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COMPARISON OF DIFFERENT ESTIMATION METHODS FOR THE INVERSE WEIGHTED LINDLEY DISTRIBUTION

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Abstract. In this paper, different estimation methods are considered for the parameters of the inverse weighted Lindley (IWL) distribution introduced by Ramos et al.(2019). Parameters of the IWL are estimated by the method of maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramér-von Mises (CVM) and Anderson Darling (AD). The performances of the estimators are compared using Monte Carlo simulation study via bias, mean square error and deficiency (Def) criteria. Finally, a real data set is analyzed for illustrative purposes.

1. Introduction

Lindley distribution presented by Lindley [7] is an important distribution in statistics and many applied areas because of its flexible mathematical properties. Furthermore, Lindley distribution is more preferable than the exponential distribution in many cases (see [5]). Different generalizations are considered in the literature such as given in [15], [1], [3] to add more flexibility to Lindley distribution. Weighted distributions can extend and provide more flexibility to standard distributions (see [11]). Two-parameter weighted Lindley (WL) distribution is introduced by Ghitany et al. [4]. Mazucheli et al. [3] study on the finite sample properties of the parameters of the WL distribution using four methods. Wang and Wang [14] propose bias-corrected maximum likelihood (bias-corrected ML) estimators for the parameters of the WL distribution. Ramos and Louzada [13] introduce three parameters generalized weighted Lindley distribution. Ramos et al. [12] propose the inverse weighted Lindley (IWL) distribution. The IWL distribution is a component of two mixture model with upside-down bathtub hazard rate function. The IWL distribution is flexible to model data sets in the presence of heterogeneity (see [12]).

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For example, if we are interested in life time of products in a group, it can be considered that the group is heterogeneous. Since the observed failure times of products could be different. In this case, the IWL distribution can be appropriate to describe the heterogeneity in the data.

The IWL distribution is specified by the probability density function (pdf)

$$f(t) = \frac{\lambda^{\phi+1}}{(\phi+\lambda)\Gamma(\phi)} t^{-\phi-1}\left(1+\frac{1}{t}\right)e^{-\lambda t^{-1}}, \tag{1}$$

for all $t > 0$, $\phi > 0$ and $\lambda > 0$ where $\Gamma(\phi)$ is the gamma function which is computed by $\Gamma(\phi) = \int_0^\infty e^{-x}x^{\phi-1}dx$ is the gamma function. The corresponding cumulative distribution function (cdf) is given by

$$F(t) = \frac{\Gamma(\phi, \lambda t^{-1})(\lambda+\phi) + (\lambda t^{-1})^{\phi} e^{-\lambda t^{-1}}}{(\lambda+\phi)\Gamma(\phi)} \tag{2}$$

where $\Gamma(x, y) = \int_x^\infty w^{y-1}e^{-x}dw$ is the upper incomplete gamma. The survival function and hazard function of the IWL distribution are defined as follows

$$S(t) = \frac{\gamma(\phi, \lambda t^{-1})(\lambda+\phi) - (\lambda t^{-1})^{\phi} e^{-\lambda t^{-1}}}{(\lambda+\phi)\Gamma(\phi)}, \tag{3}$$

$$h(t) = \frac{\lambda^{\phi+1}t^{-\phi-1}(1+t^{-1})e^{-\lambda t^{-1}}}{\gamma(\phi, \lambda t^{-1})(\lambda+\phi) - (\lambda t^{-1})^{\phi} e^{-\lambda t^{-1}}}, \tag{4}$$

respectively. Here $\gamma(y, x) = \int_0^x w^{y-1}e^{-w}dw$ is the lower incomplete gamma function. Hazard function plots of the IWL distribution for some selected values of parameters (ϕ, λ) are presented in Figure 1.

We refer to [12] for the further details about the IWL distribution.

Ramos et al. [12] present the ML and Bias-corrected ML estimators for the parameters of the IWL distribution for both complete and censored data and examine the efficienct of bias correction via Monte Carlo simulation.

To the best of our knowledge, parameters of the IWL distribution have not been estimated using different methods, namely, least square (LS), weighted least squares (WLS), Cramér-von Mises (CVM) and Anderson Darling (AD) methods.

