MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

https://doi.org/10.36753/msaen.931071
10 (3) 114-124 (2022) - Research Article
ISSN: 2147-6268

Fibonacci Lacunary Ideal Convergence of Double Sequences in Intuitionistic Fuzzy Normed Linear Spaces

Ömer Kişi

Abstract
The purpose of this article is to research the concept of Fibonacci lacunary ideal convergence of double sequences in intuitionistic fuzzy normed linear spaces (IFNS). Additionally, a new concept, called Fibonacci lacunary convergence, is examined. Fibonacci lacunary $\mathcal{I}_2$-limit points and Fibonacci lacunary $\mathcal{I}_2$-cluster points for double sequences in IFNS have been defined and the significant results have been given. Additionally, Fibonacci lacunary Cauchy and Fibonacci lacunary $\mathcal{I}_2$-Cauchy double sequences in IFNS are worked.

Keywords: *Fibonacci sequence; intuitionistic fuzzy normed linear space; limit point; cluster point.*

AMS Subject Classification (2020): *Primary: 40A30; Secondary: 40G15.*

1. Introduction and Background

Statistical convergence of single and double real sequences was firstly studied by Fast [5] and Mursaaleen and Edely [17], respectively.

$\mathcal{I}$-convergence idea was firstly considered by Kostyrko et al. [16]. Tripathy et al. [29] gave the concept of ideal convergence of double sequences in a metric space and examined fundamental features.

Using lacunary sequence, Fridy and Orhan [6] examined lacunary statistical convergence. Lacunary statistical convergence of double sequences was worked at initial stage by Savaş and Patterson [24]. Lacunary ideal convergence of real sequences was introduced by Tripathy et al. [28]. This kind of convergence extended from single to double sequences with the study of Hazarika [8]. For different studies on these topic we refer to [3, 4, 21].

After the original study of Zadeh [30], a huge number of research works have appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy sets (FS), have been extensively applied in different disciplines and technologies. The theory of intuitionistic fuzzy sets (IFS) was presented by Atanassov [1]. The FS and IFS have been extensively to analyse many complex problems associated with different fields, particularly in decision-making. In IFS, membership degrees are described with a pair of a membership degree and

Received : *01-05-2021,* ***Accepted :*** *03-03-2022*

a nonmembership degree. Intuitionistic fuzzy metric space was investigated by Park [22]. In [23], motivated by Park's definition of an IF-metric, Lael and Nourouzi first defined an IF-normed space. Statistical convergence of single and double sequences in IFNS was defined by Karakuş et al [11] and Mursaleen et al [18], respectively. Some researches of convergence of sequences in some normed linear spaces in a fuzzy settings can be found in [2, 19, 20]. Also, similar works worked by some authors, see [26, 27].

Fibonacci gave Fibonacci sequences which was published in the book 'Liber Abaci'. This sequences were earlier stated as Virahanka numbers by Indian mathematics [7]. The sequence

$$(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...)$$

is known as Fibonacci sequence [15].

Kara and Başarır [9] prensented the first applications of Fibonacci sequence in the sequence spaces. Then, Kara [10] acquired the Fibonacci difference matrix $\widehat{F}$ via Fibonacci sequence (f_n) and described some new sequence spaces in this connection. Recently, Kirişci [12] thought the Fibonacci statistical convergence on IFNS. Kişi and Tuzcuoğlu [13] examined Fibonacci lacunary statistical convergence on IFNS. Additionally, Fibonacci ideal convergence of double sequences in IFNS was worked by Kişi and Güler [14].

Let us start with fundamental definitions from the literature.

Let $\emptyset \neq S$ be a set, and then $\emptyset \neq \mathcal{I} \subseteq P(S)$ is said to be an ideal on S iff (i) $\emptyset \in \mathcal{I}$, (ii) $\mathcal{I}$ is additive under union, (iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we get $B \in \mathcal{I}$. A non-empty family of sets $\mathcal{F}$ is called filter on S iff (i) $\emptyset \notin \mathcal{F}$, (ii) for each $A, B \in \mathcal{F}$ we get $A \cap B \in \mathcal{F}$, (iii) for every $A \in \mathcal{F}$ and each $B \supseteq A$, we obtain $B \in \mathcal{F}$. Correlation between ideal and filter is specified as follows:

$$\mathcal{F}(\mathcal{I}) = \{K \subset S : K^c \in \mathcal{I}\},$$

where $K^c = S - K$.

A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is named as strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

Throughout the paper, we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Let (X, ρ) be a metric space. A double sequence $x = (x_{mn})$ is named as $\mathcal{I}_2$-convergent to ξ, if for any $\varepsilon > 0$ we get $P(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, \xi) \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we write

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = \xi.$$

A double sequence $\overline{\theta} = \theta_{us} = \{(k_u, l_s)\}$ is named as double lacunary sequence if there are two increasing sequences of integers (k_u) and (l_s) such that

$$k_0 = 0, h_u = k_u - k_{u-1} \to \infty \text{ and } l_0 = 0, \overline{h}_s = l_s - l_{s-1} \to \infty, \quad u, s \to \infty.$$

We utilize the subsequent notations

$$k_{us} := k_u l_s, h_{us} := h_u \overline{h}_s$$

and θ_{us} is determined by

$$J_{us} := \{(k, l) : k_{u-1} < k \leq k_u \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_u := \frac{k_u}{k_{u-1}}, \overline{q}_s := \frac{l_s}{l_{s-1}} \text{ and } q_{us} := q_u \overline{q}_s.$$

Throughout the paper, by $\theta_2 = \theta_{us} = \{(k_u, l_s)\}$ we will indicate a double lacunary sequence.

