of the mean function. In addition, the basic idea in the definition of convex function is to make a comparison between the appearance of the linear composition of two points under the function and the linear composition of the images of those points. Let's start with the definition of this function class.

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Definition 1.1. A function $f: I \to \mathbb{R}$ is said to be convex if the following inequality holds:

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

for all $x,y \in I$ and $t \in [0,1]$ (Pečarić et al. 1992).

Different types of convexity have been defined as a generalization of convex functions, which are widely used in the fields of statistics, engineering, convex programming and inequality theory. We will now try to briefly introduce these function classes.

Definition 1.2. (Varošanec 2007) Let $h: J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f: I \to \mathbb{R}$ is h −convex function, or that f belongs to the class $SX(h,I)$, if f is non-negative and for all $x,y \in I$, $0<\alpha<1$ we have

$$f(\alpha x+(1-\alpha)y) \leq h(\alpha)f(x)+h(1-\alpha)f(y).$$

Definition 1.3. Let $m \in [0,1]$. The function $f:[0,b] \to \mathbb{R}$ is said to be m −convex if

$$f(tx+m(1-t)y) \leq tf(x)+m(1-\alpha)f(y)$$

is satisfied for every $x,y \in [0,b]$ and $t \in [0,1]$ (Toader 1988).

Definition 1.4. (Özdemir *et al.* 2016) Let $h: J \to \mathbb{R}$ be a non-negative function. We say that $f:[0,b] \to \mathbb{R}$ is (h,m) −convex function, if f is non-negative and for all $x,y \in [0,b], m \in [0,1]$, $\alpha \in (0,1)$, we have the following inequality

$$f(\alpha x+m(1-\alpha)y) \leq h(\alpha)f(x)+mh(1-\alpha)f(y).$$

If the above inequality is reversed, then f is said to be (h,m) −concave function on $[0,b]$. Obviously, if one choose $h(\alpha)=\alpha$, then the definition reduces to non-negative m −convex functions. If we set $m=1$, the definition overlap with the definiton of h −convex functions. It is clear that we can obtain non-negative convex functions, P−functions, Godunova-Levin functions and s −convex functions (in the second sense) by selecting special values of the parameters.

Although there are many more generalizations of the concept of convexity in the literature, the function classes we have presented above have attracted the attention of many researchers and have been taken as the main motivation point in many studies. One of the main consequences for convex functions is the Hermite-Hadamard inequality. This famous inequality not only produces lower and upper limits for the Cauchy mean value of a convex function, it has also been subject to important applications in numerical analysis, statistics, and approximation theory. The following double inequality is called Hermite-Hadamard inequality in the literature.

Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and let $a,b \in I$ with $a<b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

The above inequality is in the reversed direction if f is concave.

To explore more details related to different kinds of convexity and further integral inequalities see the papers Azpetia 1994, Beckenbach 1948, Breckner 1978, Dragomir and Pearce 1998, Dragomir and Fitzpatrick 1999, Hadamard 1893, Mitrinovic 1970, Mitrinovic et al. 1993 and Orlicz 1961.

The main purpose of this study is to obtain integral inequalities containing new generalizations of Hadamard type for second order derivative (h,m) −convex functions. To prove the main inequalities, an integral identity for twice differentiable functions and some classical inequalities are used.

Main Results

In order to prove our main theorems, we need the following integral identity that was proved by Özdemir et al. in 2013:

Lemma 2.1. (Özdemir et al. 2013) Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I°$ where $a,b \in I$ with $a<b$. If $f'' \in L[a,b]$, then the following equality holds:

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{16}\left[\int_0^1 t^2 f''\left(t\frac{a+b}{2} + (1-t)a\right)dt + \int_0^1 (t-1)^2 f''\left(tb + (1-t)\frac{a+b}{2}\right)dt\right].$$

Theorem 2.2. Let $f: I \subset [0,d) \to \mathbb{R}$ be a differentiable mapping on $I°$, such that $f'' \in L[a,b]$, *where* $a,b \in I$ with $0 \le a < b < \infty$. If $|f''|$ is (h,m) −convex on $[a,b]$, for some fixed $m \in (0,1]$, $t \in [0,1]$, then the following inequality holds:

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{16}\left[\left|f''\left(\frac{a+b}{2}\right)\right|\int_0^1 t^2 h(t)dt + m\left|f''\left(\frac{a}{m}\right)\right|\int_0^1 t^2 h(1-t)dt\right] + \frac{(b-a)^2}{16}\left[|f''(b)|\int_0^1 (t-1)^2 h(t)dt + m\left|f''\left(\frac{a+b}{2m}\right)\right|\int_0^1 (t-1)^2 h(1-t)dt\right].$$

Proof. From Lemma 2.1, we can write

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)^2}{16}\left[\int_0^1 t^2 \left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right| dt + \int_0^1 (t-1)^2 \left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right| dt\right]$$

