Eastern Anatolian Journal of Science
Volume VII, Issue I, 2021, 25-29

Eastern Anatolian Journal of Science

The Hyperbolic Quadrapell Sequences

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Abstract

In this paper, we extend Quadrapell numbers to Hyperbolic Quadarapell numbers, respectively. Moreover we obtain Binet-like formulas, generating functions and some identities related with Hyperbolic Quadarpell numbers.

Keywords: Pell numbers, Quadrapell numbers, Hyperbolic numbers, Hyperbolic Quadrapell numbers.

1. Introduction

Hyperbolic numbers have applications in different areas of mathematics and theoretical physics. In particular, they are related to Lorentz-Minkowski (Space-time) geometry in the plane as well as complex numbers are to Euclidean one (Catoni 2008). The work on the function theory for hyperbolic numbers can be found in (Aydın 2019, Barreira 2016, Gargoubi 2016, Khadjiev 2016, Motter 2016, Güncan 2012). The set of hyperbolic numbers H can be described in the form as

$$\mathbb{H} = \{z = x + hy \mid h \notin \mathbb{R}, h^2 = 1, x, y \in \mathbb{R}\}$$

Addition, substruction and multiplication of two hyperbolic numbers z_1 and z_2 are defined by

$$z_1 \pm z_2 = (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2)$$

$$z_1 \times z_2 = (x_1 + hy_1) \times (x_2 + hy_2) = (x_1x_2) + (y_1y_2) + h(x_1y_2 + y_1x_2)$$

On the other hand, the division of two hyperbolic numbers are given by

$$\frac{z_1}{z_2} = \frac{x_1 + hy_1}{x_2 + hy_2}$$

$$\frac{(x_1 + hy_1)(x_2 - hy_2)}{(x_2 + hy_2)(x_2 - hy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 - y_2^2} + h\frac{(x_1y_2 + y_1x_2)}{x_2^2 - y_2^2}$$

If $x_2^2 - y_2^2 \neq 0$, then the division $\frac{z_1}{z_2}$ is possible. The hyperbolic conjugation of $z = x + hy$ is defined by $\bar{z} = x - hy$.

2. Materials and Methods

Many studies have been done on pell sequence in the past. Some of these are (Voet 2012, Atanassov 2009, Çağman and Polat 2021, Çağman 2021a, Çağman 2021b, Deveci 2015, Deveci 2018, Deveci 2020, Shannon 2006, Tas 2014, Berzsenyi 1977, Horadam 1963). The Quadrapell sequence is studied by Dursun Taşçı (Taşçı 2018).

The QuadraPell sequence is the sequence of integers D_n defined by the initial values $D_0 = D_1 = D_2 = 1, D_3 = 2$ and the recurrence relation

$$D_n = D_{n-2} + 2D_{n-3} + D_{n-4}$$

for all $n \geq 4$. The first few values of D_n are $1, 1, 1, 2, 4, 5, 9, 15, 23, 38, 12, 62, 99, 161, 261, 421$.

3. Results

Definition 3.1. The Hyperbolic Quadrapell numbers $HD_n$are defined by the initial values $HD_0 = HD_1 = 1 + h, HD_2 = 1 + 2h, HD_3 = 2 + 4h$ and the recurrence relation

$$HD_n = D_n + hD_{n+1}$$

$$HD_n = HD_{n-2} + 2HD_{n-3} + HD_{n-4}$$

for all $n \geq 4$.

Received:20.04.2021
Accepted:30.05.2021
Published:30.05.2021
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Cite this article as: S. Taş, The Hyperbolic Quadrapell Sequences, *Eastern Anatolian Journal of Science, Vol. 7, Issue 1, 25-29, 2021.*

The first few values of HD_n are $1+h, 1+h, 1+2h, 2+4h, 4+5h, 5+9h, 9+15h, 15+23h, 23+38h, 38+62h, 62+99h$.

Theorem 3.2. The generating function of the Hyperbolic Quadrapell sequnce is

$$g(x)=\frac{1+h+(1+h)x+hx^2+(-1+h)x^3}{1-x^2-2x^3-x^4}$$

Proof. Let

$$g(x)=\sum_{n=0}^{\infty} HD_n x^n = HD_0+HD_1x+HD_2x^2+HD_3x^3+\cdots+HD_nx^n+\cdots$$

be generating function of the Hyperbolic Quadrapell sequence. On the other hand, since

$$x^2g(x)=HD_0x^2+HD_1x^3+HD_2x^4+HD_3x^5+\cdots+HD_{n-2}x^n+\cdots$$

$$2x^3g(x)=2HD_0x^3+2HD_1x^4+2HD_2x^5+2HD_3x^6+\cdots+2HD_{n-3}x^n+\cdots$$

and

$$x^4g(x)=HD_0x^4+HD_1x^5+HD_2x^6+HD_3x^7+\cdots+HD_{n-3}x^n+\cdots$$

we write

$$(1-x^2-2x^3-x^4)g(x) = HD_0+HD_1x+(HD_2-HD_0)x^2+(HD_3-HD_1-2HD_0)x^3+\cdots+(HD_n-HD_{n-2}-2HD_{n-3}-HD_{n-4})x^n+\cdots$$