Schweizer and Sklar [25] defined continuous t-norm and t-conorm. Using the continuous t-norm and t-conorm, Lael and Nourouzi [23] defined the concept of IFNS as follows:

The five-tuple $(X, \phi, \omega, *, \diamond)$ is named as IFNS if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and ϕ, ω are fuzzy sets on $X \times (0, \infty)$ fulfilling the subsequent conditions: For every $a, b \in X$ and $p, q > 0$:

(i) $\phi(a,q) + \omega(a,q) \leq 1$,

(ii) $\phi(a,q) > 0$,

(iii) $\phi(a, q) = 1$ if and only if $a = 0$,

(iv) $\phi(ca, q) = \phi\left(a, \frac{q}{|c|}\right)$ if $c \neq 0$,

(v) $\phi(a, q) * \phi(b, p) \leq \phi(a + b, q + p)$,

(vi) $\phi(a,.) : (0, \infty) \to [0, 1]$ is continuous in q;

(vii) $\lim_{q\to\infty} \phi(a, q) = 1$ and $\lim_{q\to 0} \phi(a, q) = 0$,

$(viii)$ $\omega(a,q) < 1$,
(ix) $\omega(a,q) = 0$ if and only if $a = 0$,
(x) $\omega(ca,q) = \omega\left(a, \frac{q}{|c|}\right)$ if $c \neq 0$,
(xi) $\omega(a,q) \diamond \omega(b,p) \geq \omega(a+b, q+p)$,
(xii) $\omega(a,.) : (0,\infty) \to [0,1]$ is continuous in q;
$(xiii)$ $\lim_{q\to\infty} \omega(a,q) = 0$ and $\lim_{q\to 0} \omega(a,q) = 1$.

2. Main Results

Definition 2.1. Let $(X, \phi, \omega, *, \diamond)$ be an IFNS. A double sequence $x = (x_{kl})$ in X is named as Fibonacci lacunary convergent to ξ with regards to the IFN (ϕ, ω) if, for every $t > 0$ and $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) > 1 - \varepsilon \text{ and } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) < \varepsilon$$

for all $u, s \geq r_0$. In this case, we write $(\phi,\omega)^{\theta_{us}} - \lim Fx = \xi$.

Theorem 2.1. *If $(\phi,\omega)^{\theta_{us}} - \lim Fx = \xi$, then $(\phi,\omega)^{\theta_{us}} - \lim Fx$ is unique.*

Proof. Presume that $(\phi,\omega)^{\theta_{us}} - \lim Fx = \xi_1$ and $(\phi,\omega)^{\theta_{us}} - \lim Fx = \xi_2$. Given $\varepsilon > 0$, select $\gamma \in (0,1)$ such that $(1-\gamma) * (1-\gamma) > 1 - \varepsilon$ and $\gamma \diamond \gamma < \varepsilon$. Now, for all $t > 0$, there is $r_1 \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_1, t\right) > 1 - \varepsilon \text{ and } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi_1, t\right) < \varepsilon$$

for all $u, s \geq r_1$. Also, there is $r_2 \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_2, t\right) > 1 - \varepsilon \text{ and } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi_2, t\right) < \varepsilon$$

for all $u, s \geq r_2$. Consider $r_0 = \max\{r_1, r_2\}$. Then, for $u, s \geq r_0$, we take a $(m,p) \in \mathbb{N} \times \mathbb{N}$ such that

$$\phi\left(\widehat{F}x_{mp} - \xi_1, \frac{t}{2}\right) > \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_1, \frac{t}{2}\right) > 1 - \gamma$$

and

$$\phi\left(\widehat{F}x_{mp} - \xi_2, \frac{t}{2}\right) > \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_2, \frac{t}{2}\right) > 1 - \gamma.$$

Then, we obtain

$$\begin{aligned}\phi(\xi_1 - \xi_2, t) &\geq \phi\left(\widehat{F}x_{mp} - \xi_1, \tfrac{t}{2}\right) * \phi\left(\widehat{F}x_{mp} - \xi_2, \tfrac{t}{2}\right) \\ &> (1-\gamma) * (1-\gamma) > 1 - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is abritrary, we have $\phi(\xi_1 - \xi_2, t) = 1$ for every $t > 0$, which gives that $\xi_1 = \xi_2$. □

Definition 2.2. A double sequence $x = (x_{kl})$ in IFNS is named as Fibonacci lacunary $\mathcal{I}_2$-convergent to ξ with regards to the IFN (ϕ, ω) if, for each $\varepsilon > 0$ and $t > 0$, the set

$$\left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) \leq 1 - \varepsilon \\ \text{or } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) \geq \varepsilon \end{array} \right\} \in \mathcal{I}_2.$$

ξ is named the Fibonacci $\mathcal{I}_\theta$-limit of the sequence of (x_{kl}), and we note $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi$.

