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Research Article

Mappings that transform helices from Euclidean space to Minkowski space

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Abstract

In this study, we introduce mappings that transform helices in Euclidean n-space to non-null helices in Minkowski n-space or Minkowski $(n+1)$-space. Furthermore, we show that these mappings preserve the axes of the helices, and we also obtain the invariants of the mappings. Especially, by using these mappings, we give some examples of non-null helices which are constructed in Minkowski 3-space or Minkowski 4-space from some helices in Euclidean 3-space.

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1. Introduction

Helices are widely studied geometric objects that found relevancy in many fields, including but not limited to biology, computer aided design, architecture, or mechanical engineering. For example, the shape of the twisted-ladder structure of deoxyribonucleic acid (DNA) is a double helix [6, 13, 15].

In Euclidean 3-space, a helix is defined by the property that its tangent vector field makes a fixed angle with a fixed direction which is the axis of the helix . This well-known result was stated by M. A. Lancret in 1802 [9] and first proved by B. de Saint Venant in 1845. A necessary and sufficient condition for a curve to be a general helix is to have the ratio of its curvature to torsion constant. If both curvature and torsion are non-zero constants, then the curve is a circular helix [2, 9, 14]. We can adapt the helix notion to the Minkowski 3-space by using the angle notions in this space. Helix notion can similarly be extended to any n-dimensional $(n > 3)$ Euclidean or Minkowski spaces see [4, 10].

In [3], Altunkaya and Kula studied mappings that preserve helices in the n-dimensional Euclidean space, and in [1], Altunkaya studied mappings that preserve helices in the n-dimensional Minkowski space. These special mappings have been further characterized in these works.

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The papers mentioned above led us to study mappings from n-dimensional Euclidean space to n-dimensional Minkowski space that transform helices. We found some special mappings which can be associated with special relativity with a suitable change of the first coordinate to time (ct). Also, with these mappings, one can find the correlation of the helicoid motion of a particle in these spaces.

This paper is organized as follows. In section 2, we give basic theory of curves in Euclidean n-space and Minkowski n-space. Also, similar to well-known the notion of helix in Euclidean n-space, we give definition of non-null helix with non-null axis by using notion of angle between two non-null vector in Minkowski n-space.

In section 3, we define mappings that transform helices with the axis e_1 or e_n from Euclidean n-space $\mathbb{R}^n$ to Minkowski n-space $\mathbb{R}^n_1$, and vice versa. While one mapping that transforms a helix with the axis e_1 in $\mathbb{R}^n$ to a non-null helix with the timelike axis e_1 in $\mathbb{R}^n_1$, the other mappings that transform a helix with axis e_n in $\mathbb{R}^n$ to a non-null helix with the spacelike axis e_n in $\mathbb{R}^n_1$. After, we give some examples about spacelike (or timelike) helix in Minkowski 3-space which is generated by a helix in Euclidean 3-space and illustrate them.

In section 4, we also introduce mappings that transform a helix with the axis e_1 (or e_n) in $\mathbb{R}^n$ to a non-null helix with the timelike axis $(e_1, 0)$ (or the spacelike axis $(0, e_n)$) in $\mathbb{R}^{n+1}_1$. Finally, by using these mappings, we give some examples for non-null helices are constructed in Minkowski 4-space, which plays an important role in the theory of relativity, from some helices in Euclidean 3-space.

2. Preliminary

2.1. Euclidean space

Let $\mathbb{R}^n$ denote the Euclidean n-space, that is, the n-dimensional real vector space endowed with the standard inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i,$$

for all $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. Also, the norm of a vector $x \in \mathbb{R}^n$ is defined by $\|x\| = \sqrt{\langle x, x\rangle}$. Let $\{e_1, e_2, ..., e_n\}$ be the orthonormal basis where $e_j = (\delta_{1j}, \delta_{2j}, ..., \delta_{nj})$ is a unit vector in $\mathbb{R}^n$ for $j = 1, 2, ..., n$.

Let the curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ be a regular curve of order n (i.e. that $\{\gamma'(t), \gamma''(t), ..., \gamma^{(n)}(t)\}$ is a linearly independent subset of $\mathbb{R}^n$ for any $t \in I$). Now, let $\{V_1, V_2, ..., V_n\}$ be the moving Frenet frame along the regular curve γ where V_i $(i = 1, 2, ..., n)$ denotes the ith Frenet vector field. Then, the Frenet formulae are given by

$$\begin{aligned} V_1' &= \nu\kappa_1 V_2 \\ V_i' &= -\nu\kappa_{i-1}V_{i-1} + \nu\kappa_i V_{i+1}, \quad i = 2, 3, ..., n-1 \\ V_n' &= -\nu\kappa_{n-1}V_{n-1} \end{aligned}$$

where $\nu = ||\gamma'||$ and κ_i is the curvature functions of γ [5, 7].

Definition 2.1. The curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ is a helix if its tangent vector field V_1 makes the fixed angle θ with a fixed direction U which is the axis. That is, $\langle V_1, U\rangle = \cos\theta$ where $\theta \in (0, \pi) \setminus \frac{\pi}{2}$ is a constant [12].

Definition 2.2. The $(n + 1) -$ *helix* mapping $\mathcal{G} : \mathbb{R}^n \setminus N \to \mathbb{R}^{n+1}$ is defined by

$$\mathcal{G}(x_1, x_2, ..., x_n) = \frac{c}{d^2 + x_1^2 + x_2^2 + ... + (1 - a^2)x_n^2}(d, x_1, x_2, ..., x_n)$$

where $N = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | \quad x_1^2 + x_2^2 + \ldots + x_{n-1}^2 - (a^2 - 1)\, x_n^2 + d^2 \neq 0\}$, $a > 1$, $c \neq 0$ and $d \neq 0$ [3].

Theorem 2.3. *The curve $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) : I \subset R \to \mathbb{R}^n$ is a helix in $\mathbb{R}^n$ whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis e_n iff*

$$\mathcal{G}(\gamma) = \frac{c}{d^2 + \gamma_1^2 + \gamma_2^2 + ... + (1 - a^2)\gamma_n^2} (d, \gamma_1, \gamma_2, ..., \gamma_n) \tag{2.1}$$

is a helix in $\mathbb{R}^{n+1}$ whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis $(0, e_n)$ where $c \neq 0$, $d \neq 0$, $a > 1$ and $d^2 + \gamma_1^2 + \gamma_2^2 + ... + (1 - a^2)\gamma_n^2 \neq 0$ [3].