Now consider $HD_0=HD_1=1+h$, $HD_2=1+2h, HD_3=2+4h$ and $HD_n=HD_{n-2}+2HD_{n-3}+HD_{n-4}$. Thus, we obtain

$$(1-x^2-2x^3-x^4)g(x) = HD_0+HD_1x+(HD_2-HD_0)x^2+(HD_3-HD_1-2HD_0)x^3$$

$$(1-x^2-2x^3-x^4)g(x)=1+h+(1+h)x+hx^2+(-1+h)x^3$$

or

$$g(x)=\frac{1+h+(1+h)x+hx^2+(-1+h)x^3}{1-x^2-2x^3-x^4}$$

So, the proof is complete.

Now we give Binet-like formula for the Hyperbolic Quadrapell sequence.

Theorem 3.3. Binet-like formula for the Hyperbolic Quadrapell sequence is

$$HD_n=\left(\frac{1+h\alpha}{2}\right)\alpha^n+\left(\frac{1+h\beta}{2}\right)\beta^n+\left(\frac{1+h\gamma}{2\sqrt{3}i}\right)\gamma^n+\left(\frac{1+h\delta}{2\sqrt{3}i}\right)\delta^n$$

where

$$\alpha=\frac{1+\sqrt{5}}{2}, \beta=\frac{1-\sqrt{5}}{2}$$

and

$$\gamma=\frac{-1+\sqrt{3}i}{2}, \delta=\frac{-1-\sqrt{3}i}{2}$$

are the roots of the equation $x^4-x^2-2x-1=0$.

Proof. It is easily seen that

$$HD_n=D_n+hD_{n+1}$$

On the other hand, we know that the Binet-like formula for the Quadrapell sequence is

$$D_n=\frac{\alpha^n+\beta^n}{2}+\frac{\gamma^n-\delta^n}{2\sqrt{3}i}.$$

Theorem 3.4.

$$\sum_{j=0}^{3k+1} HD_j = HD_{3k+3}-2h.$$

Proof. We use the principle of mathematical induction. Since

$$HD_0+HD_1=2+2h=HD_3-2h$$

clearly the result is true for $k=0$.

Now assume it is true for an arbitrary positive integer $k>1$

$$\sum_{j=0}^{3k+1} HD_j = HD_{3k+3}-2h.$$

Then we have

$$\sum_{j=0}^{3k+4} HD_j = \sum_{j=0}^{3k+1} HD_j + HD_{3k+2}+HD_{3k+3}+HD_{3k+4}.$$

$$= HD_{3k+3}-2h+HD_{3k+2}+HD_{3k+3}+HD_{3k+4}$$

$= HD_{3k+4} + 2HD_{3k+3} + HD_{3k+2} - 2h$

$= HD_{3k+6} - 2h$

So the formula works for $k+1$. Thus, by the principle of mathematical induction the formula holds for every integer $k \geq 0$.

Theorem 3.5.

$$\sum_{j=0}^{3k+2} HD_j = HD_{3k+4} - (1+h).$$

Proof. We proceed by induction on k. Since

$$HD_0 + HD_1 + HD_2 = 3 + 4h = HD_4 - (1+h)$$

the statement is true for $k = 0$.
Now assume it is true for $k > 1$

$$\sum_{j=0}^{3k+2} HD_j = HD_{3k+4} - (1+h).$$

Then, we show that the formula holds for $k+1$. Indeed,

$$\sum_{j=0}^{3k+5} HD_j = \sum_{j=0}^{3k+2} HD_j + HD_{3k+3} + HD_{3k+4} + HD_{3k+5}.$$

$= HD_{3k+4} - (1+h) + HD_{3k+3} + HD_{3k+4} + HD_{3k+5}$

$= HD_{3k+5} + 2HD_{3k+4} + HD_{3k+3} - (1+h)$

$= HD_{3k+7} - (1+h)$

So the formula works for $k+1$. Thus, by the principle of mathematical induction the formula holds for every integer $k \geq 0$.

