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Shortest Confidence Intervals of Weibull Modulus for Small Samples in Materials Reliability Analysis

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Highlights

- Weibull modulus is an important measure of the engineering reliability of a material used in design.
- Two types of pivotal quantities were proposed for confidence intervals of Weibull modulus.
- These types are equal-tails and shortest pivotal quantity approaches.
- This study compares the performances of the equal-tailed and shortest pivotal quantity models.

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Abstract

The Weibull distribution has been widely used to model strength properties of brittle materials. Estimation of confidence intervals for Weibull shape parameter has been an important concern, since small sample sizes in materials science experiments bring about large intervals. Many methods have been proposed in the literature for constructing shorter intervals; the methods of maximum likelihood, least square, and Menon are among the most extensively studied methods. However, they all use an equal-tails approach. The pivotal quantities used for constructing confidence intervals have right-skewed and unimodal distributions, thus, they clearly do not produce the shortest intervals for a given confidence level in equal tail form. This study constructs the shortest confidence intervals for the three aforementioned methods and compares their performances by their equal-tails counterparts. To this end, a comprehensive simulation study has been conducted for the shape parameter values between 1 to 80 and the sample sizes between 3 to 20. The comparison criterion is chosen as the expected interval length. The results show that the shortest confidence intervals in each of three methods have yielded considerably narrower intervals. Further, the unknown parameter values are more centered in these intervals.

1. INTRODUCTION

The Weibull distribution [1] has been widely used to model failure times and analyze the strength properties of many advanced materials such as fracture strength of ceramics, metallic matrix composites and ceramic matrix composites [2], flexural strength of brittle materials [3], fracture toughness behavior of steels in ductile-brittle transition region [4].

The two-parameter Weibull probability density function (pdf) is given by:

$$f(t) = \frac{m}{\sigma_0}\left(\frac{t}{\sigma_0}\right)^{m-1} e^{-\left(\frac{t}{\sigma_0}\right)^m} \tag{1}$$

where F is the probability of fracture at uniaxial tensile stress T, m is the shape parameter and σ_0 is the scale parameter. The shape parameter m is alternatively referred to as the Weibull modulus. Weibull modulus is used as a measure of the variability of the strength of materials [4] or as a measure of component and system reliability [5].

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Confidence intervals of the parameters m and σ_0 can be estimated by using classical methods such as the methods of maximum likelihood, the generalized linear least squares or moments. In practice, high experimental costs limit the number of samples to be tested; and in general, the smaller the sample size the worse the statistical properties of Weibull modulus. As a result, confidence interval lengths increase as the sample sizes decrease [4].

In classical methods (maximum likelihood, moments etc.), the most common procedure for constructing exact confidence intervals (CI's) is to employ a pivotal quantity [6]. Two general types of intervals based on pivotal quantity are available: The Equal Tails Confidence Interval (ETCI) and the Shortest Confidence Interval (SCI). The ETCIs have been predominantly used in research and in practice, while the SCIs have rather been a subject of theoretical interest, see for example [7,8]. There has been no study in the literature investigating the performance of the SCIs for the Weibull modulus to the authors' best knowledge.

When the Maximum Likelihood Estimation (MLE), Weighted Linear Least Squares (WLS) or Menon's (MN) methods are used to calculate an estimate of $m, \widehat{m}$, McCool [9] showed that $\widehat{m}/m$ is a pivotal variable, that is, independent of σ_0 and m. The distribution of $\widehat{m}/m$ has been shown to be unimodal and asymmetric for a given sample size (n) and confidence level $(1-\alpha)$ by Monte Carlo simulations. The $100(1-\alpha)\%$ ETCI of the Weibull modulus can be obtained from the percentage points (d_l, d_u) of the distribution with the following conditions [10]:

$$P\left(d_l < \frac{\widehat{m}}{m} < d_u\right) = 1-\alpha,\ \ P\left(\frac{\widehat{m}}{m} < d_l\right) = \alpha/2\,,\ P\left(\frac{\widehat{m}}{m} > d_u\right) = \alpha/2. \tag{2}$$

The probability statement in Equation (2) can be easily extended to derive a confidence interval minimizing the interval length while maintaining the $1-\alpha$ coverage.

If the distribution of a pivotal statistic is unimodal and symmetric, the ETCI and the SCI are the same, hence have the same length [11]. On the other hand, for unimodal and asymmetric distributions, such as the distribution of $\widehat{m}/m$, the SCI is shorter than the ETCI [12,13]. Therefore, it would be of interest to evaluate the amount reduction in interval length if the SCI is used instead of the ETCI, particularly for small sample sizes.

In literature, many authors have proposed various methods for point and interval estimation of Weibull modulus; for a general overview, see [14-16]. For specific examples, Barbero et al [2,4] obtained the percentage points of the estimator of the Weibull modulus in modelling the mechanical properties of composite materials and published in tabular form. McCool [17] computed confidence intervals based on the MLE; his procedure requires certain tabled quantities determined from simulation studies. The most common methods for constructing confidence intervals of m are the MLE and WLS methods, which are based on the simulation of pivotal quantities [12,18-21]. The MLE and WLS methods are frequently employed for obtaining ETCIs in the materials science literature. They have been shown to be the best methods according to different comparison criteria for confidence interval and point estimation of the Weibull modulus [2,3,22-29]. For example, Wu et al [21] studied on the interval estimation precision of the Weibull modulus determined by the linear regression, WLS and MLE for analyzing effects of the number of testing specimens. Phan and McCool [28] have computed exact confidence limits for the shape parameter by using Menon's method. Bütikofer et al [29] have studied comparison of point and interval estimation for Weibull modulus by two least squares (LS) methods with interchanged axes for reliability of dental materials. Most of the above referenced studies are related to the estimation of the ETCIs of Weibull modulus [2,3,21,26,28,29], and the remaining are about point estimation or analyzing distribution of Weibull statistics. To our best knowledge, for the SCI of Weibull modulus, there are two theoretical studies by Guenther [7,8] who proposed a method to obtain SCIs and unbiased confidence intervals for Weibull, normal, gamma, Laplace, uniform distribution parameters. In addition, Ferentinos and Karakostas [30] clarified and commented on methods for finding the SCI and the ETCI for any distribution in general, but did not give any practical details for the Weibull distribution. Joula [31] developed confidence intervals for a single unknown parameter by using a pivotal quantity and presented an elementary method for deriving shortest intervals; but there is no theoretical or practical study for case of the Weibull distribution. Besides ETCI and SCI, researchers considered several confidence intervals for estimating Weibull statistics; but they are approximate intervals [32, 33, 34]. This paper aims at the calculation of the exact shortest confidence interval for Weibull modulus, and shows that the width of the confidence interval is shortest when pivotal quantities has unimodal, asymmetric and right-skewed distributions. The motivation of this study comes from the fact that contrary to the previous studies, the performances of SCI and ETCI for m can be shown by Monte Carlo simulation and a real data.