2.2. Minkowski space

Let $\mathbb{R}_1^n$ denote the Minkowski n-space, that is, the n-dimensional real vector space $\mathbb{R}^n$ endowed with the scalar product

$$\langle x, y \rangle_\star = -x_1 y_1 + \sum_{i=2}^{n} x_i y_i,$$

for all $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. Also, the norm of a vector $x \in \mathbb{R}_1^n$ is defined by $\|x\|_\star = \sqrt{|\langle x, x \rangle_\star|}$. A vector $x \in \mathbb{R}_1^n$ is said to be spacelike (resp. timelike, null) if $\langle x, x \rangle_\star > 0$ or $x = 0$ (resp. $\langle x, x \rangle_\star < 0$, $\langle x, x \rangle_\star = 0$).

A curve $\alpha : I \to \mathbb{R}_1^n$ is said to be spacelike (resp. timelike, null) if $\alpha' = \frac{d\alpha}{dt}$ is a spacelike (resp. timelike, null) vector at any $t \in I$ [10].

Definition 2.4. Let U, W be any two non-null vectors in $\mathbb{R}_1^n$.

(1) Assume that U and W are spacelike vectors, then
  (a) if $Sp\{U, W\}$ is a spacelike plane, then there is a unique number $0 \leq \overline{\theta} \leq \pi$ such that $\langle U, W \rangle_\star = \|U\|_\star \|W\|_\star \cos \overline{\theta}$,
  (b) if $Sp\{U, W\}$ is a timelike plane, then there is a unique number $\overline{\theta} \geq 0$ such that $\langle U, W \rangle_\star = \varepsilon \|U\|_\star \|W\|_\star \cosh \overline{\theta}$ where $\varepsilon = 1$ or $\varepsilon = -1$ according to $sgn(U_2) = sgn(W_2)$ or $sgn(U_2) \neq sgn(W_2)$, respectively,

(2) Assume that U and W are timelike vectors, then there is a unique number $\overline{\theta} \geq 0$ such that $\langle U, W \rangle_\star = \varepsilon \|U\|_\star \|W\|_\star \cosh \overline{\theta}$ where $\varepsilon = 1$ or $\varepsilon = -1$ according to U and W have different time-orientation or same time-orientation, respectively,

(3) Assume that U is spacelike and W is timelike, then there is a unique number $\overline{\theta} \geq 0$ such that $\langle U, W \rangle_\star = \varepsilon \|U\|_\star \|W\|_\star \sinh \overline{\theta}$ where $\varepsilon = 1$ or $\varepsilon = -1$ according to $sgn(U_2) = sgn(W_1)$ or $sgn(U_2) \neq sgn(W_1)$, respectively,

where $\overline{\theta}$ is angle between U and W [11].

Let $\alpha : I \to \mathbb{R}_1^n$ be a non-null (spacelike or timelike) curve. We assume that $\{\alpha'(t), \alpha''(t), ..., \alpha^{(n)}(t)\}$ are linearly independent at any $t \in I$. Now, let $\left\{\overline{V}_1, \overline{V}_2, ..., \overline{V}_n\right\}$ be the moving Frenet frame along the regular curve α where $\overline{V}_i$ $(i = 1, 2, ..., n)$ denotes the ith Frenet vector field. Then, the Frenet formulae are given by

$$\begin{aligned} \overline{V}_1' &= \nu_\star \varepsilon_2 k_1 \overline{V}_2, \\ \overline{V}_i' &= -\nu_\star \varepsilon_{i-1} k_{i-1} \overline{V}_{i-1} + \nu_\star \varepsilon_{i+1} k_i \overline{V}_{i+1}, \quad i = 2, 3, ..., n-1 \\ \overline{V_n}' &= -\nu_\star \varepsilon_{n-1} k_{n-1} \overline{V}_{n-1} \end{aligned}$$

where, k_i $(i = 1, 2, ..., n-1)$ denotes the ith curvature, $\nu_\star = \|\alpha'\|_\star$ and $\varepsilon_i = \left\langle \overline{V}_i, \overline{V}_i \right\rangle_\star$ for $1 \leq i \leq n$ [8].

By means of Definition 2.4, we can give the following two definitions of non-null helices with non-null axis in $\mathbb{R}_1^n$.

Definition 2.5. A regular curve $\alpha : I \to \mathbb{R}_1^n$ is a spacelike helix if its tangent vector field V_1 makes the fixed angle $\overline{\theta}$ with a non-null unit vector $U \in \mathbb{R}_1^n$ which is the axis. That is, $\varepsilon \langle V_1, U \rangle_\star = f(\overline{\theta})$ is a constant where,

(1) If U is a timelike vector, then $f(\overline{\theta}) = \sinh\overline{\theta}, \quad \overline{\theta} \geq 0$,
(2) If U is a spacelike vector and $Sp\{V_1, U\}$ is a spacelike plane in $\mathbb{R}^n_1$, then $f(\overline{\theta}) = \cos\overline{\theta}, \quad \overline{\theta} \in (0,\pi) \setminus \frac{\pi}{2}$,
(3) If U is a spacelike vector and $Sp\{V_1, U\}$ is a timelike plane in $\mathbb{R}^n_1$, then $f(\overline{\theta}) = \cosh\overline{\theta}, \quad \overline{\theta} \geq 0$.

Definition 2.6. A regular curve $\alpha : I \to \mathbb{R}^n_1$ is a timelike helix if its tangent vector field V_1 makes the fixed angle $\overline{\theta}$ with a non-null unit vector $U \in \mathbb{R}^n_1$ which is the axis. That is, $\varepsilon\langle V_1, U\rangle_\star = g(\overline{\theta})$ is a constant where,

(1) If U is a timelike vector, then $g(\overline{\theta}) = \cosh\overline{\theta}, \quad \overline{\theta} \geq 0$,
(2) If U is a spacelike vector, then $g(\overline{\theta}) = \sinh\overline{\theta}, \quad \overline{\theta} \geq 0$.

Remark 2.7. Throughout this study, all curves are regular and the mappings are built for helices with the axes e_1 or e_n. Moreover, by using similar method, the mappings can be constructed by helices with different axis.

3. Mappings that transform helices from $\mathbb{R}^n$ to $\mathbb{R}^n_1$

In this section, we introduce mappings that transform helices with axes e_1 or e_n from Euclidean n-space to Minkowski n-space, and vice versa.

3.1. A mapping for helices with axis e_1

In this subsection, we introduce a mapping that transforms a helix with axis e_1 in $\mathbb{R}^n$ to a non-null helix with the timelike axis e_1 in $\mathbb{R}^n_1$.