In this paper, we propose ML, LS, WLS, CVM and AD estimators for parameters of the IWL distribution. CVM and AD estimators are in the class of minimum distance estimators which are based on minimizing distance between the estimated and empirical cdf with respect to the parameters of interest. Minimum distance estimators are also called as goodness of fit estimators. See [2] and [8] for the further details of goodness of fit estimators. We carry out Monte Carlo simulation study in order to compare performances of the proposed estimators in terms of bias, mean squared error (MSE) and deficiency (Def) criteria.

The rest of paper is organized as follows. Brief descriptions of ML, LS, WLS, CVM and AD methods are given in Section 2. In Section 3, an extensive Monte

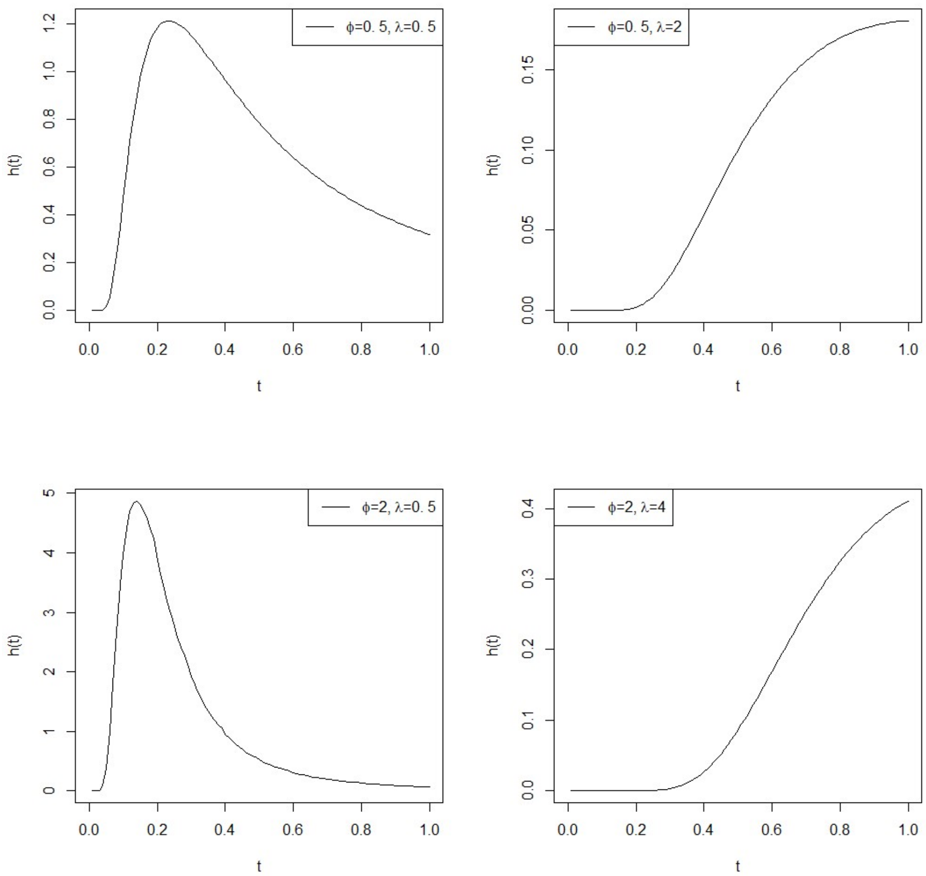

Figure 1. Hazard function plots of the IWL distribution for some selected values of parameters (ϕ, λ).

Carlo simulation study is carried out to compare the performances of the estimators for parameters of the IWL distribution. In Section 4, we give real data application to illustrate the implementation of the proposed methodology. In the final section, the concluding remarks are given.

2. Estimation methods

In this section, we give a brief information of the estimation methods, called as ML, LS, WLS, CVM and AD used to estimate parameters of the IWL in this study.