The pivotal quantities $\widehat{m}/m$ for the MLE and WLS methods has unimodal, asymmetric and right-skewed distributions [12,13], which implies the potential for significant decreases in confidence interval lengths if SCIs are used instead of ETCIs. In this paper, the exact confidence intervals of Weibull modulus have been estimated by using ETCIs and SCIs based on the MLE method, the MN method and the WLS method with the Faucher and Tyson weight factor and the hazen probability estimator [21]. For comparison of the methods, Monte Carlo simulations have been designed and run in the C++ language with large simulation run numbers. Small sample sizes between 3 and 20 have been used in the simulations.

The rest of the paper is organized as follows. Section 1 presents a literature review on interval estimation of the Weibull modulus. Section 2 includes a detailed description of the ETCI and the SCI of the Weibull modulus and their properties and discusses the classical estimation approaches. A simulation study is conducted in Section 3. Finally, a brief discussion of the findings is provided in Section 4.

2. MATERIAL METHOD

2.1. Estimation of Equal-Tailed and Shortest Confidence Intervals of Weibull Modulus

In classical inference, the standard method for deriving exact confidence intervals for a parameter is the pivotal quantity method [6,7,35]. A pivotal quantity is generally defined as a function of observations and unobservable parameters such that the function's probability distribution does not depend on the unknown parameters (including nuisance parameters) [36].

The distribution of the pivotal quantity $\widehat{m}/m$, (f_d), can be found for a given n and α value by Monte Carlo simulations. The probability statement,

$$P\left(d_l < \frac{\widehat{m}}{m} < d_u\right) = 1 - \alpha \tag{3}$$

is converted to

$$P\left(\frac{\widehat{m}}{d_u} < m < \frac{\widehat{m}}{d_l}\right) = P(m_l < m < m_u) = 1 - \alpha \tag{4}$$

to obtain a $1 - \alpha$ level confidence interval of m.

If the pivotal critical values d_l and d_u in Equation (3) are calculated in such a way that $P\left(\frac{\widehat{m}}{m} < d_u\right) = \frac{\alpha}{2}$ and $\left(\frac{\widehat{m}}{m} > d_l\right) = \frac{\alpha}{2}$, the interval $[m_l, m_u]$ is said to be the equal-tailed confidence interval (ETCI) of m. If pivotal critical values d_l and d_u in Equation (3) are calculated to minimize the distance $|m_u - m_l|$, then the interval $[m^*{}_l, m^*{}_u]$ is said to be the shortest $1 - \alpha$ level confidence interval (SCI) based on $\widehat{m}/m$ [6,30]. For the SCI, the following constrained minimization problem is solved by using optimization algorithms: (α is confidence level)

$$\begin{aligned} &Minimize, with\ respect\ to\ a\text{: } \frac{1}{d_l} - \frac{1}{d_u} \\ &subject\ to: \int_{d_l}^{d_u} f_d(x)\ dx = 1 - \alpha. \end{aligned} \tag{5}$$

This can be implemented, for example, by leveraging Python library "pymc3" or R library "HDInterval" [37,38]. The SCI of m in classical inference needs only the assumption that the distribution of $\hat{m}/m$, f_d, is unimodal in which there exists a x^* such that $f_d(x)$ is non-decreasing for $x \leq x^*$ and $f_d(x)$ is non-increasing for $x \geq x^*$ [6]. For any symmetric unimodal pdf, the SCI and the ETCI are the same.

In this study, the distribution of the pivotal quantity $\hat{m}/m$ has been obtained by Monte Carlo simulations with 10,000 replications. The flowchart of the simulation procedure for each of the MLE, WLS and MN methods is given in Figure 1.

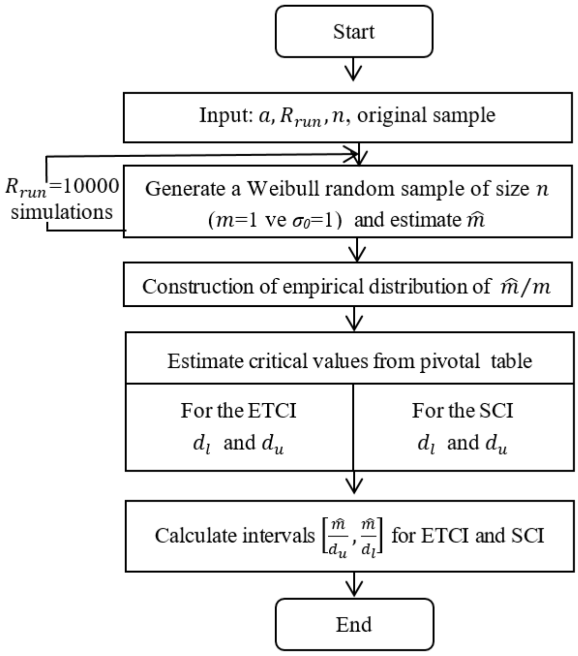


Figure 1. *Flow chart of the simulation procedure for the classical estimation methods*

2.2. Maximum Likelihood Method

The maximum likelihood estimates of Weibull parameters, $\hat{m}$ and $\hat{\sigma}_0$, can be obtained by solving the set of Equations (6) and (7) [39]

$$\frac{n}{\hat{m}} - nIn\hat{\sigma}_0 + \sum_{i=1}^{n} t_i - \sum_{i=1}^{n} \left(\frac{t_i}{\hat{\sigma}_0}\right)^{\hat{m}} In\left(\frac{t_i}{\hat{\sigma}_0}\right) = 0 \tag{6}$$

$$\hat{\sigma}_0 = \left(\frac{\sum_{i=1}^{n}(t_i)^{\hat{m}}}{n}\right)^{1/\hat{m}} \tag{7}$$

where $t_1, t_2, \ldots, t_n$ are an observed sample of size n. The Newton-Raphson method is usually employed for solving Equation (6) for $\hat{m}$. Then, $\hat{\sigma}_0$ is found by substituting $\hat{m}$ into Equation (7).