Since $|f''|$ is (h,m) −convex on $[a,b]$, we know that for any $t \in [0,1]$

$$\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right| = \left|f''\left(t\frac{a+b}{2}+m(1-t)\frac{a}{m}\right)\right| \leq h(t)\left|f''\left(\frac{a+b}{2}\right)\right| + mh(1-t)\left|f''\left(\frac{a}{m}\right)\right|$$

and

$$\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right| = \left|f''\left(tb+m(1-t)\frac{a+b}{2m}\right)\right| \leq h(t)|f''(b)| + mh(1-t)\left|f''\left(\frac{a+b}{2m}\right)\right|.$$

Thus, by using the facts that are given above, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\int_0^1 t^2\left[h(t)\left|f''\left(\frac{a+b}{2}\right)\right| + mh(1-t)\left|f''\left(\frac{a}{m}\right)\right|\right]dt + \frac{(b-a)^2}{16}\int_0^1 (t-1)^2\left[h(t)|f''(b)| + mh(1-t)\left|f''\left(\frac{a+b}{2m}\right)\right|\right]dt$$

$$= \frac{(b-a)^2}{16}\left[\left|f''\left(\frac{a+b}{2}\right)\right|\int_0^1 t^2 h(t)dt + m\left|f''\left(\frac{a}{m}\right)\right|\int_0^1 t^2 h(1-t)dt\right] + \frac{(b-a)^2}{16}\left[|f''(b)|\int_0^1 (t-1)^2 h(t)dt + m\left|f''\left(\frac{a+b}{2m}\right)\right|\int_0^1 (t-1)^2 h(1-t)dt\right]$$

.

This completes the proof.

Remark 2.3. Under the assumptions of Theorem 2.2, if we set $m = 1$, then we have the following new result

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\left(\left(\left|f''\left(\frac{a+b}{2}\right)\right| + |f''(b)|\right)\int_0^1 t^2 h(t)dt + \left(|f''(a)| + f''\left(\frac{a+b}{2}\right)\right)\int_0^1 t^2 h(1-t)dt\right).$$

Theorem 2.4. Let $f: I \subset [0,d) \to \mathbb{R}$ be a differentiable mapping on I° , such that $f'' \in L[a,b], where\ a,b \in I$ with $0 \leq a < b < \infty$. If $|f''|^q$ is (h,m) −convex on $[a,b]$, for some fixed $m \in (0,1]$, $t \in [0,1]$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then the following inequality holds:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 h(t)dt + m\left|f''\left(\frac{a}{m}\right)\right|^q\int_0^1 h(1-t)dt\right)^{\frac{1}{q}} + \left(|f''(b)|^q\int_0^1 h(t)dt + m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\int_0^1 h(1-t)dt\right)^{\frac{1}{q}}\right].$$

Proof. Suppose that $q > 1$. From Lemma 2.1 and using the well known Hölder inequality, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt + \int_0^1 (t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]$$

$$\leq \frac{(b-a)^2}{16}\left(\int_0^1 t^{2p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^2}{16}\left(\int_0^1 (t-1)^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$

Because $|f''|^q$ is (h,m) −convex on $[a,b]$, we have

$$\int_0^1 \left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt \leq h(t)\left|f''\left(\frac{a+b}{2}\right)\right|^q + mh(1-t)\left|f''\left(\frac{a}{m}\right)\right|^q$$

and

$$\int_0^1 \left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt \leq h(t)|f''(b)|^q + mh(1-t)\left|f''\left(\frac{a+b}{2m}\right)\right|^q.$$

By a simple computation,

$$\int_0^1 t^{2p}dt = \frac{1}{2p+1}$$

and

$$\int_0^1 (t-1)^{2p}dt = \frac{1}{2p+1}.$$

Therefore, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 h(t)dt + m\left|f''\left(\frac{a}{m}\right)\right|^q\int_0^1 h(1-t)dt\right)^{\frac{1}{q}} + \left(|f''(b)|^q\int_0^1 h(t)dt + m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\int_0^1 h(1-t)dt\right)^{\frac{1}{q}}\right].$$

This completes the proof.

Remark 2.5. Under the assumptions of Theorem 2.4, if we set $m=1$, then we have the following new result

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 h(t)dt + |f''(a)|^q\int_0^1 h(1-t)dt\right)^{\frac{1}{q}} + \left(|f''(b)|^q\int_0^1 h(t)dt + \left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 h(1-t)dt\right)^{\frac{1}{q}}\right].$$

Theorem 2.6. Let $f: I \subset [0,d) \to \mathbb{R}$ be a differentiable mapping on I° , such that $f'' \in L[a,b]$, *where* $a,b \in I$ with $0 \leq a < b < \infty$. If $|f''|^q$ is (h,m) −convex on $[a,b]$, for some fixed $m \in (0,1]$, $t \in [0,1]$ and $q \geq 1$, then the following inequality holds:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^2h(t)dt + m\left|f''\left(\frac{a}{m}\right)\right|^q\int_0^1 t^2h(1-t)dt\right)^{\frac{1}{q}} + \left(|f''(b)|^q\int_0^1 (t^2-1)h(t)dt + m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\int_0^1 (t^2-1)h(1-t)dt\right)^{\frac{1}{q}}\right].$$