2.1. **Maximum likelihood estimators.** Let $T_1, T_2, ..., T_n$ be a random sample from the IWL(ϕ, λ) distribution. Then, the log-likelihood function (l) of the observed sample is

$$l = n(\phi+1)\log\lambda - n\log(\lambda+\phi) - n\log\Gamma(\phi) - \lambda\sum_{i=1}^{n}\frac{1}{t_i} - (\phi+1)\sum_{i=1}^{n}\log(t_i). \tag{5}$$

The ML estimators of the parameters ϕ and λ are obtained from the following likelihood equations:

$$\frac{\partial l}{\partial \phi} = n\log(\lambda) - \sum_{i=1}^{n}\log(t_i) - \frac{n}{\lambda+\phi} - n\psi(\phi) = 0 \tag{6}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n(\phi+1)}{\lambda} - \sum_{i=1}^{n}\frac{1}{t_i} - \frac{n}{\lambda+\phi} = 0 \tag{7}$$

where $\psi(k) = \frac{\partial}{\partial k}\log\Gamma(k) = \frac{\Gamma'(k)}{\Gamma(k)}$ is the digamma function. The ML estimate of λ is obtained from equation (7) as

$$\hat{\lambda}_{ML} = \frac{-\hat{\phi}_{ML}(\xi(t)-1) + \sqrt{(\hat{\phi}_{ML}(\xi(t)-1))^2 + 4\xi(t)(\hat{\phi}_{ML}^2 + \hat{\phi}_{ML})}}{2\xi(t)} \tag{8}$$

where $\xi(t) = \sum_{i=1}^{n}(nt_i)^{-1}$. It is obvious that (6) cannot be solved explicitly. Therefore, for computing the ML estimator of ϕ, numerical methods should be performed. See [12] for more details about the ML estimators of the parameters of the IWL distribution.

2.2. **Least Squares Estimation Method.** Let $x_{(i)}$, $i = 1, 2, ..., n$ be the order statistics of a random sample from the IWL distribution. Since $F(x_{(i)})$ behaves as the i-th order statistic of a sample size from $U(0,1)$, expected value and variance of $F(x_{(i)})$ are given as follows:

$$E[F(x_{(i)})] = \frac{i}{n+1} \quad \text{and} \quad Var[F(x_{(i)})] = \frac{i(n-i+1)}{(n+1)^2(n+2)}, \tag{9}$$

respectively. The LS estimators of the parameters of the IWL distribution are obtained by minimizing the following function with respect to the parameters ϕ and λ.

$$S = \sum_{i=1}^{n}\left(F(x_{(i)}) - \frac{i}{n+1}\right)^2. \tag{10}$$

Here $F(.)$ is the cdf of the IWL given in (2). LS estimators of ϕ and λ are obtained by solving following equations:

$$\frac{\partial S}{\partial \phi} = \sum_{i=1}^{n}\left(F(x_{(i)};\phi,\lambda) - \frac{i}{n+1}\right)\Lambda_1(x_{(i)};\phi,\lambda) = 0,$$

$$\frac{\partial S}{\partial \lambda} = \sum_{i=1}^{n}\left(F(x_{(i)};\phi,\lambda) - \frac{i}{n+1}\right)\Lambda_2(x_{(i)};\phi,\lambda) = 0, \tag{11}$$

where

$$\Lambda_1(x_{(i)};\phi,\lambda) = \frac{\left(\Gamma(\phi,\lambda t^{-1})+\gamma_1(\phi+\lambda)+(\lambda t^{-1})^{\phi}\ln(\lambda t^{-1})e^{-\lambda t^{-1}}\right)\left((\lambda+\phi)\Gamma(\phi)\right)}{\left((\lambda+\phi)\Gamma(\phi)\right)^2} - \frac{\left(\Gamma(\phi,\lambda t^{-1})(\lambda+\phi)+(\lambda t^{-1})^{\phi}e^{-\lambda t^{-1}}\right)\left(\lambda\gamma_3+\Gamma(\phi)+\phi\gamma_3\right)}{\left((\lambda+\phi)\Gamma(\phi)\right)^2}, \tag{12}$$

$$\Lambda_2(x_{(i)};\phi,\lambda) = \frac{\left(\Gamma(\phi,\lambda t^{-1})+\gamma_2(\lambda+\phi)+t^{-1}e^{-\lambda t^{-1}}(\phi(\lambda t^{-1})^{\phi-1}-(\lambda t^{-1})^{\phi})\right)}{\left((\lambda+\phi)\Gamma(\phi)\right)^2} \times\left((\lambda+\phi)\Gamma(\phi)\right) - \frac{\left(\Gamma(\phi,\lambda t^{-1})(\lambda+\phi)+(\lambda t^{-1})^{\phi}e^{-\lambda t^{-1}}\right)\Gamma(\phi)}{\left((\lambda+\phi)\Gamma(\phi)\right)^2}, \tag{13}$$

respectively. It is obvious that, since equations given in (11) include nonlinear functions, numerical methods should be performed to obtain LS estimators of ϕ and λ.