2.3. Weighted Linear Least Squares

By taking double logarithms of the both sides of the cumulative density function (cdf) of the Weibull distribution, $F(t) = 1 - e^{-\left(\frac{t}{\sigma_0}\right)^m}$, a linear form of is obtained:

$$log\left(-log\big(1-F(t)\big)\right) = m\ log(t) - m\ log(\sigma_0). \quad (8)$$

By defining $Y = log\left(-log\big(1-F(t)\big)\right)$ and $X = log(t)$, Equation (8) becomes a linear function of the form $Y = bX + a$, where:

$$m = b, \sigma_0 = exp\left(\frac{-a}{b}\right). \quad (9)$$

Computation of the least squares regression for Y at the ordinate and X at the abscissa leads to the estimates $\hat{a}$ and $\hat{b}$ of the regression coefficients and finally to estimates $\widehat{m}$ and $\hat{\sigma}_0$ of the Weibull parameters. $F(t)$ in Equation (10) is usually calculated by applying a discrete probability estimator of the form

$$F\big(t_{(i)}\big) \quad = i - c/n + d \quad (10)$$

where c and d are constants historically selected to minimize the bias of $\widehat{m}$, and $t_{(i)}$ represents the ith value of an observation among the n ordered t-values forming the sample. The most commonly used estimators of F are [19,20]: mean ranks, median ranks and hazen ranks (See Table 1).

Table 1. *Probability Estimators*

Probability Estimator	Probability Estimator Equation	
Mn (Mean)	$\hat{F}\big(t_{(i)}\big) = i/(n+1)$	(11)
Md (Median)	$\hat{F}\big(t_{(i)}\big) = (i-0.3)/(n+0.4)$	(12)
Hn (Hazen)	$\hat{F}\big(t_{(i)}\big) = (i-0.5)/n$	(13)
Variable	$\hat{F}\big(t_{(i)}\big) = (i-3/8)/(n+1/4)$	(14)

In the least squares method, the probability estimators have different effects on the bias of $\widehat{m}$. For example, the probability estimator in Equation (11) gives the largest bias while Equation (13) results in the smallest bias for $n \geq 20$ [19-21,40,41]. For $n < 20$, Equation (14) is to be preferred [19]. However, it has been demonstrated that the coefficient of variation of m for the four estimators in Table 1 is approximately equal [12,19-21]. Gong [42] showed that values of c = 0.999 and d = 1000 in Equation (10) resulted in a low standard deviation for $\widehat{m}/m$. In another recent study, Gong [43] described a method for determining the confidence interval for $\widehat{m}/m$ as a function of the sample size, n, where c = 0.5 and d = 0. In addition to these studies, several authors proposed the use of a correction factor to adjust the bias of the estimated Weibull modulus [19,24,25]. In another study, it is shown that d = 1 provides the optimum unbiased solution for the mean, median and mode of m within the range of $0 \leq d \leq 1$ and also that the optimum values of c are c functions of N [23]. Several studies showed that the Hazen estimator is generally superior to the others given in Table 1 [12,19,20,41,44]. Therefore, in this study, only the Hazen estimator in Equation (13) is used as the probability estimator for interval estimation.

In general, estimates with better statistical properties are obtained by introducing weight functions to linear least squares estimation (Equations (16), (17) and (18)). Various weight functions have been proposed over the years [23,45-47]; the most commonly used are the functions proposed by Bergman [48], Faucher and Tyson [49] and Hung [47] as given in Table 2. The case $W_i = 1$ corresponds to classical unweighted linear least squares and always gives the worst estimation for any probability estimator and any sample size [50]. For accurate estimation of the Weibull modulus, the weight factor given in Equation (16) is preferred [50], especially with Hazen rank [21]. Therefore, in this study, only the WLS method using Faucher and Tyson weight factor and Hazen probability estimator has been chosen for constructing confidence intervals of the Weibull modulus.

***Table 2.** Probability Estimators*

Weight Functions	Weight Functions Equation
A1 (Weightless)	$W(t_{(i)}) = 1$ (15)
A2 (Faucher andTyson)	$W(t_{(i)}) = 3.3\,F(t_{(i)}) - 27.5\left[1 - \left(1 - F(t_{(i)})\right)^{0.025}\right]$ (16)
A3 (Bergman)	$W(t_{(i)}) = \left[\left(1 - F(t_{(i)})\right)\, In\left(1 - F(t_{(i)})\right)\right]^2$ (17)
A4 (Hung)	$W(t_{(i)}) = \dfrac{\left[\left(1 - F(t_{(i)})\right)\, In\left(1 - F(t_{(i)})\right)\right]^2}{\sum\left[\left(1 - F(t_{(i)})\right)\, In\left(1 - F(t_{(i)})\right)\right]^2}$ (18)

2.4. Menon's Method

The Menon's method has the advantage of simplicity, requiring only the computation of the mean and standard deviation of the logarithms of the values in a sample. The resulting estimators [51]

$$\hat{m} = \frac{(\pi/\sqrt{6})}{s_z} \tag{19}$$

$$\hat{\sigma}_0 = e^{(\bar{z}+0.450 s_z)} \tag{20}$$

where s_z denotes the standard deviation of the natural logarithms of the observations (T) in a random sample and $\bar{z}$ their mean. As in the MLE method, the pivotal statistic $\hat{m}/m$ have been used for confidence interval estimation [28].