Lemma 2.1. *For every $\varepsilon > 0$ and $t > 0$, the following demonstrations are equivalent.*

(a) $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi$,

(b) $\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(x_{kl} - \xi, t\right) \leq 1 - \varepsilon\right\} \in \mathcal{I}_2$ *and*

$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(x_{kl} - \xi, t\right) \geq \varepsilon\right\} \in \mathcal{I}_2$,

$$(c) \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) > 1 - \varepsilon \\ \text{and } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) < \varepsilon \end{array} \right\} \in \mathcal{F}(\mathcal{I}_2),$$

(d) $\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) > 1 - \varepsilon\right\} \in \mathcal{F}(\mathcal{I}_2)$ *and*

$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}_2)$ *and*

(e) $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim \phi\left(\widehat{F}x_{kl} - \xi, t\right) = 1$ *and* $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim \omega\left(\widehat{F}x_{kl} - \xi, t\right) = 0$.

Theorem 2.2. *If a sequence $x = (x_{kl})$ in IFNS is Fibonacci lacunary $\mathcal{I}_2$-convergent with regards to the IFN (ϕ, ω), then $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x$ is unique.*

Proof. Assume that $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi_1$ and $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi_2$. Given $\varepsilon \in (0,1)$, select $\gamma \in (0,1)$ such that $(1-\gamma) * (1-\gamma) > 1 - \varepsilon$ and $\gamma \Diamond \gamma < \varepsilon$. Then, for any $t > 0$, take the following sets:

$$K_{\phi,1}(\gamma,t) = \left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_1, \tfrac{t}{2}\right) \leq 1 - \gamma\right\},$$

$$K_{\phi,2}(\gamma,t) = \left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_2, \tfrac{t}{2}\right) \leq 1 - \gamma\right\},$$

$$K_{\omega,1}(\gamma,t) = \left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi_1, \tfrac{t}{2}\right) \geq \gamma\right\},$$

$$K_{\omega,2}(\gamma,t) = \left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi_2, \tfrac{t}{2}\right) \geq \gamma\right\}.$$

Since $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi_1$, applying Lemma 2.1, we get $K_{\phi,1}(\gamma,t) \in \mathcal{I}_2$ and $K_{\omega,1}(\gamma,t) \in \mathcal{I}_2$ for every $t > 0$. Using $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi_2$, we have $K_{\phi,2}(\gamma,t) \in \mathcal{I}_2$ and $K_{\omega,2}(\gamma,t) \in \mathcal{I}_2$ for all $t > 0$.

Now, take $K_{\phi,\omega}(\gamma,t) = (K_{\phi,1}(\gamma,t) \cup K_{\phi,2}(\gamma,t)) \cap (K_{\omega,1}(\gamma,t) \cup K_{\omega,2}(\gamma,t))$. Then, $K_{\phi,\omega}(\gamma,t) \in \mathcal{I}_2$. This gives that $K^c_{\phi,\omega}(\gamma,t) \neq \emptyset$ in $\mathcal{F}(\mathcal{I}_2)$. If $(u,s) \in K^c_{\phi,\omega}(\gamma,t)$, first, contemplate the case $(u,s) \in \left(K^c_{\phi,1}(\gamma,t) \cap K^c_{\phi,2}(\gamma,t)\right)$. Then, we get

$$\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_1, \tfrac{t}{2}\right) > 1 - \gamma \text{ and } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_2, \tfrac{t}{2}\right) > 1 - \gamma.$$

Now, obviously, we will get a $(p,q) \in \mathbb{N} \times \mathbb{N}$ such that

$$\phi\left(\widehat{F}x_{pq} - \xi_1, \frac{t}{2}\right) > \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_1, \frac{t}{2}\right) > 1 - \gamma$$

and

$$\phi\left(\widehat{F}x_{pq} - \xi_2, \frac{t}{2}\right) > \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_2, \frac{t}{2}\right) > 1 - \gamma$$

(That is, consider $\max\left\{\phi\left(\widehat{F}x_{kl}-\xi_1,\frac{t}{2}\right),\phi\left(\widehat{F}x_{kl}-\xi_2,\frac{t}{2}\right):(k,l)\in J_{us}\right\}$ and select that (k,l) as (p,q) for which the maximum occurs).

