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Research Article

Perfect fluid spacetimes, Gray's decomposition and $f(\mathcal{R}, T)$-gravity

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Abstract

In this paper, first we give the complete classifications of perfect fluid spacetimes under the Gray's decomposition. Then we investigate the condition under which the Ricci tensor is a conformal Killing tensor in a perfect fluid spacetime. Later, we study perfect fluid spacetimes in $f(\mathcal{R}, T)$-gravity theory. We find some relations between isotropic pressure and energy density of the Ricci semisymmetric perfect fluid spacetimes satisfying $f(\mathcal{R}, T)$-gravity equation to represent dark matter era.

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1. Introduction

In Einstein's general relativity theory, perfect fluids are of great importance for being special solutions of the Einstein's field equations having vanishing shear stresses, viscosity and heat conduction compatible with Bianchi identities. In cosmology, they represent the behaviour of the Hubble flow ranging from inflation to dark energy periods. Perfect fluids model the matter content of the interior of a star or an isotropic universe. For these reasons, the geometric studies of perfect fluids on modified gravity theories are very important areas of study that need to be explored. There are many different ways to modify the general relativity; Einstein-Hilbert gravitational action in some manner or another, specifically, $f(\mathcal{R})$-gravity [1, 7, 34, 40] which is the most straightforward generalization of general relativity, Gauss-Bonnet gravity [9], $f(G)$-gravity [28], $f(T)$ theory [5], $f(Q, T)$-gravity [44], the Weyl-type $f(Q, T)$-gravity [45] and $f(\mathcal{R}, T)$-gravity [10, 19, 32, 37, 43], the last is what we will discuss in this article.

An n-dimensional Lorentzian manifold (M^n, g) satisfying the Ricci condition

$$\mathrm{Ric} = ag + bB \otimes B, \tag{1.1}$$

where a, b are some scalar fields, U is a vector field such that $B(X) = g(X, U)$, for all $X \in \chi(M)$ and $g(U, U) = -1$ is said to be a perfect fluid spacetime. The timelike vector field U is defined as the velocity vector field of the perfect fluid spacetime. In [7], the compatibility of perfect fluid solutions on $f(\mathcal{R})$-gravity is investigated. In this paper,

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we consider perfect fluid spacetimes on the modified $f(\mathcal{R},T)$-gravity of general relativity, where a generic function of the scalar curvature and the energy-momentum tensor is considered in the Hilbert-Einstein action of gravitational field. Many studies have been done on perfect fluids in recent years by De and Suh [12], Mallick and De [22], Özen Zengin [30], Güler and Altay Demirbağ [17, 18] with many different points of view.

On the other hand, the equation of state of the perfect fluids in an isotropic universe describes the Robertson Walker spacetimes. That is, the Robertson Walker metric satisfies the Einstein's field equations and describes a homogeneous, isotropic, expanding or contracting universe. Since the scaling factor of the universe is derived as a function of time, the metric of the Robertson Walker spacetime can be expressed as a certain warped product metric. Alias et al [2] has introduced the notion of generalized Robertson Walker spacetime as a special kind of warped product as follows: A Lorentzian manifold (M^n, g) is called a generalized Robertson Walker spacetime if its metric can be expressed of the form

$$g = -\mathrm{d}t^2 \oplus f(t)^2 g^*, \tag{1.2}$$

where g^* is the metric of the $(n-1)$-dimensional Riemannian manifold M^* and $f : I \to (0, \infty)$ is the smooth function. If g^* is the metric of 3-dimensional Riemannian manifold of constant curvature, the (M^4, g) is called a Robertson Walker spacetime. From this point, generalized Robertson Walker spacetime is a generalization of the Robertson Walker spacetime. Also, these kind of spacetimes are widely used for the study of inhomogeneous spacetimes having isotropic radiation, [3, 11, 13, 23–25].

Besides, every Robertson Walker spacetime is a perfect fluid spacetime but the converse statement is not generally true. Also, a 4-dimensional generalized Robertson Walker spacetime is a perfect fluid spacetime if and only if it is a Robertson Walker spacetime. In $f(\mathcal{R})$-gravity, every conformally flat generalized Robertson Walker spacetime is a perfect fluid [7]. Moreover, if the divergence of the Weyl tensor of both Robertson Walker and generalized Robertson Walker spacetimes of dimensions greater than three vanishes, then all higher order gravitational modifications of the Hilbert-Einstein Lagrangian density have the perfect fluid distributions [8]. Thus the curvature properties of spacetimes containing the dark energy fluids can be observed, which is another area of study for this article.