2.5. Comparison Criterion: Expected Length Based on False Coverage Probability

Pratt [52] has shown that the expected length of a confidence set can be computed by integrating the false coverage probabilities. Given that we observed *T*=*t*, we set up a $(1-\alpha)$ level confidence set $C(T) = \left(L(T), U(T)\right)$; this set has the probability of false coverage $P_\theta\left(\theta' \epsilon C(T)\right)$, $\theta \neq \theta'$. Then the expected length of the set $C(T)$ can be formulated as follows:

$$E_\theta(Length\,|C(T)|) = \int Length\,|C(T)|\, f_\theta\,(t)\,dt = \int_{\theta' \neq \theta} P_\theta\left(\theta' \in C(T)\right)\,d\theta,\ \ \theta \in \Theta, \theta' \neq \theta. \tag{21}$$

The expected length is a somewhat stronger property than the false coverage probabilities calculated at specific θ' values, because the expected length is calculated as a sum over all false coverages [53]. The smaller the interval length of a method the higher the performance.

3. THE RESEARCH FINDINGS AND DISCUSSION

3.1. Simulation Inputs

An extensive Monte Carlo simulation has been performed to compare the performances of the ECTI and SCI methods based on the three classical estimation methods. Figure 2 shows the flowchart of the simulation procedure. All computations shown in Figure 2 have been implemented in the C++ language. Pivotal critical values (d_l, d_u) of the ETCI and the SCI for each n are calculated according to the flowchart in Figure 1.

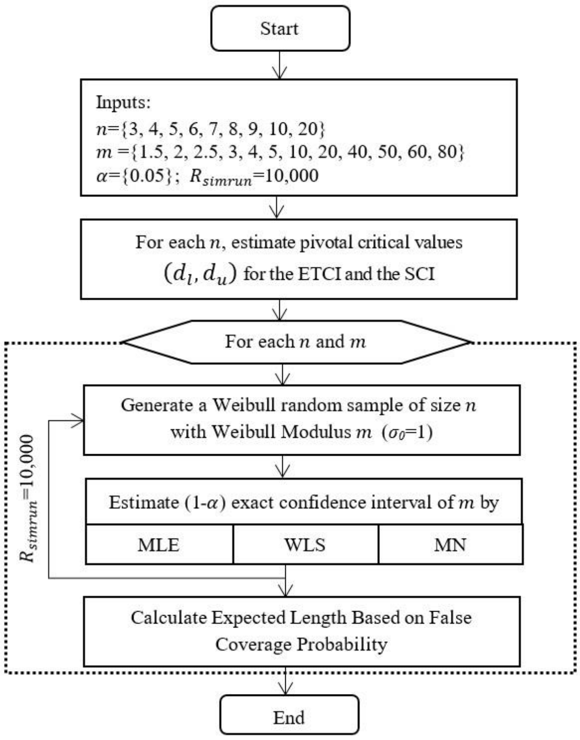


Figure 2. *Flow chart of the simulation procedure*

In the simulation, σ_0 is fixed at $\sigma_0 = 1$, since it has no effect on the results. For each set of given sample size and Weibull modulus, 10000 ETCIs and 10000 SCIs are estimated using the methods described in Section 2. The reason for selecting the small m values in Figure 2 is to examine the effect of the shape of the Weibull probability density function, which is positively skewed for $m < 2.6$, approximates the normal pdf for 2.

3.2. Simulation Results

A detailed summary of the simulation results is presented in Table 3 for each n and m according to the expected length criteria in Equation (21). In Table 3, the methods are abbreviated as WLSE, WLSS, MLEE, MLES, MNE and MNS, where the last string refer to the type of interval: S is used for the SCI and E for the ETCI. In Table 3, the result of the best method, that is the shortest interval length, for each *n* is shown in bold and the second best method is in italic.

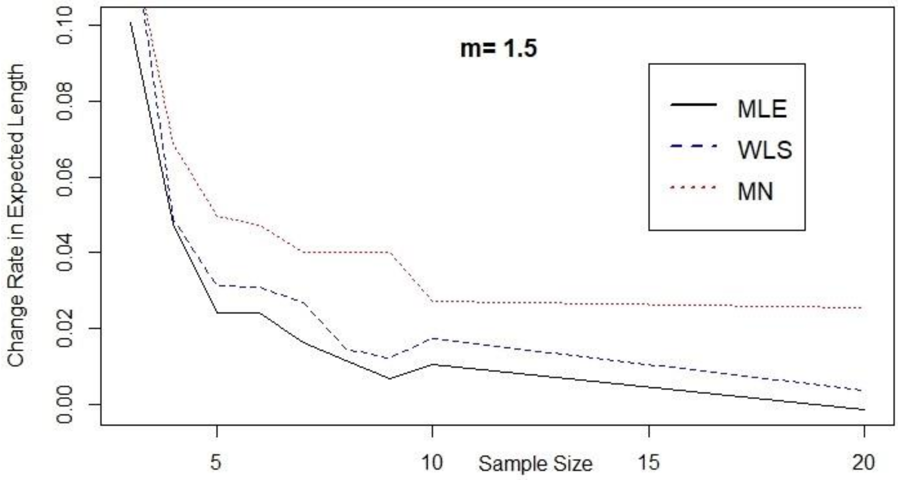
As shown in Table 3, for the almost all *n* values, except for *n* = 3, the MLES method shows the best performance in terms of the expected length. For *n* = 3, the WLSS is the best method, and also the second best for the other *n* values. The best and second methods are based on the SCI. Since the distribution of pivotal quantity $\hat{m}/m$ is asymmetric, the shortest interval has unequal tails probabilities and therefore, any method using the shortest pivotal quantity method gives narrower intervals than its the equal-tailed counterpart.

For *n* = 3, the performance ranking from best to worst is as follows: WLSS, MLES, MNS, MLEE, WLSE and MNE, hence all SCIs are always better than ETCIs. However, when *n* =4, 5 and 6, the performance ranking changes: MLES, WLSS, MLEE, WLSE, MNS and MNE. When *n*>7, it is observed the MLEE method outperforms the WLSS method and so the first three best method are changed as MLES, MLEE, WLSS; the rest remaining the same. MNS has not yielded narrower intervals than MLEE and WLSE methods for *n* > 3. But for all n values, MLES and WLSS have yielded shorter intervals than the ETCI of the all three methods.