Now, let us define the mapping $\Psi : \mathbb{R}^n \setminus \Gamma \to \mathbb{R}^n_1 \setminus \Gamma$

$$\Psi(x) = \frac{\lambda}{-a^2x_1^2 + \|x\|^2} x, \tag{3.1}$$

where $\Gamma = \left\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | \ \|x\|^2 - a^2{x_1}^2 \neq 0\right\}$ and a $\in \left(1, \sqrt{2}\right) \cup \left(\sqrt{2}, \infty\right)$, $\lambda \neq 0$.

We get easily the following corollary for the mapping Ψ.

Corollary 3.1. *Ψ is an involution. That is, $\Psi = \Psi^{-1}$.*

Lemma 3.2. *The hypercone $C = \left\{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n | \ \sum\limits_{i=2}^{n} {x_i}^2 = b^2x_1^2, \quad b^2 \neq a^2 - 1\right\}$ is invariant under the mapping Ψ where $a > 1$ is a constant.*

Proof. Let $x = (x_1, x_2, ..., x_n) \in C$ and $\Psi(x) = y$. So,

$$y_i = \frac{\lambda}{-a^2x_1^2 + \|x\|^2} x_i, \quad 1 \leq i \leq n,$$

and

$$\begin{aligned}\sum_{i=2}^{n} y_i^2 &= \left(\frac{\lambda}{-a^2x_1^2 + \|x\|^2}\right)^2 \sum_{i=2}^{n} x_i^2 \\ &= b^2\left(\frac{\lambda}{-a^2x_1^2 + \|x\|^2}\right)^2 {x_1}^2 \\ &= b^2 y_1^2.\end{aligned}$$

Thus, $y = (y_1, y_2, ..., y_n) \in C$. □

Lemma 3.3. *$\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) : I \subset R \to \mathbb{R}^n$ is a helix whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis e_1 iff $\left\|\gamma'\right\|^2 = a^2\left(\gamma_1'\right)^2$ or equivalently, $(1-a^2)\left(\gamma_1{}'\right)^2 + \sum\limits_{i=2}^{n}\left(\gamma_i{}'\right)^2 = 0$ where $a > 1$ [3].*

Lemma 3.4. *$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) : J \subset R \to \mathbb{R}_1^n$ is a timelike (spacelike) helix whose tangent vector field V_1 makes the fixed angle $\overline{\theta} = \cosh^{-1}(1/b)$, $(\overline{\theta} = \sinh^{-1}(1/b))$ with the timelike axis e_1 iff*

$$\langle \alpha', \alpha' \rangle_* = \epsilon b^2 (\alpha_1{}')^2, \tag{3.2}$$

or equivalently,

$$-(1+\epsilon b^2)(\alpha_1{}')^2 + \sum_{i=2}^{n} (\alpha_i{}')^2 = 0, \tag{3.3}$$

where b is a nonzero constant and $\epsilon = \pm 1$.

Proof. The proof can be obtained easily by using Lemma 2.1 and Lemma 2.2 in [1] □

By the following theorem, we say that the mapping Ψ transforms a helix with axis e_1 in $\mathbb{R}^n$ to a non-null helix with the timelike axis e_1 in $\mathbb{R}_1^n$.

Theorem 3.5. *Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) : I \subset \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Then, the curve γ is a helix whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$, $a \in (1,\infty)\setminus\sqrt{2}$ with axis e_1 iff the curve,*

$$\alpha : J \subset \mathbb{R} \to \mathbb{R}_1^n, \quad \alpha = \Psi(\gamma) = \frac{\lambda}{-a^2\gamma_1^2 + \|\gamma\|^2}\gamma \tag{3.4}$$

is a timelike (spacelike) helix whose tangent vector field V_1 makes the fixed angle $\overline{\theta} = \cosh^{-1}\left(\frac{\varepsilon}{\sqrt{2-a^2}}\right)$ $\left(\overline{\theta} = \sinh^{-1}\left(\frac{\varepsilon}{\sqrt{a^2-2}}\right)\right)$ in $\mathbb{R}_1^n$ with the timelike axis e_1 where $\lambda \neq 0$ and $1 < a < \sqrt{2}$ $\quad (a > \sqrt{2})$.

Proof. Suppose that the curve γ is a helix whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis e_1 where $a \in (1,\infty)\setminus\sqrt{2}$. Now, let the non-null curve α be given by $\alpha = \Psi(\gamma)$ in $\mathbb{R}_1^n$. Then, we have,

$$\alpha_i = h\,\gamma_i \quad \text{for } i = 1, 2, ..., n \tag{3.5}$$

where

$$h = \frac{\lambda}{-a^2\gamma_1^2 + \|\gamma\|^2}. \tag{3.6}$$

So,

$$\langle \alpha', \alpha' \rangle_\star = -(h\gamma_1)'^2 + \sum_{i=2}^{n} (h\gamma_i)'^2 \tag{3.7}$$

and after a straightforward calculation, we obtain

$$\sum_{i=2}^{n} (h\gamma_i)'^2 = -\left(1-a^2\right)(h\gamma_1)'^2. \tag{3.8}$$

Furthermore, since α is a non-null curve, by using (3.5),(3.7) and (3.8), we get

$$\langle \alpha', \alpha' \rangle_\star = \left(a^2 - 2\right)(\alpha_1{}')^2. \tag{3.9}$$

Therefore, from (3.2), the curve α is a non-null helix with the timelike axis e_1 where

$$\epsilon b^2 = a^2 - 2. \tag{3.10}$$

Thus, by using Lemma 3.4 and (3.10), we have two cases below.

Case 1: If $1 < a < \sqrt{2}$ then α is timelike helix ($\epsilon = -1$) whose tangent vector field V_1 makes the fixed angle $\overline{\theta} = \cosh^{-1}\left(\frac{\varepsilon}{\sqrt{2-a^2}}\right)$ with timelike axis e_1.

Case 2: If $a > \sqrt{2}$ then α is spacelike helix ($\epsilon = 1$) whose tangent vector field V_1 makes the fixed angle $\overline{\theta} = \sinh^{-1}\left(\frac{\varepsilon}{\sqrt{a^2-2}}\right)$ with timelike axis e_1.

Conversely, let us take the non-null helix $\alpha = \Psi(\gamma)$ that satisfies Case 1 or Case 2. Then, by Lemma 3.4, (3.5), (3.6) and (3.10), we get the following differential equation

$$\left(1-a^2\right)(h\gamma_1)'^2 + \sum_{i=2}^{n} (h\gamma_i)'^2 = 0. \tag{3.11}$$

After a straightforward calculation, we obtain

$$h^2\left(-a^2(\gamma_1')^2 + \|\gamma'\|^2\right) = 0. \tag{3.12}$$

Since $h \neq 0$, we have $\left\|\gamma'\right\|^2 = a^2\left(\gamma_1'\right)^2$. From Lemma 3.3, γ is a helix in $\mathbb{R}^n$. □

As a result of Theorem 3.5 and Corrollary 3.1, $\Psi^{-1}(\alpha) = \gamma$ is also a helix in $\mathbb{R}^n$.