Perfect fluid spacetimes are described by the energy momentum tensor

$$T(X,Y) = (\sigma + p)B(X)B(Y) + pg(X,Y), \tag{1.3}$$

where $g(X,U) = B(X)$, $B(U) = -1$, for all vector fields X, Y with σ the energy density and p being the isotropic pressure. The unit timelike vector field U denotes the velocity field [29]. The energy momentum tensor T points out the physical properties of spacetime while Ricci tensor controls the geometry of spacetime. In general relativity, they are associated to each other by the Einstein's field equations [29]

$$\mathrm{Ric}(X,Y) - \frac{\mathcal{R}}{2}g(X,Y) = \kappa T(X,Y) \tag{1.4}$$

where Ric is the $(0,2)$-type Ricci tensor, $\mathcal{R}$ is the scalar curvature and κ is the gravitational constant. Also, from (1.4), the energy momentum tensor is of divergence-free and a symmetric tensor [29].

In static universe, the gravitational constant can be acquired by introducing a cosmological constant depending on the principles of Einstein's theory and it gives the model of dark energy and the acceleration of the universe expansion. Moreover, σ and p are bounded under the equation $p = p(\sigma, T_0)$, where the absolute temperature is indicated by T_0 in the perfect fluid. If $p = p(\sigma)$, the perfect fluid is said to be isentropic, [20] and if $p = \sigma$, the perfect fluid is said to be a stiff matter, [41]. Perfect fluids with equation of state of the form $p = \omega\sigma$ for some constant ω attracts great attention because of the symmetry properties of the transformations between the perfect fluids [14].

Furthermore, the dark energy or dark matter theory has become active field of study in the last century as it causes the acceleration of universe expansion. However, since this theory still holds many secrets, modified gravity theories try to explain the acceleration of the universe without using the concept of dark energy [32, 43].

Gray [15] has proposed the $\mathcal{O}(n)$-invariant orthogonal irreducible decomposition of the space of all tensors of type-$(0,3)$ satisfying only the identities of the gradient of the Ricci tensor $\nabla\mathrm{Ric}$. This decomposition introduced the seven classes of Einstein-like manifolds whose Ricci tensors satisfy the defining condition of each subspace. In [27] Mantica et al obtained the form of the Ricci tensor in all the $\mathcal{O}(n)$-invariant subspaces in the generalized Robertson Walker spacetimes. According to this study, in all cases except one, the spacetime reduces to an Einstein or has the matter content of perfect fluid.

Therefore, the present paper is organized as follows: In Section 2 we examine all seven cases of the Gray's decomposition and give more general classification for perfect fluid spacetimes in each case. In Section 3, we investigate the condition under which the Ricci tensor is a conformal Killing tensor in a perfect fluid spacetime. It is proved that the Ricci tensor is conformal Killing in a perfect fluid spacetime provided that the velocity vector field is Killing and one of the associated scalar is constant. In Section 4, we study perfect fluid spacetimes in $f(\mathcal{R},T)$-gravity theory. Because according to the geometrical backgrounds of the Robertson Walker spacetimes, the matter content of the universe consists of perfect fluids. We find some relations between isotropic pressure and energy density of the Ricci semisymmetric perfect fluid spacetimes satisfying $f(\mathcal{R},T)$-gravity equation to represent dark matter era. We also observed that in $f(\mathcal{R})$-gravity a Ricci semisymmetric perfect fluid spacetime represents either dark matter era or phantom energy.

2. Gray's decompositions

In [15], Gray proposed that $\nabla\mathrm{Ric}$ can be decomposed into $\mathcal{O}(n)$-invariant terms, (for more, we refer [4, 21]). This covariant derivative can be expressed as follows [27]:

$$(\nabla_Z\mathrm{Ric})(V,W)=\tilde{R}(Z,V)W+\alpha(Z)g(V,W)+\beta(V)g(Z,W)+\beta(W)g(V,Z), \tag{2.1}$$

for all vector fields Z,V,W, where

$$\alpha(Z)=\frac{n}{(n-1)(n+2)}\nabla_Z\mathcal{R},\quad \beta(Z)=\frac{n-2}{2(n-1)(n+2)}\nabla_Z\mathcal{R}, \tag{2.2}$$

with $\tilde{R}(Z,V)W=\tilde{R}(Z,W)V$ being the trace-less tensor that can be written as a sum of its orthogonal components

$$\begin{aligned}\tilde{R}(Z,V)W=&\frac{1}{3}[\tilde{R}(Z,V)W+\tilde{R}(V,W)Z+\tilde{R}(W,Z)V]\\&+\frac{1}{3}[\tilde{R}(Z,V)W-\tilde{R}(V,Z)W]+\frac{1}{3}[\tilde{R}(Z,V)W-\tilde{R}(W,Z)V].\end{aligned} \tag{2.3}$$

The decompositions (2.1) and (2.3) provide $\mathcal{O}(n)$-invariant subspace, characterized by invariant equations that are linear in $(\nabla_Z\mathrm{Ric})(V,W)$.