2.3. **Weighted Least Squares Estimators.** The WLS estimators of the parameters ϕ and λ are obtained by minimizing the following function:

$$S_w = \sum_{i=1}^{n} w_i\left(F(x_{(i)}) - \frac{i}{n+1}\right)^2 \tag{14}$$

where w_i denotes the weight and computed by

$$w_i = \frac{1}{\mathrm{Var}(F(X_{(i)}))} = \frac{(n+1)^2(n+2)}{i(n-i-1)}, \quad i = 1, 2, ..., n.$$

The WLS estimators of ϕ and λ are obtained by solving the following nonlinear equations:

$$\begin{aligned}\frac{\partial S_w}{\partial \phi} &= \sum_{i=1}^{n} w_i \left(F(x_{(i)};\phi,\lambda) - \frac{i}{n+1} \right) \Lambda_1(x_{(i)};\phi,\lambda) = 0,\\ \frac{\partial S_w}{\partial \lambda} &= \sum_{i=1}^{n} w_i \left(F(x_{(i)};\phi,\lambda) - \frac{i}{n+1} \right) \Lambda_2(x_{(i)};\phi,\lambda) = 0,\end{aligned} \tag{15}$$

respectively. Here Λ_1 and Λ_2 are given in (13). It is clear that WLS estimators should also be obtained using numerical methods, since equations given in (15) cannot be solved explicitly.

2.4. **Cramér-von Mises estimators.** CVM estimators of the parameters of the IWL distribution are obtained by minimizing the following equation with respect to the parameters ϕ and λ.

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(x_{(i)},\phi,\lambda) - \frac{2i-1}{2n} \right)^2 \tag{16}$$

To obtain the CVM estimators of the parameters, we have to solve the following equations by using numerical methods.

$$\begin{aligned}\frac{\partial CVM}{\partial \phi} &= \sum_{i=1}^{n} \left(F(x_{(i)};\phi,\lambda) - \frac{2i-1}{2n} \right) \Lambda_1(x_{(i)};\phi,\lambda) = 0,\\ \frac{\partial CVM}{\partial \lambda} &= \sum_{i=1}^{n} \left(F(x_{(i)};\phi,\lambda) - \frac{2i-1}{2n} \right) \Lambda_2(x_{(i)};\phi,\lambda) = 0.\end{aligned} \tag{17}$$

Here, Λ_1 and Λ_2 are given in (13).

2.5. **Anderson Darling estimators.** The AD estimators of ϕ and λ are obtained by minimizing the following equation with respect to the parameters of interest.

$$A = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log \left[F(x_{(i)}) \left(1 - F(x_{(j)}) \right) \right] \right\}, \tag{18}$$

where $j = n - i + 1$. The AD estimators of ϕ and λ are obtained by solving the nonlinear equations

$$\begin{aligned}\frac{\partial A}{\partial \phi} &= \sum_{i=1}^{n} (2i-1) \left[\frac{\Lambda_1(x_{(i)},\phi,\lambda)}{F(x_{(i)},\phi,\lambda)} - \frac{\Lambda_1(x_{(j)},\phi,\lambda)}{F(x_{(j)},\phi,\lambda)} \right] = 0\\ \frac{\partial A}{\partial \lambda} &= \sum_{i=1}^{n} (2i-1) \left[\frac{\Lambda_2(x_{(i)},\phi,\lambda)}{F(x_{(i)},\phi,\lambda)} - \frac{\Lambda_2(x_{(j)},\phi,\lambda)}{F(x_{(j)},\phi,\lambda)} \right] = 0,\end{aligned} \tag{19}$$

respectively. Here, Λ_1 and Λ_2 are given in (13). Nonlinear equations given in (19) can be solved by using numerical methods.