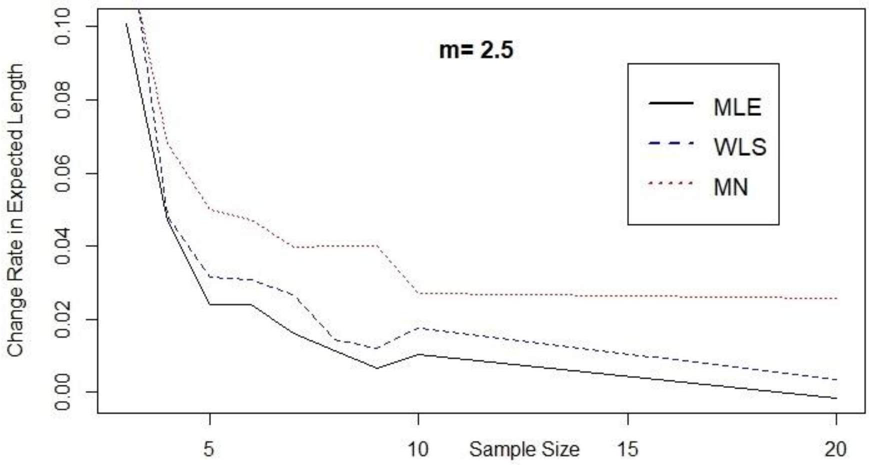
Table 3. *Expected Interval Lengths for Weibull Modulus*