Lemma 3.6.

$$HD_n + HD_{n+1} + HD_{n+3} + HD_{n+5} = HD_{n+6}.$$

Proof. By the Hyperbolic Quadrapell recurrence relation, we have

$$HD_{n+5} = HD_{n+3} + 2HD_{n+2} + HD_{n+1}.$$

Then we obtain

$HD_n + HD_{n+1} + HD_{n+3} + HD_{n+5}$

$= HD_n + 2HD_{n+1} + 2HD_{n+2} + 2HD_{n+3}$

$= HD_{n+2} + 2HD_{n+1} + HD_n + 2HD_{n+3} + HD_{n+2}$

$= HD_{n+4} + 2HD_{n+3} + HD_{n+2}$

$= HD_{n+6}.$

So the lemma is proved.

Theorem 3.7.

$$\sum_{j=0}^{3k+1} HD_{2j} = HD_{6k+3} - h.$$

Proof. We use the principle of mathematical induction. Since

$$HD_0 + HD_2 = 2 + 3h = HD_3 - h$$

clearly the result is true for $k = 0$.
Now assume it is true for an arbitrary positive integer $k > 1$

$$\sum_{j=0}^{3k+1} HD_{2j} = HD_{6k+3} - h.$$

Then we have

$$\sum_{j=0}^{3k+4} HD_{2j} = \sum_{j=0}^{3k+1} HD_{2j} + HD_{6k+4} + HD_{6k+6} + HD_{6k+8}.$$

$= HD_{6k+3} - h + HD_{6k+4} + HD_{6k+6} + HD_{6k+8}$

$= HD_{6k+9} - h$

So the formula works for $k+1$. Thus, by the principle of mathematical induction the formula holds for every integer $k \geq 0$.

Theorem 3.8.

$$\sum_{j=0}^{3k+2} HD_{2j} = HD_{6k+5} + (1-h).$$

Proof. We proceed by induction on k. Since

$$HD_0 + HD_2 + HD_4 = 6 + 8h = HD_5 + (1-h)$$

the statement is true for $k = 0$.
Now assume it is true for $k > 1$

$$\sum_{j=0}^{3k+2} HD_{2j} = HD_{6k+5} + (1-h).$$

Then, we show that the formula holds for $k+1$. Indeed,

$$\sum_{j=0}^{3k+5} HD_{2j} = \sum_{j=0}^{3k+2} HD_{2j} + HD_{6k+6} + HD_{6k+8} + HD_{6k+10}.$$

$= HD_{6k+5} + (1-h) + HD_{6k+6} + HD_{6k+8} + HD_{6k+10}$

$= HD_{6k+11} + (1-h)$

So the formula works for $k+1$. Thus, by the principle of mathematical induction the formula holds for every integer $k \geq 0$.

Theorem 3.9.

$$\sum_{j=0}^{3k} HD_{2j+1} = HD_{6k+2} - h.$$

Proof. We use the principle of mathematical induction on k. Since

$$HD_0 = 1 + h = HD_2 - h$$

clearly the result is true for $k = 0$.

Now we suppose that the statement holds for $k > 1$. Indeed,

$$\sum_{j=0}^{3k} HD_{2j+1} = HD_{6k+2} - h.$$

Then we have

$$\sum_{j=0}^{3k+3} HD_{2j+1} = \sum_{j=0}^{3k} HD_{2j+1} + HD_{6k+3} + HD_{6k+5} + HD_{6k+7}.$$

$$= HD_{6k+2} - h + HD_{6k+3} + HD_{6k+5} + HD_{6k+7}$$

$$= HD_{6k+8} - h$$

So the formula works for $k+1$. Thus, by the principle of mathematical induction the formula holds for every integer $k \geq 0$.

Theorem 3.10.

$$\sum_{j=0}^{3k+1} HD_{2j+1} = HD_{6k+4} - 1.$$

Proof. We use the principle of mathematical induction on k. Since

$$HD_1 + HD_3 = 3 + 5h = HD_2 - 1$$

clearly the result is true for $k = 0$.

Now we suppose that the statement holds for $k > 1$. Indeed,

$$\sum_{j=0}^{3k+1} HD_{2j+1} = HD_{6k+4} - 1.$$

Then we have

$$\sum_{j=0}^{3k+4} HD_{2j+1} = \sum_{j=0}^{3k+1} HD_{2j+1} + HD_{6k+5} + HD_{6k+7} + HD_{6k+9}.$$

$$= HD_{6k+4} - 1 + HD_{6k+5} + HD_{6k+7} + HD_{6k+9}$$

$$= HD_{6k+10} - 1$$

So the formula works for $k+1$. Thus, by the principle of mathematical induction the formula holds for every integer $k \geq 0$.