Therefore, the relation between $\nabla\mathrm{Ric}$ and the divergence of the Weyl conformal curvature tensor C can be given by the equation [27]

$$(\mathrm{div}\ \mathrm{C})(\mathrm{Z},\mathrm{V})\mathrm{W}=\frac{\mathrm{n}-3}{\mathrm{n}-2}[\tilde{\mathrm{R}}(\mathrm{Z},\mathrm{V})\mathrm{W}-\tilde{\mathrm{R}}(\mathrm{W},\mathrm{Z})\mathrm{V}]. \tag{2.4}$$

In Gray's decomposition we have the following subspaces:

(i) The *trivial subspace* is characterized by $\nabla\mathrm{Ric}=0$.

(ii) The *subspace* $\mathcal{I}$ is characterized by $\tilde{R}(Z,V)W=0$, i.e.,

$$(\nabla_Z\mathrm{Ric})(V,W)=\alpha(Z)g(V,W)+\omega(V)g(Z,W)+\omega(W)g(V,Z). \tag{2.5}$$

Such manifolds endowed the condition (2.5) are called *Sinyukov manifolds* [38, 39].

(iii) The *orthogonal complements* $\mathcal{I}'$ (also named as the subspace $\mathcal{A}$) is characterized by

$$(\nabla_Z \mathrm{Ric})(V,W) + (\nabla_V \mathrm{Ric})(Z,W) + (\nabla_W \mathrm{Ric})(Z,V) = 0, \tag{2.6}$$

which also yields that the scalar curvature $\mathcal{R}$ is constant.

(iv) In *the subspaces* $\mathcal{B}$ *and* $\mathcal{B}'$ the Ricci tensor is of Codazzi-type i.e.,

$$(\nabla_Z \mathrm{Ric})(V,W) = (\nabla_V \mathrm{Ric})(Z,W). \tag{2.7}$$

(v) In *the subspace* $\mathcal{I} \oplus \mathcal{A}$, the Ricci tensor satisfies the following cyclic condition

$$\begin{aligned}&(\nabla_Z \mathrm{Ric})(V,W) + (\nabla_V \mathrm{Ric})(Z,W) + (\nabla_W \mathrm{Ric})(Z,V)\\ =&2\frac{d\mathcal{R}(Z)}{n+2}g(V,W) + 2\frac{d\mathcal{R}(V)}{n+2}g(Z,W) + 2\frac{d\mathcal{R}(W)}{n+2}g(Z,V).\end{aligned} \tag{2.8}$$

(vi) In *the subspace* $\mathcal{I} \oplus \mathcal{B}$, the Ricci tensor satisfies the following Codazzi condition

$$\nabla_Z\Big[\mathrm{Ric}(V,W) - \frac{\mathcal{R}}{2(n-1)}g(V,W)\Big] = \nabla_V\Big[\mathrm{Ric}(Z,W) - \frac{\mathcal{R}}{2(n-1)}g(Z,W)\Big]. \tag{2.9}$$

which implies that the Weyl conformal curvature tensor C is of divergence-free.

(vii) In *the subspace* $\mathcal{A} \oplus \mathcal{B}$, the scalar curvature is covariant constant.

For 4-dimensional perfect fluid spacetime (M,g), contracting (1.1) over X and Y, we get

$$\mathcal{R} = 4a - b. \tag{2.10}$$

Hence $\mathcal{R}$ =constant implies that

$$4da(Z) = db(Z), \quad \forall\, Z \in \chi(M). \tag{2.11}$$

Taking the covariant derivative of (1.1), we get

$$\begin{aligned}(\nabla_Z \mathrm{Ric})(X,Y) =& da(Z)g(X,Y) + db(Z)B(X)B(Y)\\ &+ b[(\nabla_Z B)(X)B(Y) + B(X)(\nabla_Z B)(Y)],\end{aligned} \tag{2.12}$$

for all $X, Y, Z \in \chi(M)$. Since $g(U,U) = -1$, we also have $(\nabla_Z B)(U) = 0$, for all $Z \in \chi(M)$.

Now we will examine each of these seven cases:

Case (i): The condition $\nabla \mathrm{Ric} = 0$ implies the scalar curvature $\mathcal{R}$ is constant. (2.12) gives

$$\begin{aligned}da(Z)g(X,Y) + db(Z)B(X)B(Y) + b[(\nabla_Z B)(X)B(Y)&\\ + B(X)(\nabla_Z B)(Y)] = 0,&\end{aligned} \tag{2.13}$$

for all $X, Y, Z \in \chi(M)$. Putting $X = Y = U$ in (2.13) infers that

$$-da(Z) + db(Z) = 0, \quad \forall\, Z \in \chi(M). \tag{2.14}$$

(2.11) and (2.14) together entail that $da(Z) = 0$, for all $Z \in \chi(M)$ which implies a is constant and therefore (2.11) gives b is constant.