n	Method	Weibull Modulus													
		1.5	1.8	2	2.2	2.5	2.8	3	4	5	10	20	40	60	80
3	WLSE	5.59	6.33	7.48	8.11	9.21	10.56	11.37	15.06	18.42	37.51	76.03	149.15	227.00	303.05
	WLSS	**4.87**	**5.37**	**6.58**	**7.26**	**8.01**	**9.23**	**10.13**	**13.52**	**16.61**	**32.91**	**67.22**	**131.56**	**202.28**	**271.79**
	MLEE	5.44	6.19	7.29	7.89	8.97	10.27	11.07	14.63	17.88	36.40	73.84	145.31	220.55	294.38
	MLES	*4.89*	*5.43*	*6.61*	*7.28*	*8.07*	*9.27*	*10.18*	*13.54*	*16.63*	*33.01*	*67.42*	*132.45*	*202.99*	*272.45*
	MNE	5.80	6.53	7.77	8.42	9.51	10.95	11.77	15.68	19.06	38.76	78.63	155.15	235.05	313.22
	MNS	5.08	5.61	6.88	7.57	8.35	9.63	10.54	14.13	17.26	34.24	69.94	137.68	210.62	282.40
4	WLSE	3.69	4.46	4.92	5.44	6.20	6.98	7.50	9.98	12.30	24.81	49.37	99.22	146.86	196.33
	WLSS	*3.51*	*4.24*	*4.68*	*5.17*	*5.90*	*6.64*	*7.14*	*9.50*	*11.70*	*23.60*	*46.96*	*94.07*	*139.69*	*186.68*
	MLEE	3.65	4.40	4.86	5.37	6.12	6.88	7.40	9.86	12.13	24.49	48.74	97.87	145.08	193.93
	MLES	**3.47**	**4.20**	**4.63**	**5.12**	**5.84**	**6.55**	**7.05**	**9.38**	**11.55**	**23.32**	**46.42**	**93.04**	**138.15**	**184.64**
	MNE	4.01	4.83	5.33	5.89	6.70	7.56	8.16	10.83	13.30	26.97	53.54	107.72	159.69	213.04
	MNS	3.73	4.50	4.96	5.48	6.24	7.04	7.60	10.09	12.38	25.11	49.84	99.89	148.68	198.38
5	WLSE	2.94	3.55	3.92	4.28	4.94	5.52	5.95	7.82	9.75	19.83	39.22	78.52	118.34	157.41
	WLSS	*2.85*	*3.44*	*3.79*	*4.15*	*4.78*	*5.35*	*5.76*	*7.57*	*9.44*	*19.21*	*37.99*	*76.06*	*114.64*	*152.48*
	MLEE	2.86	3.44	3.80	4.15	4.79	5.36	5.77	7.59	9.47	19.28	38.09	76.19	115.02	153.00
	MLES	**2.79**	**3.36**	**3.71**	**4.05**	**4.68**	**5.23**	**5.63**	**7.41**	**9.24**	**18.82**	**37.17**	**74.37**	**112.27**	**149.33**
	MNE	3.25	3.91	4.34	4.76	5.46	6.10	6.57	8.65	10.77	21.94	43.42	86.86	131.08	174.27
	MNS	3.09	3.72	4.13	4.52	5.19	5.79	6.25	8.22	10.24	20.85	41.26	82.53	124.56	165.61
6	WLSE	2.50	3.02	3.37	3.71	4.22	4.68	5.05	6.77	8.44	16.85	33.75	67.15	100.47	134.29
	WLSS	*2.42*	*2.92*	*3.27*	*3.60*	*4.09*	*4.53*	*4.89*	*6.56*	*8.18*	*16.33*	*32.70*	*65.08*	*97.37*	*130.15*
	MLEE	2.43	2.93	3.27	3.61	4.10	4.55	4.90	6.57	8.20	16.35	32.82	65.24	97.66	130.43
	MLES	**2.37**	**2.86**	**3.20**	**3.52**	**4.00**	**4.44**	**4.78**	**6.42**	**8.00**	**15.96**	**32.03**	**63.67**	**95.31**	**127.28**
	MNE	2.82	3.40	3.80	4.20	4.75	5.28	5.68	7.62	9.51	18.95	38.20	75.70	113.01	151.30
	MNS	2.69	3.24	3.63	4.00	4.53	5.03	5.41	7.26	9.07	18.06	36.40	72.13	107.68	144.16
7	WLSE	2.24	2.67	2.97	3.29	3.72	4.15	4.45	5.95	7.33	14.90	29.58	59.37	89.13	118.53
	WLSS	2.18	2.60	2.89	3.20	3.62	4.04	4.33	5.79	7.14	14.51	28.79	57.79	86.76	115.37
	MLEE	*2.18*	*2.60*	*2.89*	*3.19*	*3.62*	*4.04*	*4.32*	*5.78*	*7.13*	*14.48*	*28.78*	*57.74*	*86.70*	*115.13*
	MLES	**2.14**	**2.56**	**2.84**	**3.14**	**3.56**	**3.97**	**4.25**	**5.69**	**7.01**	**14.24**	**28.30**	**56.79**	**85.28**	**113.24**
	MNE	2.56	3.05	3.39	3.76	4.26	4.75	5.07	6.81	8.39	17.04	33.79	67.78	102.13	135.03
	MNS	2.46	2.93	3.26	3.61	4.09	4.56	4.87	6.54	8.05	16.36	32.44	65.08	98.05	129.64
8	WLSE	2.00	2.42	2.68	2.98	3.36	3.77	4.04	5.38	6.70	13.36	26.73	53.73	80.75	107.53
	WLSS	1.97	2.38	2.64	2.94	3.31	3.72	3.98	5.30	6.60	13.16	26.34	52.95	79.58	105.96
	MLEE	*1.96*	*2.36*	*2.62*	*2.91*	*3.29*	*3.69*	*3.95*	*5.27*	*6.55*	*13.06*	*26.15*	*52.57*	*79.07*	*105.19*
	MLES	**1.93**	**2.33**	**2.59**	**2.88**	**3.25**	**3.65**	**3.90**	**5.21**	**6.48**	**12.92**	**25.85**	**51.97**	**78.17**	**103.99**
	MNE	2.34	2.82	3.12	3.47	3.93	4.39	4.69	6.29	7.82	15.56	31.27	62.85	94.28	125.58
	MNS	2.24	2.71	3.00	3.34	3.77	4.21	4.51	6.04	7.50	14.94	30.02	60.33	90.52	120.56
9	WLSE	1.85	2.21	2.46	2.72	3.07	3.45	3.68	4.90	6.16	12.39	24.59	49.33	74.05	98.72
	WLSS	1.82	2.18	2.43	2.68	3.04	3.41	3.63	4.84	6.08	12.24	24.29	48.73	73.14	97.52
	MLEE	*1.81*	*2.16*	*2.41*	*2.65*	*3.01*	*3.38*	*3.60*	*4.80*	*6.03*	*12.12*	*24.08*	*48.24*	*72.39*	*96.55*
	MLES	**1.79**	**2.15**	**2.39**	**2.64**	**2.99**	**3.36**	**3.57**	**4.77**	**5.99**	**12.04**	**23.92**	**47.92**	**71.90**	**95.90**
	MNE	2.18	2.60	2.90	3.19	3.61	4.09	4.34	5.79	7.26	14.58	28.99	58.18	87.04	116.08
	MNS	2.09	2.49	2.78	3.06	3.47	3.93	4.16	5.55	6.97	14.00	27.83	55.85	83.56	111.44
10	WLSE	1.72	2.06	2.30	2.53	2.86	3.21	3.44	4.61	5.75	11.51	23.06	45.69	68.73	91.89
	WLSS	1.69	2.03	2.26	2.48	2.81	3.16	3.38	4.53	5.65	11.31	22.66	44.90	67.53	90.30
	MLEE	*1.68*	*2.02*	*2.25*	*2.47*	*2.80*	*3.14*	*3.36*	*4.51*	*5.63*	*11.25*	*22.57*	*44.70*	*67.17*	*89.89*
	MLES	**1.66**	**2.00**	**2.22**	**2.45**	**2.77**	**3.11**	**3.33**	**4.46**	**5.57**	**11.14**	**22.33**	**44.23**	**66.46**	**88.94**
	MNE	2.03	2.45	2.72	2.99	3.38	3.79	4.07	5.46	6.80	13.55	27.28	53.94	81.32	108.90
	MNS	1.98	2.38	2.64	2.91	3.29	3.69	3.96	5.31	6.62	13.19	26.54	52.46	79.10	105.93
20	WLSE	1.13	1.36	1.50	1.65	1.88	2.11	2.26	3.01	3.76	7.52	15.05	30.14	45.10	60.38
	WLSS	1.12	1.35	1.50	1.64	1.88	2.10	2.25	3.00	3.74	7.50	15.00	30.03	44.93	60.16
	MLEE	**1.10**	**1.32**	**1.46**	**1.60**	**1.83**	**2.05**	*2.19*	*2.92*	*3.65*	*7.31*	*14.62*	*29.26*	*43.81*	*58.61*
	MLES	**1.10**	**1.32**	**1.46**	**1.60**	**1.83**	**2.05**	**2.20**	**2.93**	**3.66**	**7.32**	**14.64**	**29.30**	**43.87**	**58.70**
	MNE	1.40	1.68	1.86	2.04	2.32	2.60	2.80	3.72	4.66	9.31	18.64	37.20	55.85	74.50
	MNS	1.36	1.63	1.81	1.99	2.27	2.54	2.73	3.63	4.54	9.07	18.16	36.25	54.42	72.61

The MLES, WLSS and MNS methods result in different rates of decrease in the interval length respectively as compared to the MLEE, WLSE and MNE; the difference become significant particularly for $n \leq 10$. For n=3, the WLSS results in a decrease of 10%-13% in the interval length of WLSE. Similarly the MLES and MNE methods result in 10%-7% and 14%-10% decreases, respectively. These rates drop to approximately 6% for n=4. As the sample size increases, the estimation performances of SCI and ETCI for the MLE, WLS and MN become closer. The rates of decrease approaches to zero for the WLS and MLE methods for n=20, but remains at 3% for MN. For m=1.5, 2.5, 10 and 80, the rates of decrease achieved by the MLES, WLSS and MNS are also shown graphically in Figure 3 (a-b) and Figure 4 (a-b).