Now we investigate the new property of Hyperbolic Quadrapell numbers in relation with Quadrapell matrix formula. We consider the following matrices:

$$Q_4 = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} 2+4h & 1+2h & 1+h & 1+h \\ 1+2h & 1+h & 1+h & -1+h \\ 1+h & 1+h & -1+h & 2-h \\ 1+h & -1+h & 2-h & -2+2h \end{bmatrix}$$

and

$$M_4^n = \begin{bmatrix} HD_{n+3} & HD_{n+2} & HD_{n+1} & HD_n \\ HD_{n+2} & HD_{n+1} & HD_n & HD_{n-1} \\ HD_{n+1} & HD_n & HD_{n-1} & HD_{n-2} \\ HD_n & HD_{n-1} & HD_{n-2} & HD_{n-3} \end{bmatrix}$$

Theorem 3.11. For all $n \in \mathbb{Z}^+$ we have

$$Q_4^n K_4 = M_4^n.$$

Proof. The proof is easily seen that using the induction on n.

4. Discussion

We defined Hyperbolic Quadrapell numbers and we obtain Binet-like formulas, generating functions and some identities related with Hyperbolic Quadrapell numbers.

References

ATANASSOV, K., DIMITROV, D., SHANNON, A.(2009). A remark on ψ-function and pell-padovan's sequence, *Notes on Number Theory and Discrete Mathematics*, 15(2), 1-44.

AYDIN, F.T.(2019). Hyperbolic Fibonacci sequence, *Universal Journal of Mathematics and Applications*, 2(2), 59-64.

BARREİRA, L., POPESCU, L.H., VALLS, C.(2016). Hyperbolic sequences of linear operators and evolution maps, *Milan Journal of Mathematics*, 84(2), 203-216.

BERZSENYİ, G.(1977). Gaussian fibonacci numbers

CATONİ, F., BOCCALETTİ, D., CANNATA, R., CATONİ, V., NİCHELATTİ, E., ZAMPETTİ, P. (2008). The mathematics of Minkowski space-time: with an introduction to commutative hypercomplex numbers, *Springer Science & Business Media*.

ÇAĞMAN, A.(2021a). Repdigits as Product of Fibonacci and Pell numbers. *Turkish Journal of Science*, 6(1), 31-35.

ÇAĞMAN, A.(2021b). Explicit Solutions of Powers of Three as Sums of Three Pell Numbers Based on Baker's Type Inequalities. *Turkish Journal of Inequalities*, 5(1), 93-103.

ÇAĞMAN, A, POLAT, K.(2021). . On a Diophantine equation related to the difference of two Pell numbers. *Contributions to Mathematics*. Volume 3, 37-42

DEVECİ, Ö., KARADUMAN, E.(2015). The pell sequences in finite groups. *Util. Math*, 96, 263-276.

DEVECİ, Ö., SHANNON, A.G.(2018). The quaternion-pell sequence. *Communication in Algebra*, 46(12), 50403-5409.

DEVECİ, Ö., SHANNON, A.G.(2020). The complex-type k- Fibonacci sequences and their applications. *Communication in Algebra*, pages 1-16.

GARGOUBI, H., KOSSENTINI, S. (2016). f-algebra structure on hyperbolic numbers, *Advances in Applied Clifford Algebras*, 26(4), 1211-1233.

GÜNCAN, A., ERBIL, Y. (2012). The q-fibonacci hyperbolic functions, *In AIP Conference Proceedings*, American Institute of Physics, volume 1479, pages 946-949.

HORADAM, A.F. (1963). Complex fibonacci numbers and fibonacci quaternions, *The American Mathematical Monthly*, 70(3), 289-291.

KHADJIEV, D., GÖKSAL, Y. (2016). Applications of hyperbolic numbers to the invariant theory in two-dimensional pseudo-euclidean space, *Advances in Applied Clifford Algebras*, 26(2), 645-668.

MOTTER, A.E., ROSA, M.A.F. (2016). Hyperbolic calculus, *Advances in Applied Clifford Algebras*, 8(1), 109-128.

SHANNON, A.G., HORADAM, A.F., ANDERSON, P.G. (2006). The auxiliary equation associated with the plastic number, *Notes on Number Theory and Discrete Mathematics*, 12(1), 1-12.

SHANNON, A.G., ANDERSON, P.G., HORADAM, A.F. (2006). Properties of cordonnier, perrin and van der laan numbers, *International Journal of Mathematical Education in Science and Technology*, 37(7), 825-831.

TAS, S., DEVECI, O., KARADUMAN, E. (2014). The fibonacci-padovan sequences in fnite groups, *Maejo International Journal of Science And Technology*, 8(3), 279-287.

TAŞCI, D. (2018). Gaussian padovan and gaussian pell-padovan sequences, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 67(2), 82-88.

VOET, C. (2012). The poetics of order: Dom hans van der laan's architectonic space, *Architectural Research Quarterly*, 16(2), 137.