Combining (1.1) and (1.3) with (1.4), we get

$$a = \frac{\kappa(\sigma - p)}{2}, \quad \text{and} \quad b = \kappa(p + \sigma). \tag{2.15}$$

Hence $p - \sigma$ =constant and $p + \sigma$ =constant. Thus, both σ and p are constants.

It is known (p. 339 of [29]) that the energy and force equations for a perfect fluid are as follows:

$$U\sigma = g(grad\ \sigma, U) = -(\sigma + p)\mathrm{div}\ U \tag{2.16}$$

and

$$(\sigma+p)(\nabla_U U) = -grad_{\perp}p = -grad\ p - g(grad\ p, U)U \qquad (2.17)$$
$$= -grad\ p - (Up)U$$

where the spatial pressure gradient $grad_{\perp}p$ is the component of $gradp$ orthogonal to U.

Since σ is constant, it follows from (2.16) that either $\sigma+p=0$ or $\mathrm{div}U=0$. Again, since p is constant from (2.17) we get either $\sigma+p=0$ or $\nabla_U U=0$. (Since the scalar field b is non-zero, we may take $\sigma+p\neq 0$.) In this case, $\mathrm{div}U=0$ and $\nabla_U U=0$. But $\mathrm{div}U$ represents the expansion scalar and $\nabla_U U$ represents the acceleration vector. Thus, we have the following:

Theorem 2.1. *If a perfect fluid spacetime belongs to the trivial subspace, then the expansion scalar and the acceleration vector of the fluid vanish.*

Remark 2.2. Since the velocity vector U is of divergence-free, it is irrotational and therefore the vorticity the fluid vanishes.

Case (ii): In the subspace $\mathcal{I}$, the Ricci tensor satisfies the condition $\tilde{R}(Z,V)W=0$, for all $Z,V,W\in\chi(M)$. Since the gradient of the Ricci tensor and the divergence of the Weyl conformal tensor are connected by the relation (2.4), div C $=0$. Moreover, it is known from [36] that a 4-dimensional perfect fluid spacetime with divergence-free Weyl conformal tensor with an equation of state $p=p(\mu)$ is conformally flat, and it is endowed with the Robertson Walker metic, the flow is irrotational, geodesic and has no shear.

From the above discussion we conclude the following:

Theorem 2.3. *If a perfect fluid spacetime satisfying the equation of state $p=p(\mu)$ belongs to the class $\mathcal{I}$, then it is conformally flat, and it becomes Robertson Walker spacetime; the flow is irrotational, geodesic and has no shear.*

Case (iii): Assume that the Ricci tensor belongs to the subspace $\mathcal{A}$. Then, the condition (2.6) holds and also the scalar curvature $\mathcal{R}$ is constant. By taking the covariant derivative of the Eistein's field equtation (1.4) we get

$$(\nabla_Z \mathrm{Ric})(V,W) - \frac{d\mathcal{R}(Z)}{2}g(V,W) = \kappa(\nabla_Z T)(V,W), \qquad (2.18)$$

for all $Z,V,W\in\chi(M)$. Using $\mathcal{R}$ =constant in (2.18), we obtain

$$(\nabla_Z T)(V,W) + (\nabla_V T)(Z,W) + (\nabla_W T)(Z,V) = 0, \qquad (2.19)$$

for all $Z,V,W\in\chi(M)$. Hence the energy-momentum tensor is Killing.

Sharma and Ghosh [35] prove that:

Theorem A 1. Let (M,g) be a perfect fluid spacetime such that its energy-momentum tensor is Killig. Then

(i) M is expansion-free and shear-free and its flow is geodesic, however not necessarily vorticity-free,
(ii) its energy density and pressure are constant on M.

Thus, we can state that:

Theorem 2.4. *Let the perfect fluid spacetime belongs to the class $\mathcal{A}$. Then*

(i) *M is expansion-free and shear-free and its flow is geodesic, however not necessarily vorticity-free,*
(ii) *its energy density and pressure are constant on M.*

Case (iv): If a perfect fluid spacetime belongs to $\mathcal{B}$ and $\mathcal{B}'$, then the Ricci tensor is of Codazzi-type and so the scalar curvature $\mathcal{R}$ is constant.