Example 3.6. Let us take the helix

$$\gamma(t) = \left(\frac{\sqrt{2}t^3}{3} + \sqrt{2}t, \frac{1}{3}\left(t^2+2\right)^{3/2}, t\right)$$

and its tangent vector

$$V_1(t) = \left(\sqrt{\frac{2}{3}}, \frac{t\sqrt{t^2+2}}{\sqrt{3}\,(t^2+1)}, \frac{1}{\sqrt{3}\,(t^2+1)}\right)$$

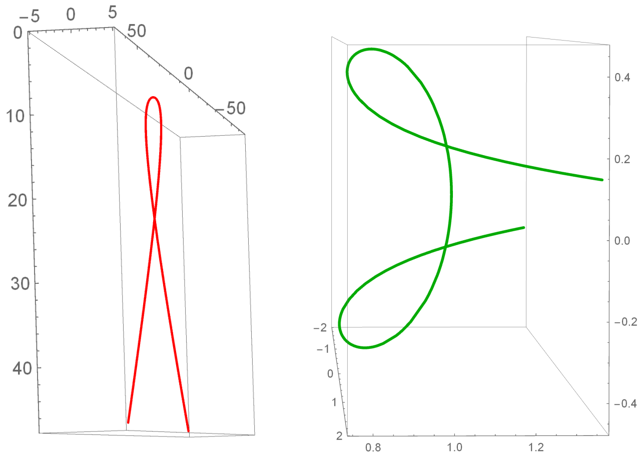
makes the fixed angle $\theta = \cos^{-1}(\sqrt{\frac{2}{3}})$ with axis e_1 in $\mathbb{R}^3$. If we choose $\lambda = 1$ in (3.4),

$$\alpha(t) = \Psi(\gamma(t)) = \left(\frac{3\sqrt{2}t\,(t^2+3)}{12t^2+8}, \frac{3(t^2+2)^{3/2}}{12t^2+8}, \frac{9t}{12t^2+8}\right)$$

is a timelike helix and its tangent vector

$$\overline{V_1}(t) = \left(\sqrt{2}, \frac{t\,(t^2-2)\sqrt{t^2+2}}{t^4-t^2+2}, \frac{2-3t^2}{t^4-t^2+2}\right)$$

makes the fixed angle $\overline{\theta} = \sinh^{-1}\left(\sqrt{2}\right)$ with timelike axis e_1 in $\mathbb{R}^3_1$ (see Figure 1).



(a) **(b)**

Figure 1. **(a)** The helix γ in $\mathbb{R}^3$, **(b)** The timelike helix $\alpha = \Psi(\gamma)$ in $\mathbb{R}^3_1$.

Example 3.7. Let us take the helix

$$\gamma(t)=\left(\frac{5\mathrm{e}^{\mathrm{t}}}{\sqrt{29}}, \mathrm{e}^{\mathrm{t}}\cos 2t, \mathrm{e}^{\mathrm{t}}\sin 2t\right)$$

and its tangent vector

$$V_1(t) = \left(\sqrt{\frac{5}{34}}, \sqrt{\frac{29}{170}}(\cos 2t - 2\sin 2t), \sqrt{\frac{29}{170}}(\sin 2t + 2\cos 2t)\right)$$

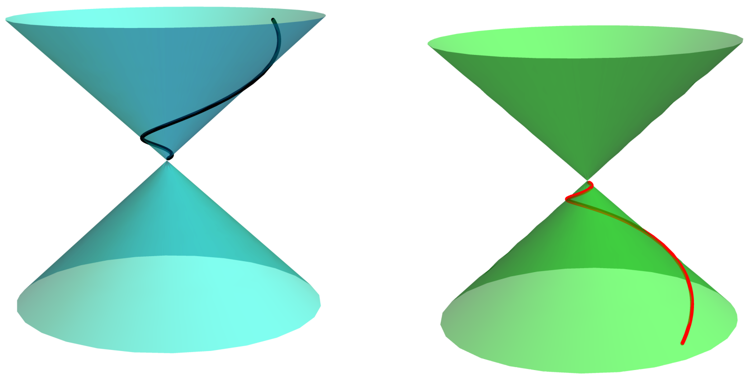
makes the fixed angle $\theta = \cos^{-1}(\sqrt{\frac{5}{34}})$ with axis e_1 in $\mathbb{R}^3$. If we choose $\lambda = 1$ in (3.4),

$$\alpha(t) = \Psi(\gamma(t)) = \left(-\frac{5e^{-t}}{4\sqrt{29}}, -\frac{1}{4}e^{-t}\cos 2t, -\frac{1}{2}e^{-t}\sin t\cos t\right)$$

is a spacelike helix and its tangent vector

$$\overline{V_1}(t) = \left(\frac{\sqrt{5}}{2\sqrt{6}}, \frac{\sqrt{29}}{2\sqrt{30}}(2\sin 2t + \cos 2t), \frac{\sqrt{29}}{2\sqrt{30}}(\sin 2t - 2\cos 2t)\right)$$

makes the fixed angle $\overline{\theta} = \cosh^{-1}\left(\frac{\sqrt{5}}{2\sqrt{6}}\right)$ with timelike axis e_1 in $\mathbb{R}^3_1$. Also, the helices γ and α lie on the surface $\left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, y^2 + z^2 = \frac{29}{25}x^2\right\}$ (see Figure 2).



(a) **(b)**

Figure 2. **(a)** The conical helix γ in $\mathbb{R}^3$, **(b)** The spacelike helix $\alpha = \Psi(\gamma)$ in $\mathbb{R}^3_1$.

3.2. Mappings for helices with axis e_n

In this subsection, we introduce mappings which transforms a helix with axis e_n in $\mathbb{R}^n$ to a non-null helix with the spacelike axis e_n in $\mathbb{R}^n_1$.

Now, let $\psi_1 : \mathbb{R}^n \to \mathbb{R}^n_1$ be the mappping defined by

$$\psi_1(x) = \left(\frac{a^2-1}{a}x_n, \frac{\sqrt{a^2-1}}{a}x_2, \frac{\sqrt{a^2-1}}{a}x_3, \ldots, \frac{\sqrt{a^2-1}}{a}x_{n-1}, \frac{1}{a}x_1\right), \tag{3.13}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $a > 1$.

Similar to Lemma 3.3 and Lemma 3.4 , we give the following two Lemmas.