Then, we obtain

$$\begin{aligned}\phi\left(\xi_1-\xi_2,t\right) &\geq \phi\left(\widehat{F}x_{pq}-\xi_1,\tfrac{t}{2}\right)*\phi\left(\widehat{F}x_{pq}-\xi_2,\tfrac{t}{2}\right)\\ &>(1-\gamma)*(1-\gamma)>1-\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we get $\phi\left(\xi_1-\xi_2,t\right)=1$ for each $t>0$, which gives that $\xi_1=\xi_2$. At the same time, if $(u,s)\in K^c_{\phi,1}(\gamma,t)\cap K^c_{\phi,2}(\gamma,t)$, then by using the similar method, it can be demonstrated that $\omega\left(\xi_1-\xi_2,t\right)<\varepsilon$, for arbitrary $\varepsilon>0$ and for every $t>0$, and so $\xi_1=\xi_2$. Hence, in all cases, we deduce that $F\mathcal{I}^{(\phi,\omega)}_{\theta_{us}}-\lim x$ is unique. □

Theorem 2.3. *If $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi$, then $F\mathcal{I}^{(\phi,\omega)}_{\theta_{us}}-\lim x=\xi$.*

Proof. Let $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi$. Then, for every $t>0$ and $\varepsilon\in(0,1)$, there is $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)>1-\varepsilon \text{ and } \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)<\varepsilon$$

for all $u,s\geq r_0$. Therefore, we obtain

$$\begin{aligned}A&=\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq1-\varepsilon\\ \text{or }\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right\}\\ &\subseteq\left\{(1,1),(2,2),...,(k_0-1,k_0-1)\right\}.\end{aligned}$$

But, with $\mathcal{I}_2$ being admissible ideal, we get $A\in\mathcal{I}_2$. Hence, $F\mathcal{I}^{(\phi,\omega)}_{\theta_{us}}-\lim x=\xi$. □

Theorem 2.4. *If $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi$, then there exists a subsequence $\left(x_{k'(u)l'(s)}\right)$ of x such that $(\phi,\omega)^{\theta_{us}}-\lim Fx_{k'(u)l'(s)}=\xi$.*

Proof. Let $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi$. Then, for every $t>0$ and $\varepsilon\in(0,1)$, there exists $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)>1-\varepsilon \text{ and } \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)<\varepsilon$$

for all $u,s\geq r_0$. Obviously, for each $u,s\geq r_0$, we can select $(k'(u),l'(s))\in J_{us}$ such that

$$\phi\left(\widehat{F}x_{k'(u)l'(s)}-\xi,t\right)>\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)>1-\varepsilon$$

$$\text{and }\omega\left(\widehat{F}x_{k'(u)l'(s)}-\xi,t\right)<\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)<\varepsilon.$$

It follows that $(\phi,\omega)^{\theta_{us}}-\lim Fx_{k'(u)l'(s)}=\xi$. □

Definition 2.3. A double sequence $x=(x_{jk})$ in IFNS is named as Fibonacci lacunary Cauchy with regards to the IFN (ϕ,ω) if, for each $\varepsilon>0$ and $t>0$, there exist $N=N(\varepsilon)$ and $M=M(\varepsilon)$ such that, for all $j,p\geq N$, $k,q\geq M$,

$$\frac{1}{h_{us}}\sum_{(j,k),(p,q)\in J_{us}}\phi\left(\widehat{F}x_{jk}-\widehat{F}x_{pq},t\right)>1-\varepsilon \text{ and } \frac{1}{h_{us}}\sum_{(j,k),(p,q)\in J_{us}}\omega\left(\widehat{F}x_{jk}-\widehat{F}x_{pq},t\right)<\varepsilon.$$

Definition 2.4. A double sequence $x=(x_{jk})$ in IFNS is named as Fibonacci lacunary $\mathcal{I}_2$-Cauchy with regards to the IFN (ϕ,ω) if, for every $\varepsilon\in(0,1)$ and $t>0$, there exists $(p,q)\in\mathbb{N}\times\mathbb{N}$ fulfilling

$$\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(j,k)\in J_{us}}\phi\left(\widehat{F}x_{jk}-\widehat{F}x_{pq},t\right)>1-\varepsilon\\ \text{and }\frac{1}{h_{us}}\sum_{(j,k)\in J_{us}}\omega\left(\widehat{F}x_{jk}-\widehat{F}x_{pq},t\right)<\varepsilon\end{array}\right\}\in\mathcal{F}(\mathcal{I}_2).$$

Definition 2.5. A double sequence $x = (x_{jk})$ in IFNS is named as Fibonacci lacunary $\mathcal{I}_2^*$-Cauchy with regards to the IFN (ϕ, ω) if there is a subset $M = \{(j_m, k_m) : j_1 < j_2 < ...; k_1 < k_2 < ...\}$ of $\mathbb{N} \times \mathbb{N}$ such that the set $M' = \{(u, s) \in \mathbb{N} \times \mathbb{N} : (j_u, k_s) \in J_{us}\} \in \mathcal{F}(\mathcal{I}_2)$ and the subsequence $(x_{j_u k_s})$ is a Fibonacci lacunary Cauchy sequence with regards to the IFN (ϕ, ω).