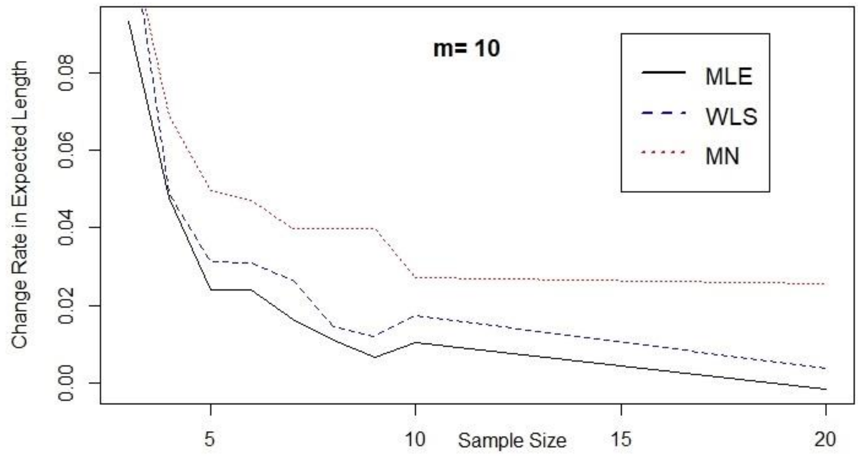


(**a**)

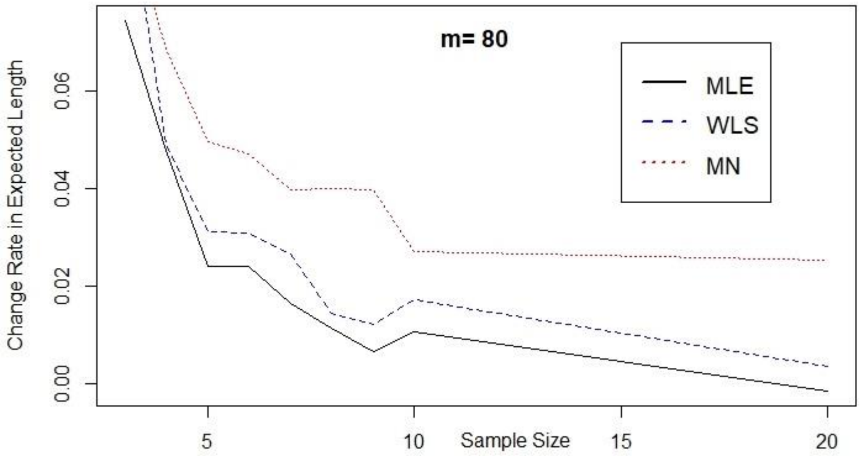


(**b**)

Figure 3. *Rates of decrease in expected length achieved by SCIs as compared to ETCIs for : (a) m=1.5 (b) m=2.5*



(**a**)



(**b**)

Figure 4. *Rates of decrease in expected length achieved by SCIs as compared to ETCIs for : (a) m=10 (b) m=80*

As it is well known, for a 95% ETCI, the area in each tail is equal to 0.025. However, the pivotal statistic $\hat{m}/m$ has a positively skewed distribution and so the SCI gets different area value in each tail differently from the ETCI. The left tail probabilities, $P(\hat{m}/m < d_l)$, and right tail probabilities, $P(\hat{m}/m > d_u)$, of WLSS, MLES and MNS are summarized in Table 4. For n=3, the WLSS, the MLES and the MNS yield the SCI with % 0.3 left tailed-% 4.7 left tailed probability. By increasing sample size, these methods show a non-monotonic increase on the left tail; but, even when n=20, left tail probabilities for the SCIs do not reach at 0.025. As a result, lower and upper bounds of SCIs are always smaller than those of ETCIs.

Table 4. *The left tail and right left tail probabilities of WLSS, MLES and MNS*

METHOD	**TAILS**	**Weibull shape parameter m**								
		3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**20**
WLSS	$P(\hat{m}/m < d^*_l)$	0.003	0.006	0.011	0.010	0.013	0.016	0.013	0.016	0.016
	$P(\hat{m}/m > d^*_u)$	0.047	0.044	0.039	0.040	0.037	0.034	0.037	0.034	0.034
MLES	$P(\hat{m}/m < d^*_l)$	0.003	0.008	0.013	0.013	0.012	0.014	0.016	0.016	0.019
	$P(\hat{m}/m > d^*_u)$	0.047	0.042	0.037	0.037	0.038	0.036	0.034	0.034	0.031
MNS	$P(\hat{m}/m < d^*_l)$	0.003	0.006	0.009	0.009	0.011	0.009	0.013	0.012	0.014
	$P(\hat{m}/m > d^*_u)$	0.047	0.044	0.041	0.041	0.039	0.041	0.037	0.038	0.036

Figure 5 and 6 plots average confidence intervals obtained from the 6 methods for m= 1.5 and m=80, that is, any lower or upper bound is the average of the lower or upper bounds of 10,000 confidence intervals estimated by Monte-Carlo simulations. They show that the WLSS, the MLES and the MNS methods generate confidence intervals that are more centered around m as compared to the WLSE, the MLEE and the MNE methods. This phenomenon is particularly noticeable for small samples.