Lemma 3.8. *$\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) : I \subset R \to \mathbb{R}^n$ is a helix whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis e_n iff $\|\gamma'\|^2 = a^2(\gamma_n')^2$ or equivalently, $(1-a^2)(\gamma_n')^2 + \sum_{i=1}^{n-1}(\gamma_i')^2 = 0$ where $a > 1$.*

Lemma 3.9. *$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) : J \subset R \to \mathbb{R}^n_1$ is a non-null helix whose tangent vector field V_1 makes the fixed angle with the spacelike axis e_n iff*

$$\langle \alpha', \alpha' \rangle_* = \epsilon b^2 (\alpha_n')^2, \tag{3.14}$$

or equivalently,

$$-(\alpha_1')^2 + \sum_{j=2}^{n-1}\left(\alpha_j'\right)^2 + (1-\epsilon b^2)(\alpha_n')^2 = 0, \tag{3.15}$$

where b is a nonzero constant and $\epsilon = \pm 1$.

By the following theorem, we say that the mapping ψ_1 transforms a helix with axis e_n in $\mathbb{R}^n$ to a non-null helix with the spacelike axis e_n in $\mathbb{R}^n_1$.

Theorem 3.10. *Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) : I \subset \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Then, the curve γ is a helix whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis e_n where $a \in \left(1, \sqrt{2}\right) \cup \left(\sqrt{2}, \infty\right)$ iff the curve*

$$\alpha : J \subset \mathbb{R} \to \mathbb{R}^n_1, \quad \alpha = \psi_1(\gamma) \tag{3.16}$$

is a timelike (spacelike) helix whose tangent vector field V_1 makes the fixed angle $\overline{\theta} = \sinh^{-1}\left(\frac{\varepsilon}{\sqrt{-2+a^2}}\right)$ $\left(\overline{\theta} = \cosh^{-1}\left(\frac{\varepsilon}{\sqrt{2-a^2}}\right)\right)$ with the spacelike axis e_n in $\mathbb{R}^n_1$, where $a > \sqrt{2}$ $(1 < a < \sqrt{2})$.

Proof. Suppose that the curve γ is a helix whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with e_n where $a \in (1,\infty)\setminus\sqrt{2}$. Now, by using (3.16), we have

$$\alpha_1 = \frac{a^2-1}{a}\gamma_n, \tag{3.17}$$

$$\alpha_j = \frac{\sqrt{a^2-1}}{a}\gamma_j, \ j = 2,3,...,n-1, \tag{3.18}$$

$$\alpha_n = \frac{1}{a}\gamma_1. \tag{3.19}$$

So,

$$\langle \alpha', \alpha' \rangle_\star = (\alpha_n')^2\left(2-a^2\right), \tag{3.20}$$

and from (3.14), the curve α is a non-null helix with the spacelike axis e_n where

$$\epsilon b^2 = 2 - a^2. \tag{3.21}$$

Thus, by using Lemma 3.9 and (3.21),

Case 1: If $1 < a < \sqrt{2}$ then α is spacelike helix ($\epsilon = 1$) whose tangent vector field V_1 makes the fixed angle $\overline{\theta} = \cosh^{-1}\left(\frac{\varepsilon}{\sqrt{2-a^2}}\right)$ with spacelike axis e_n.

Case 2: If $a > \sqrt{2}$ then α is timelike helix ($\epsilon = -1$) whose tangent vector field V_1 makes the fixed angle $\overline{\theta} = \sinh^{-1}\left(\frac{\varepsilon}{\sqrt{a^2-2}}\right)$ with spacelike axis e_n.

Conversely, let us take the non-null helix α with spacelike axis e_n in $\mathbb{R}^n_1$ that satisfies Case 1 or Case 2. Then, it is clear that the curve γ is a helix with axis e_n in $\mathbb{R}^n$. □

Let $\psi_2 : \mathbb{R}^n \to \mathbb{R}^n_1$ be the mappping defined by

$$\psi_2(x) = \left(\sqrt{a^4-1}\,x_n, \sqrt{a^2+1}\,x_2, \sqrt{a^2+1}\,x_3, \ldots, \sqrt{a^2+1}\,x_{n-1}, ax_1\right), \tag{3.22}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $a > 1$.

By the following theorem, we say that the mapping ψ_2 transforms a helix with axis e_n in $\mathbb{R}^n$ to a timelike helix with the spacelike axis e_n in $\mathbb{R}^n_1$.

Theorem 3.11. *Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) : I \subset \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Then, the curve γ is a helix whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis e_n where $a > 1$ iff the curve,*

$$\alpha : J \subset \mathbb{R} \to \mathbb{R}^n_1, \quad \alpha = \psi_2(\gamma) \tag{3.23}$$

is a timelike helix whose tangent vector field V_1 makes the fixed angle $\overline{\theta} = \sinh^{-1}(\varepsilon a)$ with the spacelike axis e_n in $\mathbb{R}^n_1$.

Proof. We omit the proof since it is analogous to the proof of Theorem 3.10. □

Let $\psi_3 : \mathbb{R}^n \to \mathbb{R}^n_1$ be the mappping defined by

$$\psi_3(x) = \left(a\sqrt{a^2-1}x_n, ax_2, ax_3, \ldots, ax_{n-1}, \sqrt{a^2-1}x_1\right), \tag{3.24}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $a > 1$.

By the following theorem, we say that the mapping ψ_3 transforms a helix with axis e_n in $\mathbb{R}^n$ to a timelike helix with the spacelike axis e_n in $\mathbb{R}^n_1$.

Theorem 3.12. *Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) : I \subset \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Then, the curve γ is a helix whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis e_n where $a > 1$ iff the curve,*

$$\alpha : J \subset \mathbb{R} \to \mathbb{R}^n_1, \quad \alpha = \psi_3(\gamma) \tag{3.25}$$

is a timelike helix whose tangent vector field V_1 makes the fixed angle $\overline{\theta} = \sinh^{-1}(\varepsilon\sqrt{a^2-1})$ with the spacelike axis e_n in $\mathbb{R}^n_1$.