Theorem 2.5. *A double sequence $x = (x_{jk})$ is Fibonacci lacunary $\mathcal{I}_2$-convergent with regards to the IFN (ϕ, ω) iff it is Fibonacci lacunary $\mathcal{I}_2$-Cauchy with regards to (ϕ, ω).*

Proof. Let $x = (x_{jk})$ be Fibonacci lacunary $\mathcal{I}_2$-convergent to ξ with regards to the IFN (ϕ, ω). Then

$$\left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \phi\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) \leq 1 - \varepsilon \\ \text{or } \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \omega\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) \geq \varepsilon \end{array} \right\} \in \mathcal{I}_2.$$

Specifically, for $j = M$, $k = N$

$$\left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{h_{us}} \sum\limits_{(M,N) \in J_{us}} \phi\left(\widehat{F}x_{MN} - \xi, \frac{t}{2}\right) \leq 1 - \varepsilon \\ \text{or } \dfrac{1}{h_{us}} \sum\limits_{(M,N) \in J_{us}} \omega\left(\widehat{F}x_{MN} - \xi, \frac{t}{2}\right) \geq \varepsilon \end{array} \right\} \in \mathcal{I}_2.$$

Since

$$\begin{aligned} \phi\left(\widehat{F}x_{jk} - \widehat{F}x_{MN}, t\right) &= \phi\left(\widehat{F}x_{jk} - \xi - \widehat{F}x_{MN} + \xi, \frac{t}{2} + \frac{t}{2}\right) \\ &\geq \phi\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) * \phi\left(\widehat{F}x_{MN} - \xi, \frac{t}{2}\right) \end{aligned}$$

and

$$\omega\left(\widehat{F}x_{jk} - \widehat{F}x_{MN}, t\right) \leq \omega\left(x_{jk} - \xi, \frac{t}{2}\right) \diamond \omega\left(x_{MN} - \xi, \frac{t}{2}\right),$$

we obtain

$$\left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \mu\left(\widehat{F}x_{jk} - \widehat{F}x_{MN}, t\right) \leq 1 - \varepsilon \\ \text{or } \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \omega\left(\widehat{F}x_{jk} - \widehat{F}x_{MN}, t\right) \geq \varepsilon \end{array} \right\} \in \mathcal{I}_2.$$

That is, x is Fibonacci $\mathcal{I}_2$-lacunary Cauchy with regards to (ϕ, ω).

In contrast, let $x = (x_{jk})$ be Fibonacci $\mathcal{I}_2$-lacunary Cauchy but not Fibonacci lacunary $\mathcal{I}_2$-convergent with regards to the IFN (ϕ, ω). Then, there are N and M such that the set $A(\varepsilon, t) \in \mathcal{I}_2$, where

$$A(\varepsilon, t) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \phi(x_{jk} - x_{MN}, t) \leq 1 - \varepsilon \\ \text{or } \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \omega(x_{kl} - x_{MN}, t) \geq \varepsilon \end{array} \right\}$$

and also $B(\varepsilon, t) \in \mathcal{I}_2$, where

$$B(\varepsilon, t) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \phi\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) \leq 1 - \varepsilon \\ \text{or } \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \omega\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) \geq \varepsilon \end{array} \right\}.$$

Since

$$\phi\left(\widehat{F}x_{jk} - \widehat{F}x_{MN}, t\right) \geq 2\phi\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) > 1 - \varepsilon,$$

and

$$\omega\left(\widehat{F}x_{jk}-\widehat{F}x_{MN},t\right)\leq 2\omega\left(\widehat{F}x_{jk}-\xi,\frac{t}{2}\right)<\varepsilon,$$

if $\phi\left(\widehat{F}x_{jk}-\xi,\frac{t}{2}\right)>\frac{(1-\varepsilon)}{2}$ and $\omega\left(\widehat{F}x_{jk}-\xi,\frac{t}{2}\right)<\frac{\varepsilon}{2}$. Therefore,

$$\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\dfrac{1}{h_{us}}\sum\limits_{(j,k)\in J_{us}}\phi\left(\widehat{F}x_{jk}-\widehat{F}x_{MN},t\right)>1-\varepsilon\\ \text{or }\dfrac{1}{h_{us}}\sum\limits_{(j,k)\in J_{us}}\omega\left(\widehat{F}x_{jk}-\widehat{F}x_{MN},t\right)<\varepsilon\end{array}\right\}\in\mathcal{I}_2,$$

that is, $A^c(\varepsilon,t)\in\mathcal{I}_2$ and hence, $A(\varepsilon,t)\in\mathcal{F}(\mathcal{I}_2)$, which leads to a contradiction. Hence x must be Fibonacci lacunary $\mathcal{I}_2$-convergent with regards to the IFN (ϕ,ω). □

Theorem 2.6. *If (ρ_{us}) is a double lacunary refinement of θ_{us} and $F\mathcal{I}_{\rho_{us}}^{(\phi,\omega)}-\lim x=\xi$, then $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)}-\lim x=\xi$.*