On a 4-dimensional Lorentzian manifold (M, g), if the Yang's equation

$$(\nabla_Z \mathrm{Ric})(V, W) = (\nabla_V \mathrm{Ric})(Z, W) \tag{2.20}$$

holds, then it is called "Yang's pure space". In [16], Guilfoyle and Nolan proved the following:

Theorem A 2. A 4-dimensional perfect fluid spacetime (M, g) with $\sigma + p \neq 0$ is a Yang pure space if and only if (M, g) is a Robertson Walker spacetime.

Therefore we obtain:

Theorem 2.5. *If a perfect fluid spacetime belongs to classes $\mathcal{B}$ and $\mathcal{B}'$, then it becomes a Robertson Walker spacetime.*

Case (v): In this case, we have the relation (2.8), from which we infer the scalar curvature $\mathcal{R}$ is constant. From (2.11), we get $4a - b =$constant and so $p = \frac{\sigma}{3}$+constant. Hence we can state that:

Theorem 2.6. *If a perfect fluid spacetime belongs to the subspace $\mathcal{I} \oplus \mathcal{A}$, then the spacetime satisfies the state eqution $p = \frac{\sigma}{3}$+constant.*

Remark 2.7. If the constant given in Theorem 2.6 is equal to zero, then the perfect fluid spacetime represents radiation era. It may be mentioned that in a perfect fluid Yang pure space, the equation of state is given by $p = \frac{\sigma}{3}$+constant [16].

Case (vi): In this case div C $= 0$. Now, we can apply Theorem A 1 and state the same result as in Theorem 2.3.

Case (vii): In the subspace $\mathcal{A} \oplus \mathcal{B}$, the scalar curvature is covariant constant. Hence from (2.10) and (2.11), again we have $p = \frac{\sigma}{3}$+constant. Therefore, in this case also we get Theorem 2.6.

3. Conformal Killing tensor in a perfect fluid spacetime

In [26], the authors characterize a perfect fluid spacetime whose Ricci tensor is conformal Killing tensor and prove that the perfect fluid spacetime is a generalized Robertson Walker spacetime subject to the condition the velocity vector field is irrotational. Also they obtain a state equation $p = p(\mu)$, $\sigma + p \neq 0$. In this section, we explore the condition under which the Ricci tensor is a conformal Killing tensor in a perfect fluid spacetime (M, g).

Before going to the main result we define conformal Killing tensor.

Definition 3.1. [33] A symmetric tensor $A(X, Y)$ is named conformal Killing tensor if it satisfies the condition

$$\begin{aligned}(\nabla_X A)(Y, Z) + (\nabla_Y A)(Z, X) + (\nabla_Z A)(X, Y)&\\ = d\alpha(X) g(Y, Z) + d\alpha(Y) g(Z, X) + d\alpha(Z) g(X, Y)&,\end{aligned} \tag{3.1}$$

being α, a non-vanishing 1-form.

Conformal Killing tensors define first integrals for null geodesics. In [42], Walker and Penrose proved that a conformal Killing tensor is the first integral of the null geodesic equations of every type of $\{2,2\}$ vacuum solutions of Einstein's field equations in dimension 4. Conformal Killing tensors have also applications in Einstein-Weyl geometry.

Suppose the velocity vector field U of the perfect fluid spacetime is Killing, which gives

$$(\nabla_X B)(Y) + (\nabla_Y B)(X) = 0, \quad \forall\ X, Y \in \chi(M). \tag{3.2}$$

Using (2.12), we obtain

$$\begin{aligned}(\nabla_X \mathrm{Ric})(Y,Z) &+ (\nabla_Y \mathrm{Ric})(Z,X) + (\nabla_Z \mathrm{Ric})(X,Y) \\ =& da(X)g(Y,Z) + da(Y)g(Z,X) + da(Z)g(X,Y) \\ &+ db(X)B(Y)B(Z) + db(Y)B(X)B(Z) + db(Z)B(X)B(Y) \\ =& da(X)g(Y,Z) + da(Y)g(Z,X) + da(Z)g(X,Y),\end{aligned} \tag{3.3}$$

provided that $b = \kappa(\sigma + p)$ =constant, which entails that the Ricci tensor is conformal Killing.

Theorem 3.2. *The Ricci tensor is conformal Killing in a perfect fluid spacetime provided that the velocity vector field is Killing and the scalar b is constant.*

Now, we present the following:

Lemma 3.3. *In a spacetime satisfying the Einstein's field equations, if the Ricci tensor is Killing, then the energy-momentum tensor is Killing.*