Proof. We omit the proof since it is analogous to the proof of Theorem 3.10. □

Example 3.13. Let us take the helix

$$\gamma(t) = \left(\sqrt{11}\cos\frac{t}{6}, \sqrt{11}\sin\frac{t}{6}, \frac{5t}{6}\right)$$

and its tangent vector

$$V_1(t) = \left(-\frac{\sqrt{11}}{6}\sin\frac{t}{6}, \frac{\sqrt{11}}{6}\cos\frac{t}{6}, \frac{5}{6}\right)$$

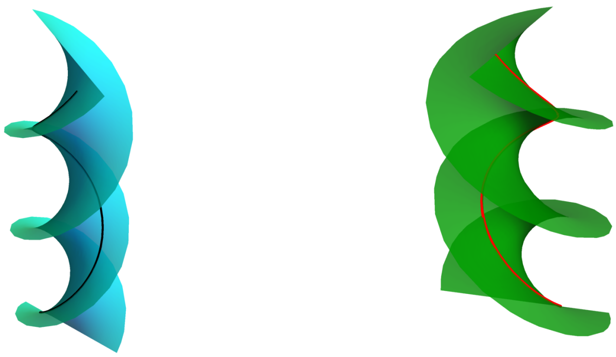
makes the fixed angle $\theta = \cos^{-1}(\frac{5}{6})$ with axis e_3 in $\mathbb{R}^3$. Also, the curve

$$\alpha(t) = \psi_1(\gamma(t)) = \left(\frac{11t}{36}, \frac{11}{6}\sin\frac{t}{6}, \frac{5\sqrt{11}}{6}\cos\frac{t}{6}\right)$$

is a spacelike helix and its tangent vector

$$\overline{V_1}(t) = \left(\sqrt{\frac{11}{14}}\csc\frac{t}{6}, \sqrt{\frac{11}{14}}\cot\frac{t}{6}, -\frac{5}{\sqrt{14}}\right)$$

makes the fixed angle $\bar{\theta} = \cosh^{-1}\left(\frac{5}{\sqrt{14}}\right)$ with the spacelike axis e_3 in $\mathbb{R}^3_1$. Moreover, the curve γ lies on the helicoid $\{(x, y, z) \in \mathbb{R}^3 \,|\, \frac{y}{x} = \tan\frac{z}{5}\}$ in $\mathbb{R}^3$ and the curve α lies on the surface $\left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, \frac{y}{z} = \frac{\sqrt{11}}{5}\tan\frac{6x}{11}\right\}$in $\mathbb{R}^3_1$ (see Figure 3).



(a) **(b)**

Figure 3. **(a)** The helix γ in $\mathbb{R}^3$, **(b)** The spacelike helix $\alpha = \psi_1(\gamma)$ in $\mathbb{R}^3_1$.

Example 3.14. Let us take the spherical helix

$$\gamma(t) = \left(-\frac{3}{5}\cos 4t - \frac{2}{5}\cos 6t, -\frac{3}{5}\sin 4t - \frac{2}{5}\sin 6t, \frac{2\sqrt{6}}{5}\sin t\right)$$

and its tangent vector

$$V_1(t) = \left(\frac{2\sqrt{6}}{5}\sin 5t, \frac{-2\sqrt{6}}{5}\cos 5t, \frac{1}{5}\right)$$

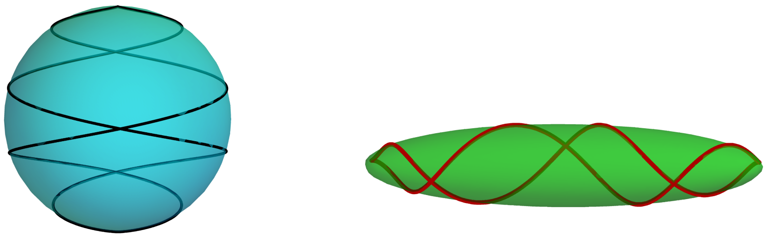
makes the fixed angle $\theta = \cos^{-1}(\frac{1}{5})$ with axis e_3 in $\mathbb{R}^3$. Also, the curve

$$\alpha(t) = \psi_2(\gamma(t)) = \left(\frac{24\sqrt{26}}{5}\sin t, -\frac{\sqrt{26}}{5}(3\sin 4t + 2\sin 6t), -3\cos 4t - 2\cos 6t\right)$$

is a timelike helix and its tangent vector

$$\overline{V_1}(t) = \left(\frac{2\sqrt{26}\cos t}{\sin 4t + \sin 6t}, -\frac{\sqrt{26}(\cos 4t + \cos 6t)}{\sin 4t + \sin 6t}, 5 \right)$$

makes the fixed angle $\overline{\theta} = \sinh^{-1}(5)$ with spacelike axis e_3 in $\mathbb{R}^3_1$. Moreover, the curve γ lies on the unit sphere in $\mathbb{R}^3$ and the curve α lies on the surface $\left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, \frac{x^2}{624} + \frac{y^2}{26} + \frac{z^2}{25} = 1 \right\}$ in $\mathbb{R}^3_1$ (see Figure 4).



(a) (b)

Figure 4. **(a)** The spherical helix γ in $\mathbb{R}^3$, **(b)** The timelike helix $\alpha = \psi_2(\gamma)$ in $\mathbb{R}^3_1$.

Example 3.15. Let us take the helix

$$\gamma(t) = \left(\frac{2t\sin t + 2\cos t}{\sqrt{3}}, \frac{2\sin t - 2t\cos t}{\sqrt{3}}, \frac{t^2+1}{3} \right)$$

and its tangent vector

$$V_1(t) = \left(\frac{\sqrt{3}}{2}\cos t, \frac{\sqrt{3}}{2}\sin t, \frac{1}{2} \right)$$

makes the fixed angle $\theta = \cos^{-1}(\frac{1}{2})$ with axis e_3 in $\mathbb{R}^3$. Also, the curve

$$\alpha(t) = \psi_3(\gamma(t)) = \left(\frac{2t^2+2}{\sqrt{3}}, \frac{4\sin t - 4t\cos t}{\sqrt{3}}, 2t\sin t + 2\cos t \right)$$

is a timelike helix and its tangent vector

$$\overline{V_1}(t) = \left(2\sec t, 2\tan t, \sqrt{3} \right)$$

makes the fixed angle $\overline{\theta} = \sinh^{-1}(\sqrt{3})$ with spacelike axis e_3 in $\mathbb{R}^3_1$. Moreover, the curve γ lies on the paraboloid $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 4z\}$ in $\mathbb{R}^3$ and the curve α lies on the surface $\left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, \frac{3y^2}{4} + z^2 = 2\sqrt{3}\,x \right\}$ in $\mathbb{R}^3_1$ (see Figure 5).

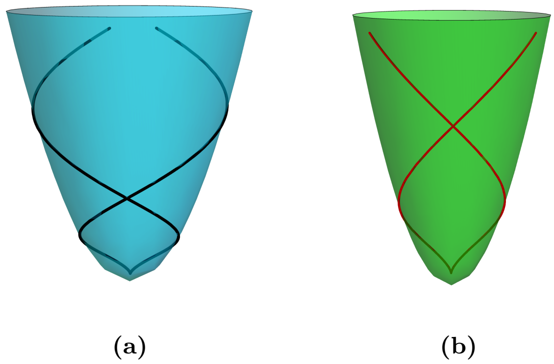


Figure 5. **(a)** The paraboloidal helix γ in $\mathbb{R}^3$, **(b)** The timelike helix $\alpha = \psi_3(\gamma)$ in $\mathbb{R}^3_1$.