Proof. Presume that each $\mathcal{I}_{us}$ of θ_{us} involves the points $\left(\overline{k}_{u,i},\overline{l}_{s,j}\right)_{i,j=1}^{v(u),w(s)}$ of (ρ_{us}) so that

$$k_{u-1}<\overline{k}_{u,1}<\overline{k}_{u,2}<...<\overline{k}_{u,v(u)}=k_u,\ \text{where } \overline{I}_{u,i}=\left(\overline{k}_{u,i-1},\overline{k}_{u,i}\right],$$

$$l_{s-1}<\overline{l}_{s,1}<\overline{l}_{s,2}<...<\overline{l}_{s,w(s)}=l_s,\ \text{where } \overline{J}_{s,j}=\left(\overline{l}_{s,j-1},\overline{l}_{s,j}\right],$$

and

$$\overline{J}_{u,s,i,j}=\left\{(k,l):\overline{k}_{u,i-1}<k\leq\overline{k}_u;\overline{l}_{s,j-1}<l\leq\overline{l}_s\right\}$$

for all u,s and $v(u)\geq 1, w(s)\geq 1$ this gives that $(k_u,l_s)\subseteq\left(\overline{k}_u,\overline{l}_s\right)$. Let $\left(\overline{J}_{ij}\right)_{i,j=1,1}^{\infty,\infty}$ be the sequence of abutting blocks of $\left(\overline{J}_{u,s,i,j}\right)$ ordered by increasing a lower right index points. Since $F\mathcal{I}_{\rho_{us}}^{(\phi,\omega)}-\lim x=\xi$, we obtain the following for each $t>0$ and $\varepsilon\in(0,1)$

$$\left\{\begin{array}{c}(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\overline{h}_{ij}}\sum\limits_{\overline{J}_{ij}\subset J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \text{or }\frac{1}{\overline{h}_{ij}}\sum\limits_{\overline{J}_{ij}\subset J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right\}\in\mathcal{I}_2. \tag{2.1}$$

As before, we take $h_{us}=h_u\overline{h}_s;\overline{h}_{ui}=\overline{k}_{ui}-\overline{k}_{u,i-1};\overline{h}_{sj}=\overline{l}_{s,j}-\overline{l}_{s,j-1}$.

For each $t>0$ and $\varepsilon\in(0,1)$ we get

$$\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum\limits_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \text{or }\frac{1}{h_{us}}\sum\limits_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right\}$$

$$\subseteq\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum\limits_{(k,l)\in J_{us}}\{(i,j)\in\mathbb{N}\times\mathbb{N}:\\ \left.\begin{array}{c}\frac{1}{\overline{h}_{ij}}\sum\limits_{\overline{J}_{ij}\subset J_{us};(k,l)\in\overline{J}_{ij}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \text{or }\frac{1}{\overline{h}_{ij}}\sum\limits_{\overline{J}_{ij}\subset J_{us};(k,l)\in\overline{J}_{ij}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right\}\end{array}\right\}.$$

By (2.1), for each $t>0$ and $\varepsilon\in(0,1)$ if we define

$$t_{ij}=\left(\begin{array}{c}\frac{1}{\overline{h}_{ij}}\sum\limits_{(k,l)\in\overline{J}_{ij}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \text{or }\frac{1}{\overline{h}_{i,j}}\sum\limits_{\overline{J}_{ij}\subset J_{us};(k,l)\in\overline{J}_{ij}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right)_{i,j=1}^{\infty,\infty},$$

then $(t_{i,j})$ is a Pringsheim null sequence. The transformation

$$(At)_{us} = \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \left(\begin{array}{c} \overline{h}_{ij} \frac{1}{\overline{h}_{ij}} \sum\limits_{(k,l)\in \overline{J}_{ij}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) \leq 1-\varepsilon \\ \text{or } \overline{h}_{ij} \frac{1}{\overline{h}_{ij}} \sum\limits_{\overline{J}_{ij}\subset J_{us};(k,l)\in \overline{J}_{ij}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) \geq \varepsilon \end{array} \right)$$

fulfills all situations for a matrix transformation to map a Pringsheim null sequence into a Pringsheim null sequence. Hence, $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi$. □

Definition 2.6. Let $(X,\phi,\omega,*,\diamond)$ be an IFNS.

(a) An element $\xi \in X$ is named as Fibonacci lacunary $\mathcal{I}_2$-limit point of $x = (x_{kl})$ if there is set $M = \{(k_1,l_1) < (k_2,l_2) < ... < (k_u,l_s) < ...\} \subset \mathbb{N}\times\mathbb{N}$ such that the set

$$M' = \{(u,s) \in \mathbb{N}\times\mathbb{N} : (k_u,l_s) \in J_{us}\} \notin \mathcal{I}_2$$

and $(\phi,\omega)^{\theta_{us}} - \lim Fx_{k_u l_s} = \xi$.

(b) $\xi \in X$ is named as Fibonacci lacunary $\mathcal{I}_2$-cluster point of $x = (x_{kl})$ if, for every $t > 0$ and $\varepsilon \in (0,1)$, we get

$$\left\{ \begin{array}{c} (u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum\limits_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) > 1-\varepsilon \\ \text{and } \frac{1}{h_{us}} \sum\limits_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) < \varepsilon \end{array} \right\} \notin \mathcal{I}_2.$$

$\Lambda^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x)$ and $\Gamma^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x)$ indicate the set of all Fibonacci lacunary $\mathcal{I}_2$-limit points and the set of all Fibonacci lacunary $\mathcal{I}_2$-cluster points in IFNS, respectively.