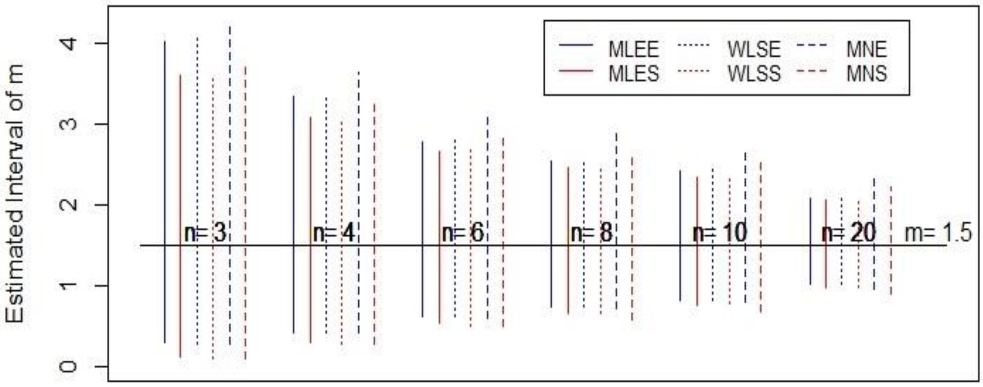


Figure 5. *The position of estimated confidence intervals according to m=1.5*

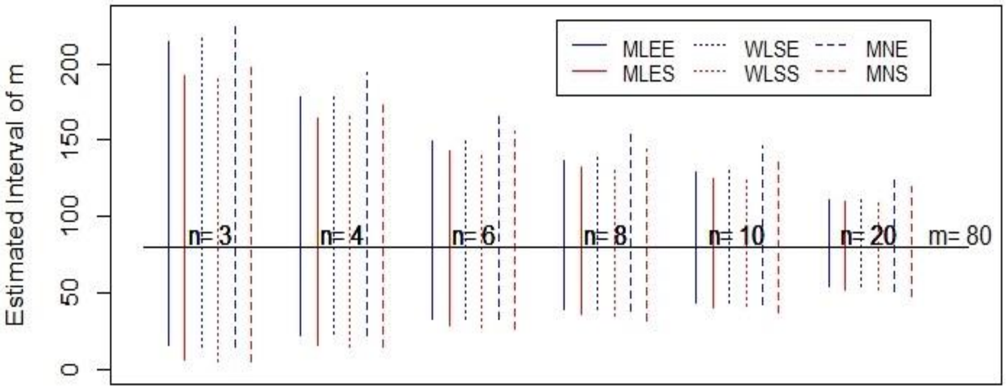


Figure 6. *The position of estimated confidence intervals according to m=80*

For verification of the simulation results, an exercise estimating confidence intervals of Weibull modulus of a composite component was performed using the previously presented simulation findings [54]: 19 identical composite specimens were prepared from quasi-isotropic carbon-epoxy sheets with (0°)3 configuration, 0.89 mm thickness, and 295 gr/m2 weight and the tension experiments were carried out using an Instron 8516+ universal testing machine. The fracture strength values measured are presented in Table 5. In order to diversify the application study in terms of sample size, in addition to an application to the entire sample, random subsets of size of 3, 5 and 10 were drawn from the sample of 19 observations: These subset consists of { 473, 442, 502.7 } for n=3, { 450.9, 442, 476.5, 521.6, 439 } for n=5 and { 513.6, 552,519, 521.6, 439,450.9, 463.5, 497.5, 476.5, 477 } for n=10. For each data set, the p-values of the Kolmogorov-Smirnov test are more than 0.05 which means that the assumption of following Weibull distribution cannot be rejected. The six methods were carried out to estimate the confidence intervals of m for each data set. The result of the best method for each n was shown in bold in Table 6.

Table 5. *Fracture strength of carbon-epoxy composite material specimens (megapascals)*

Test No.	1	2	3	4	5	6	7	8	9	10
Fracture strength [MPa]	532.7	502.5	442	473	519	502.7	477	510	522	552
Test No.	11	12	13	14	15	16	17	18	19	
Fracture strength [MPa]	522	439	513.6	497.5	521.6	450.9	476.5	507.3	463.5	

As can be seen in Table 6, the experimental results are quite similar to simulation results and the MLES model outperforms the other methods.

***Table 6**. Experimental results: Interval lengths, their lower and upper limits*

Method	**n=3**	**n=5**	**n=10**	**n=19**
MLES	**33,484 [0,909 34,393]**	**18,568 [3,525 22,093]**	**14,752 [7,214 21,966]**	**13,153 [11,725 24,877]**
MLEE	*35,853 [2,637 38,49]*	*19,207 [4,217 23,424]*	*15,037 [7,663 22,7]*	*13,191 [11,954 25,145]*
WLSS	36,975 [1,177 38,152]	*19,432 [3,277 22,709]*	16,137 [7,461 23,599]	13,891 [11,668 25,559]
WLSE	40,169 [2,92 43,09]	20,27 [4,182 24,452]	16,425 [8,02 24,445]	13,999 [12,079 26,078]
MNS	36,239 [0,757 36,996]	26,751 [3,904 30,655]	19,961 [7,318 27,279]	16,939 [11,181 28,121]
MNE	39,731 [2,748 42,479]	28,722 [5,34 34,062]	20,687 [8,498 29,185]	17,511 [11,893 29,404]

As can be seen in Table 6, the experimental results are quite similar to simulation results and the shortest pivotal quantity technique based on maximum likelihood (MLES) outperforms the other methods. As n increases, performances of the WLS and MLES methods become closer to each other, but the MLES maintains its superiority by a small difference in estimated interval length.

4. RESULTS

This study compares the performances of the equal-tailed and shortest pivotal quantity models based on the maximum likelihood (MLE), weighted linear least squares (WLS) or Menon's (MN) methods for estimating exact confidence intervals of the Weibull modulus (m) in small samples. These models of methods are abbreviated as WLSE, WLSS, MLEE, MLES, MNE and MNS, where the last string S and E refers to the shortest and the equal-tailed pivotal quantity, respectively. For a comparison, an extensive Monte-Carlo simulation study has been conducted and the results have been assessed in terms of the expected interval length.

The simulation results show that since the pivotal statistic $\widehat{m}/m$ has a positively skewed distribution, any of the MLES, the WLSS or the MNS methods gives better results in expected interval length than its the equal-tailed counterpart. However, MNS has not even yielded narrower intervals than MLEE and WLSE methods for $n > 3$. Among the six methods, MLES yields the narrowest expected interval for the almost all n values, except for $n = 3$ where WLES is best. In addition, any of the MLES, the WLSS or the MNS methods generates more centered confidence intervals around m as compared to its the equal-tailed counterpart. This result is very important for engineers that want the unknown parameter to lie in or near the center of the estimated interval, especially having narrower length.

In this study, for the WLS method, only the Faucher and Tyson weight factor and the hazen probability estimator have been choosen based on the literature; but any WLS method on the combination of different weight factors and probability estimators will generate a positively skewed distribution for the pivotal statistic $\widehat{m}/m$ and so any WLSS will give again better results in the expected interval length as compared to its the equal-tailed counterpart. However, for future studies, different pivotal statistics can be used to estimate exact confidence intervals of m; and within this framework the performances of the MLE, the MN and different WLS methods can be compared.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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