4. Mappings that transform helices from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}_1$

Now, we introduce a mapping that transforms a helix with axis e_1 in $\mathbb{R}^n$ to a non-null helix with the timelike axis $(e_1, 0)$ in $\mathbb{R}^{n+1}_1$.

Let $\Phi : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^{n+1}_1$ be the mappping defined by

$$\Phi\left(x_1, x_2, \ldots, x_n\right) = \frac{\mu}{d^2 - a^2 x_1^2 + \|x\|^2}\left(x_1, x_2, \ldots, x_n, d\right), \tag{4.1}$$

where $\Omega = \left\{x \in \mathbb{R}^n \mid \ \|x\|^2 - a^2 {x_1}^2 + d^2 \neq 0\right\}$, $\mu \neq 0$, $d \neq 0$ and $a > 1$.

Analogously to the proof of Theorem 3.5, we can prove that the following theorem.

Theorem 4.1. *Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) : I \subset \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Then, the curve γ is a helix whose tangent vector field V_1 makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis $e_1 \in \mathbb{R}^n$ where $a \in (1, \infty) \setminus \sqrt{2}$ iff the curve $\beta : J \subset \mathbb{R} \to \mathbb{R}^{n+1}_1$,*

$$\beta = \Phi(\gamma) = \frac{\mu}{d^2 - a^2\gamma_1^2 + \|\gamma\|^2}\left(\gamma_1, \gamma_2, ..., \gamma_n, d\right) \tag{4.2}$$

is a timelike (spacelike) helix whose tangent vector field V_1 makes the fixed angle $\overline{\theta} = \cosh^{-1}\left(\frac{\varepsilon}{\sqrt{2-a^2}}\right)\left(\overline{\theta} = \sinh^{-1}\left(\frac{\varepsilon}{\sqrt{a^2-2}}\right)\right)$ with the timelike axis $(e_1, 0) \in \mathbb{R}^{n+1}_1$, where $\mu \neq 0$, $d \neq 0$ and $1 < a < \sqrt{2}$ $(a > \sqrt{2})$.

Example 4.2. Let us take the helix

$$\gamma(t) = \left(\frac{t^3}{3} + t, \frac{2}{3}\left(t^2 + 2\right)^{3/2}, 2t\right)$$

and its tangent vector

$$V_1(t) = \left(\frac{1}{\sqrt{5}}, \frac{2t\sqrt{t^2+2}}{\sqrt{5}\left(t^2+1\right)}, \frac{2}{\sqrt{5}\left(t^2+1\right)}\right)$$

makes the fixed angle $\theta = \cos^{-1}(\frac{1}{\sqrt{5}})$ with axis e_1 in $\mathbb{R}^3$. If we choose $\mu = 2$ and $d = 2$ in (4.2),

$$\beta(t) = \Phi(\gamma(t)) = \left(\frac{3t\left(t^2+3\right)}{24t^2+34}, \frac{3\left(t^2+2\right)^{3/2}}{12t^2+17}, \frac{9t}{12t^2+17}, \frac{9}{12t^2+17}\right)$$

is a spacelike helix and its tangent vector

$$\overline{V_1}(t) = \left(\frac{1}{\sqrt{3}}, \frac{2t\left(4t^2+1\right)\sqrt{t^2+2}}{\sqrt{3}\left(4t^4+5t^2+17\right)}, \frac{34-24t^2}{\sqrt{3}\left(4t^4+5t^2+17\right)}, -\frac{16\sqrt{3}t}{4t^4+5t^2+17}\right)$$

makes the fixed angle $\overline{\theta}=\cosh^{-1}\left(\frac{1}{\sqrt{3}}\right)$ with timelike axis $(e_1,0)=(1,0,0,0)$ in $\mathbb{R}^4_1$.

Example 4.3. Let us take the cylindrical helix

$$\gamma(t)=\left(\frac{\sqrt{3}}{\sqrt{2}}t,\cos\frac{t}{\sqrt{2}},\sin\frac{t}{\sqrt{2}}\right)$$

and its tangent vector

$$V_1(t)=\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\sin\frac{t}{\sqrt{2}},\frac{1}{2}\cos\frac{t}{\sqrt{2}}\right)$$

makes the fixed angle $\theta=\cos^{-1}(\frac{\sqrt{3}}{2})$ with axis e_1 in $\mathbb{R}^3$. If we choose $\mu=1$ and $d=\frac{1}{2}$ in (4.2),

$$\beta(t)=\Phi(\gamma(t))=\left(\frac{2\sqrt{6}t}{5-2t^2},\frac{4}{5-2t^2}\cos\frac{t}{\sqrt{2}},\frac{4}{5-2t^2}\sin\frac{t}{\sqrt{2}},\frac{2}{5-2t^2}\right)$$

is a timelike helix and its tangent vector

$$\overline{V_1}(t)=\left(\sqrt{\frac{3}{2}},\frac{\sqrt{2}\,(2t^2-5)\sin\frac{t}{\sqrt{2}}+8t\cos\frac{t}{\sqrt{2}}}{4t^2+10},\frac{\sqrt{2}\,(5-2t^2)\cos\frac{t}{\sqrt{2}}+8t\sin\frac{t}{\sqrt{2}}}{4t^2+10},\frac{2t}{2t^2+5}\right)$$

makes the fixed angle $\overline{\theta}=\sinh^{-1}\left(\sqrt{\frac{3}{2}}\right)$ with timelike axis $(e_1,0)=(1,0,0,0)$ in $\mathbb{R}^4_1$.

Now, we define mappings that transforms a helix with e_n in $\mathbb{R}^n$ to a non-null helix with the spacelike axis $(0,e_n)$ in $\mathbb{R}^{n+1}_1$.

Let $\varphi_i=\psi_i\circ\mathcal{G}:\mathbb{R}^n\to\mathbb{R}^{n+1}_1$ be a mapping for $i=1,2,3$. Then, the mapping ψ_i transforms a helix from with axis e_n in $\mathbb{R}^n$ to another helix with the spacelike axis $(0,e_n)$ in $\mathbb{R}^{n+1}_1$.