3. Simulation study

In this section, we conduct a Monte-Carlo simulation study to compare the performance of the different estimation methods discussed in the previous section. The bias, MSE and Def criteria are used in the comparisons. The bias and MSE are respectively formulated as follows:

$$\text{Bias}(\hat{\theta}) = E(\theta - \hat{\theta}) \quad \text{and} \quad \text{MSE}(\hat{\theta}) = E(\theta - \hat{\theta})^2$$

where $\theta = (\phi, \lambda)$. The mathematical expression of the Def criterion used in this study to compare joint efficiencies of the parameters is given as

$$\text{Def} = \text{MSE}(\hat{\phi}) + \text{MSE}(\hat{\lambda}),$$

see [6] for the further details on DEF. In simulation study, we generate random data from the IWL distribution using the algorithm given by Ramos et al. [12]. The simulation study is performed considering the values: $(\phi, \lambda) = (0.5, 0.5)$, $(0.5, 2)$, $(2, 0.5)$, $(2, 4)$ and $n = (20, 50, 100, 200, 500)$.
For all the numerical computations, we use the R statistical software environment. The ML, LS, WLS, CVM and AD estimators of the parameters are obtained by using "optim" function. Simulation results are given in Table 1-Table 4.

It is observed from Table 1 and Table 2 that the ML estimators of ϕ and λ have the smallest bias for all sample sizes. The ML estimator is also the most efficient one for both ϕ and λ parameters with the smallest MSE values for all cases. The AD estimators of ϕ and λ outperform LS, WLS and CVM estimators in terms of bias and MSE criteria. Overall, the ML estimators of parameters of the IWL distribution is the best estimator in terms of Def criterion. It is followed by AD estimators.

It is observed from Table 3 that the ML estimators of ϕ and λ perform better than the others in terms of bias and MSE criterion in most cases. However AD estimators of ϕ and λ outperform the ML, LS, WLS and CVM estimators in terms of both bias and MSE criteria, when $n = 20$. According to Def, the AD estimator has the best performance for $n = 20$. Otherwise the ML estimator can be preferred.

It is observed from Table 4 that the ML estimators of ϕ and λ have the smallest bias and MSE values in most cases. On the other hand, the bias values of all estimators are close to each other. The AD is the best for $n = 20$ and followed by WLS and LS estimators.

The simulation results show that ML has the best performance with the lowest deficiency almost in all cases. However, AD has a little bit smaller deficiency than the ML when $n = 20$, $\phi = 2$ and $\lambda = 4$. Also, ML has higher deficiency than LS, WLS and AD when $n = 20$, $\phi = 2$ and $\lambda = 0.5$.

Overall, we suggest using the ML methodology for estimating the parameters of the IWL distribution because of its superior performance. Also for the small sample size, the AD estimators can be preferred. It can be also said that CVM estimators of ϕ and λ demonstrate the weakest performance for all cases.

Table 1. Simulated biases, MSEs and Def values of the ML, LS, WLS, CVM and AD estimators for $\phi = 0.5$, $\lambda = 0.5$.