Proof. Let us suppose that the Ricci tensor is Killing. Then

$$(\nabla_X \mathrm{Ric})(Y,Z) + (\nabla_Y \mathrm{Ric})(Z,X) + (\nabla_Z \mathrm{Ric})(X,Y) = 0, \tag{3.4}$$

for all $X, Y, Z \in \chi(M)$ which implies $\mathcal{R}$ is constant. Hence from the Einstein's field equations, we get

$$(\nabla_X T)(Y,Z) + (\nabla_Y T)(Z,X) + (\nabla_Z T)(X,Y) = 0, \tag{3.5}$$

for all X, Y, Z which means that the energy-momentum tensor is Killing. □

The converse is also true. If we suppose that the energy-momentum tensor is Killing, then Einstein's field equations infer that

$$\begin{aligned}(\nabla_X \mathrm{Ric})(Y,Z) &+ (\nabla_Y \mathrm{Ric})(Z,X) + (\nabla_Z \mathrm{Ric})(X,Y) \\ =& \frac{1}{2}\Big[dr(X)g(Y,Z) + dr(Y)g(Z,X) + dr(Z)g(X,Y)\Big]\end{aligned} \tag{3.6}$$

for all $X, Y, Z \in \chi(M)$, which implies after contraction $\mathcal{R}$ =constant. Hence the above equation reduces to

$$(\nabla_X \mathrm{Ric})(Y,Z) + (\nabla_Y \mathrm{Ric})(Z,X) + (\nabla_Z \mathrm{Ric})(X,Y) = 0. \tag{3.7}$$

Moreover, if the energy-momentum tensor of a perfect fluid spacetime is Killig, then it has vanishing expansion scalar and vanishing shear tensor but not necessarily vanishing vorticity; also its flow is geodesic, it has constant energy density and pressure. Thus, from Theorem A1 and the above Lemma, we conclude the following:

Proposition 3.4. *In a perfect fluid spacetime (M, g) if the Ricci tensor is Killing, then*

(i) *M has vanishing expansion scalar and shear tensor and it has geodesic flow, but its vorticity does not necessarily vanish,*

(ii) *the energy density and the pressure of M are constant.*

4. Perfect fluids satisfying $f(\mathcal{R},T)$-gravity

In this section we study Ricci semisymmetric perfect fluid spacetimes satisfying $f(\mathcal{R},T)$-gravity, represents the physical character of the matter distribution to get a theoretical model. We choose

$$f(\mathcal{R},T) = \mathcal{R} + 2f(T), \tag{4.1}$$

where $f(T)$ is a function on the trace T of the energy-momentum tensor and the term $2f(T)$ modifies the gravitational mutual effects between matter distribution and geometry of the spacetime.

In [19], Harko et al proposed the theory of $f(\mathcal{R}, T)$-gravity which is the alteration of the theory of general relativity. In this deviated theory, the gravitational Lagrangian consists of an arbitrary function of the scalar curvature $\mathcal{R}$ and the trace T of the energy-momentum tensor. In their approach, the stress-energy tensor is not considered conservative. In [37], Singh et al have reconstructed a model of $f(\mathcal{R}, T)$-gravity in which the scale factor evolve exponentially. Later, in Chakraborty's approach [10], the arbitrary function $f(\mathcal{R}, T)$ is obtained under the conservativeness of the energy momentum tensor, although the form of the field equations remain similar.

The modified Einstein-Hilbert action term is defined by

$$\mathcal{H} = \frac{1}{16\pi}\int [f(\mathcal{R}, T) + \mathcal{L}_m]\sqrt{(-g)}\mathrm{d}^4x, \tag{4.2}$$

where $f(\mathcal{R}, T)$ is an arbitrary function of the scalar curvature $\mathcal{R}$ and the trace T of the energy-momentum tensor and $\mathcal{L}_m$ is the matter Lagrangian of the scalar field. The matter distribution is described by the stress energy momentum tensor

$$T_{ab} = \frac{-2\delta(\sqrt{-g})\mathcal{L}_m}{\sqrt{-g}\delta^{ab}}. \tag{4.3}$$

Let us consider that the matter Lagrangian of the scalar field depends only on the metric tensor g, and not on its derivatives.

By the variation of action (4.2) with respect to the metric tensor g, the field equations of $f(\mathcal{R}, T)$-gravity is given by

$$\begin{aligned} f_{\mathcal{R}}(\mathcal{R}, T)\mathrm{Ric}(X, Y) - &\frac{1}{2}f(\mathcal{R}, T)g(X, Y) + (g(X, Y)\nabla_{e_i}\nabla^{e_i} - \nabla_X\nabla_Y)f_{\mathcal{R}}(\mathcal{R}, T) \\ =&8\pi T(X, Y) - f_T(\mathcal{R}, T)T(X, Y) - f_T(\mathcal{R}, T)\psi(X, Y), \end{aligned} \tag{4.4}$$

where $f_{\mathcal{R}}$ and f_T denote the partial derivatives of $f(\mathcal{R}, T)$ with respect to $\mathcal{R}$ and T, respectively. Here, $\square \equiv \nabla_{e_i}\nabla^{e_i}$ denotes the d'Alembert operator for the orthonormal frame field $\{e_i\}_i$ and ψ is defined by

$$\psi(X, Y) = -2T(X, Y) + g(X, Y)\mathcal{L}_m - 2g^{lk}\frac{\partial^2\mathcal{L}_m}{\partial g^{ab}\partial g^{lk}}. \tag{4.5}$$

If $f(\mathcal{R}, T) = f(\mathcal{R})$, then the equations (4.2) and (4.3) give the field equations of $f(\mathcal{R})$-gravity.