Theorem 2.7. *For every sequence $x = (x_{kl})$ in IFNS, we have $\Lambda^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x) \subseteq \Gamma^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x)$.*

Proof. Let $\xi \in \Lambda^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x)$. Then, there is a set $M \subset \mathbb{N}\times\mathbb{N}$ such that the set $M' \notin \mathcal{I}_2$, where M and M' are as in Definition 2.6, fulfills $(\phi,\omega)^{\theta_{us}} - \lim Fx_{k_u l_s} = \xi$. Hence, for every $t > 0$ and $\varepsilon \in (0,1)$, there are $u_0, s_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{k_u l_s} - \xi, t\right) > 1-\varepsilon \text{ and } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{k_u l_s} - \xi, t\right) < \varepsilon$$

for all $u \geq u_0$, $s \geq s_0$. Therefore,

$$B = \left\{ \begin{array}{c} (u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum\limits_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) > 1-\varepsilon \\ \text{and } \frac{1}{h_{us}} \sum\limits_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) < \varepsilon \end{array} \right\}$$
$$\supseteq M' \backslash \{(k_1,l_1) < (k_2,l_2) < ... < (k_{u_0}, l_{s_0})\}.$$

Now, with $\mathcal{I}_2$ being admissible, we must have $M' \backslash \{(k_1,l_1) < (k_2,l_2) < ... < (k_{u_0}, l_{s_0})\} \notin \mathcal{I}_2$ and as such $B \notin \mathcal{I}_2$. Hence, $\xi \in \Gamma^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x)$. □

Theorem 2.8. *The following observations are equivalent.*

(a) ξ is Fibonacci lacunary $\mathcal{I}_2$-limit point of x.

(b) There are two sequences $y = (y_{kl})$ and $z = (z_{kl})$ in IFNS such that $x = y + z$ and $(\phi,\omega)^{\theta_{us}} - \lim Fy = \xi$ and

$$\left\{(u,s) \in \mathbb{N}\times\mathbb{N} : (k,l) \in J_{us},\, z_{kl} \neq \overline{0}\right\} \in \mathcal{I}_2.$$

Proof. Presume that (a) holds. Then there are M and M' are as in Definition 2.6 such that $M' \notin \mathcal{I}_2$ and $(\phi,\omega)^{\theta_{us}} - \lim Fx = \xi$. Take the sequences y and z as follows:

$$y_{kl} = \begin{cases} x_{kl}, & \text{if } (k,l) \in J_{us},\ (u,s) \in M' \\ \xi, & \text{otherwise.} \end{cases}$$

and

$$z_{kl} = \begin{cases} \overline{0}, & \text{if } (k,l) \in J_{us},\ (u,s) \in M' \\ x_{kl} - \xi, & \text{otherwise.} \end{cases}$$

It is adequate to think the case $(k,l) \in J_{us}$ such that $(u,s) \in \mathbb{N} \times \mathbb{N} \setminus M'$. Then, for each $t > 0$ and $\varepsilon \in (0,1)$. Then, we have $\phi\left(\widehat{F}y_{kl} - \xi, t\right) = 1 > 1 - \varepsilon$ and $\omega\left(\widehat{F}y_{kl} - \xi, t\right) = 0 < \varepsilon$. Thus, we write

$$\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}y_{kl} - \xi, t\right) = 1 > 1 - \varepsilon \text{ and } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}y_{kl} - \xi, t\right) = 0 < \varepsilon.$$

Hence, $(\phi,\omega)^{\theta_{us}} - \lim y = \xi$. Now, we have

$$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : (k,l) \in J_{us},\ z_{kl} \neq \overline{0}\right\} \subset \mathbb{N} \times \mathbb{N} \diagdown M'.$$

But $\mathbb{N} \times \mathbb{N} \diagdown M' \in \mathcal{I}_2$, and so

$$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : (k,l) \in J_{us},\ z_{kl} \neq \overline{0}\right\} \in \mathcal{I}_2.$$

Now, presume that (b) holds. Let $M' = \left\{(u,s) \in \mathbb{N} \times \mathbb{N} : (k,l) \in J_{us},\ z_{kl} = \overline{0}\right\}$. Then, obviously $M' \in \mathcal{F}(\mathcal{I}_2)$ and so it is an infinite set. Construct the set

$$M = \{(k_1,l_1) < (k_2,l_2) < \ldots < (k_u,l_s) < \ldots\} \subset \mathbb{N} \times \mathbb{N}$$

such that $(k_u,l_s) \in J_{us}$ and $z_{k_u l_s} = \overline{0}$. Since $x_{k_u l_s} = y_{k_u l_s}$ and $(\phi,\omega)^{\theta_{us}} - \lim Fy = \xi$ we obtain $(\phi,\omega)^{\theta_{us}} - \lim Fx_{k_u l_s} = \xi$. □

Acknowledgment.

We thank the reviewer for their insightful comments and suggestions that helped us improve the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Atanassov, K.T.: *Intuitionistic fuzzy sets.* Fuzzy Sets and Syst. **20**, 87-96 (1986).

[2] Debnath, P.: *Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces*. Comput. Math. Appl. **63**, 708-715 (2012).