		ϕ		λ		
n	Method	Bias	MSE	Bias	MSE	Def
20	ML	-0.1784	0.0019	-1.5279	0.3212	0.3231
	LS	-0.2345	0.0028	-2.3822	0.4274	0.4302
	WLS	-0.2291	0.0025	-2.7214	0.3867	0.3892
	CVM	-0.2608	0.0031	-2.5854	0.6440	0.6471
	AD	-0.2096	0.0025	-1.9553	0.3501	0.3526
50	ML	-0.1784	0.0018	-1.4903	0.3138	0.3157
	LS	-0.2341	0.0026	-2.3477	0.4117	0.4144
	WLS	-0.2291	0.0023	-2.6822	0.3645	0.3668
	CVM	-0.2519	0.0029	-2.4238	0.6248	0.6277
	AD	-0.2073	0.0022	-1.9289	0.3421	0.3443
100	ML	-0.1669	0.0017	-1.4630	0.3103	0.3120
	LS	-0.2314	0.0025	-2.3501	0.4013	0.4038
	WLS	-0.2240	0.0023	-2.5933	0.3528	0.3551
	CVM	-0.2497	0.0028	-2.3878	0.6209	0.6237
	AD	-0.2072	0.0021	-1.9823	0.3419	0.3441
200	ML	-0.1518	0.0015	-1.4334	0.3067	0.3082
	LS	-0.2294	0.0023	-2.3326	0.4002	0.4025
	WLS	-0.2233	0.0022	-2.4987	0.3312	0.3334
	CVM	-0.2474	0.0026	-2.3512	0.6076	0.6102
	AD	-0.2022	0.0021	-1.9663	0.3353	0.3374
500	ML	-0.1364	0.0011	-1.4280	0.2952	0.2964
	LS	-0.2234	0.0020	-2.3293	0.3982	0.4002
	WLS	-0.2212	0.0021	-2.4574	0.3166	0.3187
	CVM	-0.2469	0.0024	-2.3367	0.5825	0.5849
	AD	-0.2019	0.0019	-1.9356	0.3285	0.3304

4. Application

In this section, we analyse a real data set taken from the literature to show the implementation of the proposed methods. The data set in Table 5 consist of the failure stresses (in GPa) of 65 single carbon fiber of length 50mm. This data set is taken from Mazucheli et al. [9] in which weighted Lindley (WL) distribution is used.

To fit the IWL distribution to the data set, we use Q-Q plot technique which is one of the well-known and widely used graphical techniques. It is observed from Figure (2) that IWL distribution provides good fit to model the failure stresses data set.

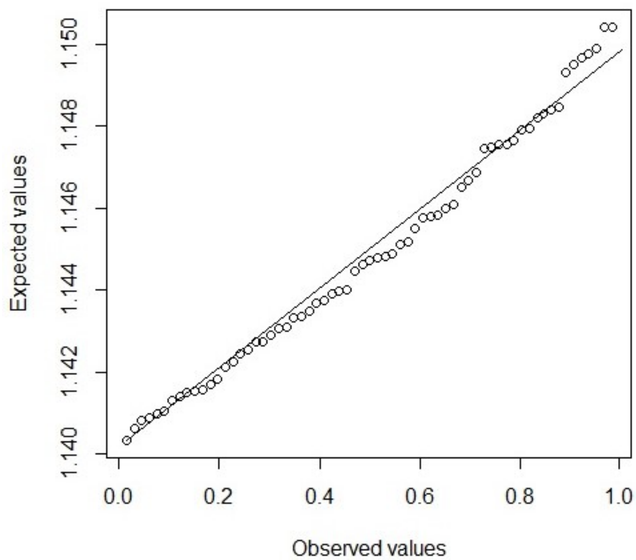

FIGURE 2. IWL QQ plot for the failure stresses data set.

In this study, we use Kolmogorov-Simirnov (KS) test which is a well-known goodness of fit test to test whether the IWL distribution is appropriate for the data.

Furthermore, to identify the parameter estimation methods providing a better fit to the data set, we use Akaike information criterion (AIC), Bayesian information criterion (BCI), the root mean square error (RMSE) and coefficient of determination (R^2) criteria.

We present the estimates of the IWL parameters, AIC, BIC, RMSE, R^2 and p-values obtained from Kolmogrov-Smirnov test are given in Table 6 for the failure stresses data set.

Acording to the results of the KS test given in Table 6, it can be concluded that the IWL distribution with the ML, LS, WLS, CVM and AD estimators of ϕ and λ works quite well to fit the failure stresses data set. However, It is clear from Table 6 that the ML is more desirable according to p-values for the IWL distribution.

It is also obvious from Table 6 that the ML estimates is the most appropriate model among the others. They are followed by the AD estimates. Since it is known that the model having the lowest AIC, the lowest BIC, the lowest RMSE and the highest R^2 value among the models provides better fitting to the data.

Table 2. Simulated biases, MSEs and Def values of the ML, LS, WLS, CVM and AD estimators for $\phi = 0.5$, $\lambda = 2$.