It is known that the value of Lagrangian does not uniquely defined, so we may take $\mathcal{L}_m = -p$ and using (1.3), we get

$$T(X, Y) = -pg(X, Y) + (\sigma + p)\eta(X)\eta(Y), \tag{4.6}$$

for all X, Y, where $\eta(X) = g(X, U)$, $g(U, U) = 1$ and so $(\nabla_X\eta)(U) = 0$. By virtue of (4.6), the variation of stress energy momentum tensor is given by

$$\psi(X, Y) = -pg(X, Y) - 2T(X, Y). \tag{4.7}$$

By using (4.1) and (4.4), we get

$$\begin{aligned} \mathrm{Ric}(X, Y) =&\frac{1}{2}g(X, Y) - 2f'(T)T(X, Y) - 2f'(T)\psi(X, Y) \\ &+f(T)g(X, Y) + 8\pi T(X, Y) \end{aligned} \tag{4.8}$$

for all X, Y.

To derive the field equations Harko et al [19] have not considered conservation of the energy-momentum tensor. However Chakraborty assumed the conservation of the energy momentum tensor in his paper [10]. In this section, we consider perfect fluid spacetime solution of $f(\mathcal{R},T)$-gravity equation assuming the conservation of the energy momentum tensor. Then we have

$$\mathrm{Ric}(X,Y)=\Big[\frac{1}{2}\mathcal{R}+f(T)-8p\pi\Big]g(X,Y)+[(\sigma+p)(8\pi+2f'(T))]\eta(X)\eta(Y). \tag{4.9}$$

The above equation implies that in $f(\mathcal{R},T)$-gravity theory the Ricci tensor of the perfect fluid spacetime is of the above form. We present (4.9) as

$$\mathrm{Ric}(X,Y)=\alpha g(X,Y)+\beta\eta(X)\eta(Y), \tag{4.10}$$

where $\alpha=\frac{1}{2}\mathcal{R}+f(T)-8p\pi$ and $\beta=(\sigma+p)(8\pi+2f'(T))$. The contraction of (4.10) gives

$$\mathcal{R}=4\Big[\frac{1}{2}\mathcal{R}+f(T)-8p\pi\Big]+(\sigma+p)(8\pi+2f'(T)), \tag{4.11}$$

which yields

$$\alpha=-[f(T)-8p\pi]-(\sigma+p)[4\pi+f'(T)]. \tag{4.12}$$

Now, we assume that the perfect fluid spacetime in $f(\mathcal{R},T)$-gravity is Ricci semisymmetric (i.e. $R\cdot\mathrm{Ric}=0$). From (4.10), we infer

$$\begin{aligned}&[WZ(\alpha)-ZW(\alpha)]g(X,Y)+[WZ(\beta)-ZW(\beta)]\eta(X)\eta(Y) \\ &\beta[\{(\nabla_W\nabla_Z\eta)(X)-(\nabla_Z\nabla_W\eta)(X)\}\eta(Y)+\{(\nabla_W\nabla_Z\eta)(Y) \\ &-(\nabla_Z\nabla_W\eta)(Y)\}\eta(X)]=0,\quad\forall\ \ X,Y,Z,W\in\chi(M).\end{aligned} \tag{4.13}$$

Since α and β are scalars, the above equation entails that

$$\beta[R(U,X,Z,W)\eta(Y)+R(U,Y,Z,W)\eta(X)]=0,\quad\forall\ \ X,Y,Z,W\in\chi(M), \tag{4.14}$$

which implies either $\beta=0$ or $R(U,X,Z,W)\eta(Y)+R(U,Y,Z,W)\eta(X)=0$.

Case (i) If $\beta=0$, then either $p+\sigma=0$ or $8\pi+2f'(T)=0$. Thus $p+\sigma=0$ which represents dark matter area, provided $8\pi+2f'(T)\neq0$.