[3] Dündar, E., Ulusu, U., Pancaroğlu, N.: *Strongly $\mathcal{I}_2$-lacunary convergence and $\mathcal{I}_2$-lacunary Cauchy double sequences of sets*. Aligarh Bull. Math. **35** (1-2), 1-15 (2016).

[4] Dündar, E., Ulusu, U., Aydın, B.: *$\mathcal{I}_2$-lacunary statistical convergence of double sequences of sets*. Konuralp J. Math. **5** (1), 1-10 (2017).

[5] Fast, H.: (1951). *Sur la convergence statistique*. Colloq. Math. **2**, 241-244 (1951).

[6] Fridy, J.A., Orhan, C.: *Lacunary statistical convergence.* Pacific J. Math. **160** (1), 43-51 (1993).

[7] Goonatilake S.: Toward a Global Science. Indiana University Press. (1998).

[8] Hazarika, B.: *Lacunary ideal convergence of multiple sequences*. J. Egyptian Math. Soc. 24, 54-59 (2016).

[9] Kara, E.E., Başarır, M.: *An application of Fibonacci numbers into infinite Toeplitz matrices*. Casp. J. Math. Sci. **1** (1), 43-47 (2012).

[10] Kara, E.E.: *Some topological and geometrical properties of new Banach sequence spaces*. J. Inequal. Appl. **38** (2013), (2013).

[11] Karakuş, S., Demirci, K., Duman, O.: *Statistical convergence on intuitionistic fuzzy normed spaces*. Chaos Solitons Fractals. **35**, 763-769 (2008).

[12] Kirişçi, M., Karaisa, A.: *Fibonacci statistical convergence and Korovkin type approximation theorems*. J. Inequal. Appl. **2017** (229), 1-15 (2017).

[13] Kişi, Ö., Tuzcuoğlu, İ.: *Fibonacci lacunary statistical convergence in intuitionistic fuzzy normed linear spaces*. J. Progress Res. Soc. Sci. **16** (3), 3001-3007 (2020).

[14] Kişi, Ö., Güler, E.: *On Fibonacci ideal convergence of double sequences in intuitionistic fuzzy normed linear spaces*. Turk. J. Math. Comput. Sci. **11**, 46-55 (2019).

[15] Koshy, T.: Fibonacci and Lucas Numbers with Applications. Wiley, New York. (2001).

[16] Kostyrko, P., Salat, T., Wilczynsski, W.: *$\mathcal{I}$-convergence*. Real Anal. Exchange. **6** (2), 669-686 (2001).

[17] Mursaleen, M., Edely, O.H.: *Statistical convergence of double sequences*. J. Math. Anal. Appl. **288**, 223-231 (2003).

[18] Mursaleen, M., Mohiuddine, S.A.: *Statistical convergence of double sequences in intuitionistic fuzzy normed space*. Chaos Solitons Fractals. **41**, 2414-2421 (2009).

[19] Mursaleen, M., Mohiuddine, S.A., Edely, O.H.H.: *On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces.* Comput. Math. Appl. **59**, 603-611 (2010).

[20] Mursaleen, M., Mohiuddine, S.A.: *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*. J. Comput. Appl. Math. **233** (2), 142-149 (2009).

[21] Nuray, F., Ulusu, U., Dündar, E.: *Lacunary statistical convergence of double sequences of sets*. Soft Comput. **20**, 2883-2888 (2016).

[22] Park, J.H.: *Intuitionistic fuzzy metric spaces*. Chaos Solitons Fractals. **22**, 1039-1046 (2004).

[23] Lael, F., Nourouzi, K.: *Some results on the IF-normed spaces*. Chaos Solitons Fractals. **37**, 931-939 (2008).

[24] Savaş, E., Patterson, R.F.: *Lacunary statistical convergence of double sequences*. Math. Commun. **10**, 55-61 (2005).

[25] Schweizer, B., Sklar, A.: *Statistical metric spaces*. Pacific J. Math. **10**, 314-334. (1960).

[26] Temizer Ersoy, M., Furkan, H.: *Distinguished supspaces in topological sequence spaces theory*. AIMS Math. **5** (4), 2858-2868 (2020).

[27] Temizer Ersoy, M.: Some abelian, tauberian and core theorems related to the (V, λ) summability. Univers. J. Math. Appl. **4** (2), 70-75.

[28] Tripathy, B.C., Hazarika, B., Choudhary, B.B.: *Lacunary $\mathcal{I}$-convergent sequences*. Kyungpook Math. J. **2** (4), 473-482 (2012).

[29] Tripathy, B.K., Tripathy, B.C.: *On $\mathcal{I}$-convergent double sequences*. Soochow J. Math. **31** (4), 549-560 (2005).

[30] Zadeh, L.A.: *Fuzzy sets*. Inf.Control. **8**, 338-353 (1965). (2021).

Affiliations

Ömer Kişi
Address: Bartın University, Dept. of Mathematics, 74100, Bartın-Turkey.
e-mail: okisi@bartin.edu.tr
ORCID ID:0000-0001-6844-3092