		ϕ		λ		
n	Method	Bias	MSE	Bias	MSE	Def
20	ML	-0.0884	0.0022	0.2936	0.1503	0.1541
	LS	-0.1597	0.0056	0.6346	0.1749	0.1805
	WLS	-0.1227	0.0038	0.3956	0.2276	0.2298
	CVM	-0.1989	0.0044	0.5366	0.2196	0.2240
	AD	-0.1346	0.0029	0.2968	0.1589	0.1617
50	ML	-0.0863	0.0021	0.2930	0.1485	0.1519
	LS	-0.1582	0.0051	0.6312	0.1702	0.1753
	WLS	-0.1223	0.0034	0.3956	0.2208	0.2229
	CVM	-0.1972	0.0041	0.5226	0.2112	0.2153
	AD	-0.1340	0.0026	0.2913	0.1429	0.1455
100	ML	-0.0855	0.0019	0.2857	0.1376	0.1396
	LS	-0.1578	0.0048	0.5947	0.1673	0.1720
	WLS	-0.1219	0.0030	0.3346	0.2189	0.2220
	CVM	-0.1906	0.0039	0.4985	0.2098	0.2138
	AD	-0.1324	0.0024	0.2791	0.1320	0.1344
200	ML	-0.0846	0.0018	0.2680	0.1296	0.1314
	LS	-0.1566	0.0046	0.5747	0.1573	0.1618
	WLS	-0.1187	0.0027	0.3298	0.2056	0.2083
	CVM	-0.1893	0.0037	0.4757	0.1945	0.1982
	AD	-0.1310	0.0022	0.2587	0.1256	0.1279
500	ML	-0.0838	0.0016	0.2297	0.1172	0.1188
	LS	-0.1487	0.0043	0.5493	0.1494	0.1536
	WLS	-0.1174	0.0025	0.3086	0.1986	0.2011
	CVM	-0.1876	0.0035	0.4328	0.1942	0.1977
	AD	-0.1306	0.0020	0.2328	0.1128	0.1148

5. Conclusion

In this paper, we focus different estimation methods of the unknown parameters of the IWL distribution. We consider ML, LS and WLS as classical methods and CVM and AD as minimum distance methods. As far as we know, LS, WLS, AD and CVM methods have not been used for estimating the parameters of the IWL distribution previously. We compare the performance of the estimators via Monte Carlo simulation study in terms of bias, MSE and Def criteria. The results of simulation study show that among the mentioned estimators, ML has the best performance in most of the cases. Also, it can be concluded that ML is followed by AD especially for small sample sizes. Overall, we suggest using ML methodology

Table 3. Simulated biases, MSEs and Def values of the ML, LS, WLS, CVM and AD estimators for $\phi = 2$, $\lambda = 0.5$.

		ϕ		λ		
n	Method	Bias	MSE	Bias	MSE	Def
20	ML	0.2831	0.0273	-0.7123	0.1688	0.1961
	LS	0.2457	0.0236	-0.6542	0.1583	0.1819
	WLS	0.2396	0.0253	-0.6288	0.1221	0.1474
	CVM	0.3139	0.0295	-0.7245	0.1747	0.2042
	AD	0.1946	0.0217	-0.6125	0.1174	0.1391
50	ML	0.1912	0.0207	-0.6073	0.1049	0.1256
	LS	0.2231	0.0225	-0.6456	0.1466	0.1691
	WLS	0.2065	0.0219	-0.6207	0.1207	0.1426
	CVM	0.3056	0.0278	-0.7098	0.1653	0.1931
	AD	0.1915	0.0212	-0.6098	0.1122	0.1334
100	ML	0.1905	0.0199	-0.5877	0.0972	0.1171
	LS	0.2178	0.0217	-0.6325	0.1352	0.1570
	WLS	0.1976	0.0205	-0.6140	0.1195	0.1400
	CVM	0.2945	0.0266	-0.6947	0.1573	0.1838
	AD	0.1911	0.0201	-0.5927	0.1002	0.1203
200	ML	0.1877	0.0188	-0.5614	0.0954	0.1141
	LS	0.2046	0.0202	-0.6245	0.1294	0.1496
	WLS	0.1912	0.0197	-0.6076	0.1124	0.1321
	CVM	0.2818	0.0242	-0.6544	0.1407	0.1648
	AD	0.1893	0.0193	-0.5706	0.0998	0.1191
500	ML	0.1763	0.0164	-0.5533	0.0826	0.0990
	LS	0.1932	0.0192	-0.6126	0.1122	0.1314
	WLS	0.1846	0.0187	-0.5973	0.1042	0.1229
	CVM	0.2666	0.0211	-0.6286	0.1376	0.1588
	AD	0.1786	0.0176	-0.5683	0.0919	0.1095

to obtain estimators of the IWL distribution. AD gives relatively good results and it is also preferable.