Case (ii) If $R(U,X,Z,W)\eta(Y)+R(U,Y,Z,W)\eta(X)=0$, for all $X,Y,Z,W\in\chi(M)$, then $R(U,Y,Z,W)=0$ since $\eta(U)=1$. Contracting the last equation over Y,Z, we get $\mathrm{Ric}(U,W)=0$, for all $W\in\chi(M)$. Hence (4.10) gives $(\alpha-\beta)\eta(W)=0$ so $\alpha=\beta$. Moreover, $\alpha=\beta$ implies

$$-[f(T)-8p\pi]=3(\sigma+p)[4\pi+f'(T)]. \tag{4.15}$$

Hence we have $\sigma+p=\frac{8p\pi-f(T)}{3(4\pi+f'(T))}$, provided $8\pi+2f'(T)\neq0$.

Therefore, we can state:

Theorem 4.1. *In $f(\mathcal{R},T)$-gravity theory satisfying* (4.1)*, a Ricci semisymmetric perfect fluid spacetime represents either dark matter era or $\sigma+p=\frac{8p\pi-f(T)}{3(4\pi+f'(T))}$, provided $8\pi+2f'(T)\neq0$.*

Remark 4.2. If $8\pi+2f'(T)=0$, then we provide $f(T)=-4\pi T+c$, for some constant c.

Remark 4.3. If $f(T)=0$, then $f(\mathcal{R},T)$-gravity becomes $f(\mathcal{R})$-gravity. Then by Theorem 4.1, in $f(\mathcal{R})$-gravity a Ricci semisymmetric perfect fluid spacetime represents either dark matter era or $3\sigma+p=0$. The equation of state is defined as the ratio of isotropic pressure to energy density, $\omega=\frac{p}{\sigma}=-3<-1$. Hence it represents the phantom energy, [6, 14]. It definitely has some interesting features. For instance, the energy density of phantom energy increases with time. It also violates the dominant energy condition.

A Lorentzian manifold is said to be Ricci recurrent [31] if the Ricci tensor Ric satisfies the condition

$$(\nabla_X\mathrm{Ric})(Y,Z)=A(X)\mathrm{Ric}(Y,Z), \tag{4.16}$$

where A is a non-zero 1-form. If $A = 0$, then the manifold reduces to a Ricci symmetric manifold. It is known that a Ricci recurrent manifold is Ricci semisymmetric.

Hence we obtain the following:

Corollary 4.4. *In $f(\mathcal{R}, T)$-gravity theory satisfying* (4.1)*, a Ricci recurrent perfect fluid spacetime represents dark matter era or* $\sigma + p = \frac{8p\pi - f(T)}{3(4\pi + f'(T))}$.

Now, we suppose that the Ricci tensor is of Codazzi type. Then we have

$$(\nabla_X \mathrm{Ric})(Y, Z) = (\nabla_Y \mathrm{Ric})(X, Z) \tag{4.17}$$

for all $X, Y, Z \in \chi(M)$. With the help of (4.10), the above equation gives

$$\begin{aligned}(X\alpha)g(Y,Z) + (X\beta)\eta(Y)\eta(Z) + \beta[(\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y)] \\ = (Y\alpha)g(X,Z) + (Y\beta)\eta(X)\eta(Z) + \beta[(\nabla_Y\eta)(X)\eta(Z) + (\nabla_Y\eta)(Z)\eta(X)],\end{aligned} \tag{4.18}$$

where $\eta(X) = g(U, X)$, for all X and $g(U, U) = 1$. Putting $Y = Z = U$ in (4.18), we get

$$(X\alpha) + (X\beta) = [(U\alpha) + (U\beta)]\eta(X) + \beta(\nabla_U\eta)(X). \tag{4.19}$$

(4.17) implies the scalar curvature $\mathcal{R}$ is constant. Therefore, from (4.10) we get $(X\mathcal{R}) = 4(X\alpha) + (X\beta)$ and so

$$4(X\alpha) = -(X\beta). \tag{4.20}$$

Let us suppose that the perfect fluid spacetime represents dark matter era, that is, $p+\sigma = 0$ which means that $\beta = 0$ and hence we get $\alpha =$constant. Hence we get from (4.12),

$$f(T) = 8\pi p + \lambda, \quad \lambda = \text{ constant} \tag{4.21}$$

so $p = \frac{f(T)}{8\pi} +$ constant. Thus we can state that:

Theorem 4.5. *In $f(\mathcal{R}, T)$-gravity theory satisfying* (4.1)*, if a perfect fluid spacetime whose Ricci tensor is of Codazzi type has dark matter era, then its pressure is* $p = \frac{f(T)}{8\pi} +$ *constant.*

If $f(T) = 0$, by (4.21) we get $p = 0$ and hence from $\sigma + p = 0$ we refer $\sigma = 0$. Thus we get:

Corollary 4.6. *In $f(\mathcal{R})$-gravity theory satisfying $f(\mathcal{R}, T) = \mathcal{R}$, a dark matter perfect fluid spacetime having Codazzi type of Ricci tensor is vacuum.*

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