Proof. Suppose that $q \geq 1$. From Lemma 2.1 and using well known the Power mean inequality, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt + \int_0^1 (t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]$$

$$\leq \frac{(b-a)^2}{16}\left(\int_0^1 t^2 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2 \left|f"\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^2}{16}\left(\int_0^1 (t-1)^2 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (t-1)^2 \left|f"\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$

Because $|f"|^q$ is (h,m) −convex, we have

$$\int_0^1 t^2 \left|f"\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt \leq \left|f"\left(\frac{a+b}{2}\right)\right|^q \int_0^1 t^2 h(t)dt + m\left|f"\left(\frac{a}{m}\right)\right|^q \int_0^1 t^2 h(1-t)dt$$

and

$$\int_0^1 t^2 \left|f"\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt \leq |f"(b)|^q \int_0^1 (t^2-1)h(t)dt + m\left|f"\left(\frac{a+b}{2m}\right)\right|^q \int_0^1 (t^2-1)h(1-t)dt.$$

Therefore, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\left|f"\left(\frac{a+b}{2}\right)\right|^q \int_0^1 t^2 h(t)dt + m\left|f"\left(\frac{a}{m}\right)\right|^q \int_0^1 t^2 h(1-t)dt\right)^{\frac{1}{q}} + \left(|f"(b)|^q \int_0^1 (t^2-1)h(t)dt + m\left|f"\left(\frac{a+b}{2m}\right)\right|^q \int_0^1 (t^2-1)h(1-t)dt\right)^{\frac{1}{q}}\right].$$

This completes the proof.

Remark 2.7. Under the assumptions of Theorem 2.6, if we set $m=1$, then we have the following new result

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\left|f"\left(\frac{a+b}{2}\right)\right|^q \int_0^1 t^2 h(t)dt + |f"(a)|^q \int_0^1 t^2 h(1-t)dt\right)^{\frac{1}{q}} + \left(|f"(b)|^q \int_0^1 (t^2-1)h(t)dt + \left|f"\left(\frac{a+b}{2}\right)\right|^q \int_0^1 (t^2-1)h(1-t)dt\right)^{\frac{1}{q}}\right].$$

Theorem 2.8. Let $f: I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f" \in L[a,b]$, where $a,b \in I$ with $0 \leq a < b < \infty$. If $|f"|^q$ is (h,m) −convex on $[a,b]$, for some fixed $m \in (0,1]$ and $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\left(\frac{1}{p(2p+1)} + \frac{1}{q}\int_0^1 h(t)\left|f"\left(\frac{a+b}{2}\right)\right|^q + mh(1-t)\left|f"\left(\frac{a}{m}\right)\right|^q dt\right) + \frac{(b-a)^2}{16}\left(\frac{1}{p(2p+1)} + \frac{1}{q}\int_0^1 h(t)|f"(b)|^q + mh(1-t)\left|f"\left(\frac{a+b}{2m}\right)\right|^q dt\right)$$

Proof. Since $|f"|^q$ is a (h,m) −convex on $[a,b]$, from Lemma 2.1 and using the Young inequality, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right.$$
$$\left.+\int_0^1 (t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]$$
$$\leq \frac{(b-a)^2}{16}\int_0^1\left(\frac{t^{2p}}{p}+\frac{\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q}{q}\right)dt$$
$$+\frac{(b-a)^2}{16}\int_0^1\left(\frac{(t-1)^{2p}}{p}\right.$$
$$\left.+\frac{\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q}{q}\right)dt$$
$$\leq \frac{(b-a)^2}{16}\left(\int_0^1\frac{t^{2p}}{p}dt\right.$$
$$\left.+\int_0^1\frac{h(t)\left|f''\left(\frac{a+b}{2}\right)\right|^q+mh(1-t)\left|f''\left(\frac{a}{m}\right)\right|^q}{q}dt\right)$$
$$+\frac{(b-a)^2}{16}\left(\int_0^1\frac{(t-1)^{2p}}{p}dt\right.$$
$$\left.+\int_0^1\frac{h(t)|f''(b)|^q+mh(1-t)\left|f''\left(\frac{a+b}{2m}\right)\right|^q}{q}dt\right).$$

This completes the proof.

Remark 2.9. Under the assumptions of Theorem 2.8, if we set $m=1$, then we have the following new result:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{p(2p+1)}\right.$$
$$+\frac{1}{q}\int_0^1 h(t)\left|f''\left(\frac{a+b}{2}\right)\right|^q$$
$$\left.+h(1-t)|f''(a)|^q dt\right)$$
$$+\frac{(b-a)^2}{16}\left(\frac{1}{p(2p+1)}\right.$$
$$+\frac{1}{q}\int_0^1 h(t)|f''(b)|^q$$
$$\left.+h(1-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q dt\right).$$

Remark 2.10. Under the assumptions of our Theorems, if we select special cases of $h(t)$, then we can provide several integral inequalities for different kinds of convex functions. We omit the details.

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