Corollary 4.4. *Let $\gamma=(\gamma_1,\gamma_2,...,\gamma_n):I\subset\mathbb{R}\to\mathbb{R}^n$ be a regular curve. Then, the curve γ is a helix whose tangent vector field V_1 makes the fixed angle $\theta=\cos^{-1}(\frac{1}{a})$ with axis $e_n\in\mathbb{R}^n$ where $a>1$. Then,*

1) *Let us take,*

$$\beta_1=\varphi_1(\gamma)=\frac{c}{d^2-a^2\gamma_n^2+\|\gamma\|^2}\left(\frac{a^2-1}{a}\gamma_n,\frac{\sqrt{a^2-1}}{a}\gamma_1,\frac{\sqrt{a^2-1}}{a}\gamma_2,\dots,\frac{\sqrt{a^2-1}}{a}\gamma_{n-1},\frac{1}{a}d\right)$$

i) $\beta_1:J\subset\mathbb{R}\to\mathbb{R}^{n+1}_1$ is a timelike helix whose tangent vector field V_1 makes the fixed angle $\overline{\theta}=\sinh^{-1}\left(\frac{\varepsilon}{\sqrt{a^2-2}}\right)$ with the spacelike axis $(0,e_n)$, where $a>\sqrt{2}$,

ii) $\beta_1:J\subset\mathbb{R}\to\mathbb{R}^{n+1}_1$ is a spacelike helix whose tangent vector field V_1 makes the fixed angle $\overline{\theta}=\cosh^{-1}\left(\frac{\varepsilon}{\sqrt{2-a^2}}\right)$ with the spacelike axis $(0,e_n)$, where $1<a<\sqrt{2}$.

2) *The curve $\beta_2:J\subset\mathbb{R}\to\mathbb{R}^{n+1}_1$,*

$$\beta_2=\varphi_2(\gamma)=\frac{c}{d^2-a^2\gamma_n^2+\|\gamma\|^2}\left(\sqrt{a^4-1}\gamma_n,\sqrt{a^2+1}\gamma_1,\sqrt{a^2+1}\gamma_2,\dots,\sqrt{a^2+1}\gamma_{n-1},ad\right)$$

is a timelike helix whose tangent vector field V_1 makes the fixed angle $\overline{\theta}=\sinh^{-1}(\varepsilon a)$ with the spacelike axis $(0,e_n)$.

3) *The curve $\beta_3:J\subset\mathbb{R}\to\mathbb{R}^{n+1}_1$,*

$$\beta_3=\varphi_3(\gamma)=\frac{c}{d^2-a^2\gamma_n^2+\|\gamma\|^2}\left(a\sqrt{a^2-1}\gamma_n,a\gamma_1,a\gamma_2,\dots,a\gamma_{n-1},d\sqrt{a^2-1}\right)$$

is a timelike helix whose tangent vector field V_1 makes the fixed angle $\overline{\theta}=\sinh^{-1}\left(\varepsilon\sqrt{a^2-1}\right)$ with the spacelike axis $(0,e_n)$.

Example 4.5. Let us take the helix

$$\gamma(t) = \left(\sqrt{11}\cos\frac{t}{6}, \sqrt{11}\sin\frac{t}{6}, \frac{5t}{6}\right)$$

and its tangent vector

$$V_1(t) = \left(-\frac{\sqrt{11}}{6}\sin\frac{t}{6}, \frac{\sqrt{11}}{6}\cos\frac{t}{6}, \frac{5}{6}\right)$$

makes the fixed angle $\theta = \cos^{-1}(\frac{5}{6})$ with axis e_3 in $\mathbb{R}^3$. If we choose $c = 1$ and $d = 1$ in Corollary 4.4, then

1) The curve

$$\varphi_1(\gamma(t)) = \left(\frac{11t}{432 - 11t^2}, \frac{66}{432 - 11t^2}\cos\frac{t}{6}, \frac{66}{432 - 11t^2}\sin\frac{t}{6}, \frac{30}{432 - 11t^2}\right)$$

is a spacelike helix in $\mathbb{R}^4_1$. Then, its tangent vector field

$$\overline{V_1}(t) = \left(\frac{11t^2 + 432}{12\sqrt{14}t}, \frac{(11t^2 - 432)\sin\frac{t}{6} + 132t\cos\frac{t}{6}}{12\sqrt{14}t}, \frac{(432 - 11t^2)\cos\frac{t}{6} + 132t\sin\frac{t}{6}}{12\sqrt{14}t}, \frac{5}{\sqrt{14}}\right)$$

makes the fixed angle $\bar{\theta} = \cosh^{-1}\left(\frac{5}{\sqrt{14}}\right)$ with the spacelike axis $(0, e_3) = (0, 0, 0, 1)$ in $\mathbb{R}^4_1$.

2) The curve

$$\varphi_2(\gamma(t)) = \left(\frac{6\sqrt{671}t}{2160 - 55t^2}, \frac{36\sqrt{671}}{2160 - 55t^2}\cos\frac{t}{6}, \frac{36\sqrt{671}}{2160 - 55t^2}\sin\frac{t}{6}, \frac{216}{2160 - 55t^2}\right)$$

is a timelike helix in $\mathbb{R}^4_1$. Then, its tangent vector field

$$\overline{V_1}(t) = \left(\sqrt{\frac{61}{11}}\frac{11t^2 + 432}{60t}, \sqrt{\frac{61}{11}}\frac{\left(11t^2 - 432\right)\sin\frac{t}{6} + 132t\cos\frac{t}{6}}{60t}, \sqrt{\frac{61}{11}}\frac{\left(432 - 11t^2\right)\cos\frac{t}{6} + 132t\sin\frac{t}{6}}{60t}, \frac{6}{5}\right)$$

makes the fixed angle $\overline{\theta} = \sinh^{-1}\left(\frac{6}{5}\right)$ with the spacelike axis $(0, e_3) = (0, 0, 0, 1)$ in $\mathbb{R}^4_1$.

3) The curve

$$\varphi_3(\gamma(t)) = \left(\frac{36\sqrt{11}t}{2160 - 55t^2}, \frac{216\sqrt{11}}{2160 - 55t^2}\cos\frac{t}{6}, \frac{216\sqrt{11}}{2160 - 55t^2}\sin\frac{t}{6}, \frac{36\sqrt{11}}{2160 - 55t^2}\right)$$

is a timelike helix in $\mathbb{R}^4_1$. Then, its tangent vector field

$$\overline{V_1}(t) = \left(\frac{11t^2 + 432}{10\sqrt{11}t}, \frac{(11t^2 - 432)\sin\frac{t}{6} + 132t\cos\frac{t}{6}}{10\sqrt{11}t}, \frac{(432 - 11t^2)\cos\frac{t}{6} + 132t\sin\frac{t}{6}}{10\sqrt{11}t}, \frac{\sqrt{11}}{5}\right)$$

makes the fixed angle $\overline{\theta} = \sinh^{-1}\left(\frac{\sqrt{11}}{5}\right)$ with the spacelike axis $(0, e_3) = (0, 0, 0, 1)$ in $\mathbb{R}^4_1$.

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