Declaration of Competing Interests The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

TABLE 4. Simulated biases, MSEs and Def values of the ML, LS, WLS, CVM and AD estimators for $\phi = 2$, $\lambda = 4$.

		ϕ		λ		
n	Method	Bias	MSE	Bias	MSE	Def
20	ML	0.1105	0.0106	2.5729	0.3218	0.3324
	LS	0.1127	0.0134	2.6473	0.3457	0.3591
	WLS	0.1110	0.0108	2.6126	0.3462	0.3571
	CVM	0.1312	0.0153	2.7390	0.3959	0.4112
	AD	0.1057	0.0097	2.4957	0.2562	0.2659
50	ML	0.1026	0.0092	2.4559	0.2452	0.2544
	LS	0.1103	0.0128	2.5927	0.3419	0.3547
	WLS	0.1098	0.0095	2.5919	0.3404	0.3499
	CVM	0.1276	0.0146	2.6514	0.3727	0.3872
	AD	0.1033	0.0095	2.4627	0.2496	0.2590
100	ML	0.0956	0.0087	2.4227	0.2383	0.2470
	LS	0.1097	0.0117	2.5569	0.3293	0.3411
	WLS	0.1024	0.0093	2.5224	0.3221	0.3314
	CVM	0.1222	0.0123	2.6007	0.3656	0.3779
	AD	0.1002	0.0090	2.4316	0.2392	0.2483
200	ML	0.0899	0.0083	2.3723	0.2251	0.2334
	LS	0.0977	0.0107	2.4928	0.3118	0.3226
	WLS	0.0965	0.0089	2.4791	0.3076	0.3165
	CVM	0.1152	0.0115	2.5817	0.3422	0.3537
	AD	0.0926	0.0087	2.3917	0.2286	0.2373
500	ML	0.0823	0.0083	2.3357	0.2119	0.2202
	LS	0.0943	0.0107	2.4129	0.3066	0.3173
	WLS	0.0931	0.0089	2.4057	0.2915	0.3004
	CVM	0.1016	0.0115	2.5517	0.3166	0.3282
	AD	0.0893	0.0087	2.3620	0.2148	0.2235

TABLE 5. The failure stresses (in GPa) of 65 single carbon fibers of length 50 mm.

1.339	1.434	1.549	1.574	1.589	1.613	1.746	1.753	1.7646	1.807	1.812	1.840	1.852
1.852	1.862	1.864	1.931	1.952	1.974	2.019	2.051	2.055	2.058	2.088	2.125	2.162
2.171	2.172	2.18	2.194	2.211	2.270	2.272	2.280	2.299	2.308	2.335	2.349	2.356
2.386	2.390	2.410	2.430	2.431	2.458	2.471	2.497	2.514	2.558	2.577	2.593	2.601
2.604	2.620	2.633	2.670	2.682	2.699	2.705	2.735	2.785	3.020	3.042	3.116	3.174

TABLE 6. Estimates of the parameters, AIC, BIC, RMSE, R^2 and D values for failure stress data set.

Method	$\hat{\phi}$	$\hat{\lambda}$	AIC	BIC	$RMSE$	R^2	p-value
ML	1.6499	3.3788	250.2429	254.5917	0.1307	0.6023	0.8919
LS	1.6361	4.8553	255.3739	259.7227	0.1393	0.5834	0.8608
WLS	1.6435	4.8799	255.2037	259.5525	0.1393	0.5830	0.8208
CVM	1.6363	4.8553	255.3570	259.7057	0.1393	0.5834	0.8301
AD	1.6196	4.5039	252.1279	256.4767	0.1350	0.5956	0.8624

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