KMU Journal of Engineering and Natural Sciences
https://dergipark.org.tr/en/pub/kmujens
3(1), 7-14, (2021) © KMUJENS
e-ISSN: 2687-5071

Dairesel-hiperbolik Fibonacci ve Lucas kuaterniyonlar üzerine bir çalışma

A study on circular-hyperbolic Fibonacci and Lucas quaternions

Nazmiye YILMAZ[1]

[1]Department of Mathematics, Kamil Özdağ Science Faculty, Karamanoğlu Mehmetbey University, Karaman, Turkey

(Received: 3 May 2021; Accepted: 2 June 2021)

Özet. Dairesel-hiperbolik Fibonacci ve Lucas kuaterniyonlarının bazı özelliklerini araştırıyoruz (kısaca $\mathbb{CH}FLQ$ ile gösterilen). Ayrıca negatif indislilerini tanıtıyoruz ve kombinatorik toplamlarını elde ediyoruz. Son olarak bu $\mathbb{CH}FLQ$ kuaterniyonlarının genel bir toplamını, üstel ve Poisson üreteç fonksiyonlarını sunuyoruz.

Anahtar Kelimeler: binom katsayısı, dairesel-hiperbolik Fibonacci kuaterniyonları, dairesel-hiperbolik Lucas kuaterniyonları, hiperbolik sayılar.

Abstract. We investigate some properties of circular-hyperbolic Fibonacci and Lucas quaternions ($\mathbb{CH}FLQ$ for short). Also, we introduce their negative subscripts and obtain combinatorial sums. Finally, we present a general summation, exponential and Poisson generating functions of the $\mathbb{CH}FLQ$.

Keywords: binomial coefficient, circular-hyperbolic Fibonacci quaternions, circular-hyperbolic Lucas quaternions, hyperbolic numbers.
MSC 2010: 11B37, 20G20, 11R52.

1. Introduction

The real quaternions were first described by Irish mathematician William Rowan Hamilton in 1843. Hamilton [10] introduced a set of real quaternions which can be represented as

$$H = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Also, there has been an increasing interest on quaternions that play an important role in various areas such as computer sciences, physics, differential geometry, quantum physics, signal, color image processing, geostatics and analysis [1, 6, 22].

Another type of numbers is hyperbolic numbers. The set including the number h which is not a real number but its square is equal to 1, is called a set of hyperbolic numbers and defined as

$$\mathbb{H} = \{z = x + y\mathbf{h} \mid x, y \in \mathbb{R}\}.$$

The work on the hyperbolic numbers can be found in [3, 5, 7, 17, 20].

Circular-hyperbolic numbers, [5], w can be expressed in the form as

$$\mathbb{CH} = \{w = z_1 + z_2\mathbf{h} \mid z_1, z_2 \in \mathbb{C}\},$$

Corresponding Author Email: yilmaznzmy@gmail.com.

where $\mathbf{i}^2 = -1$, $\mathbf{h}^2 = 1$, $\mathbf{h} \neq \pm 1$, $\mathbf{ih} = -\mathbf{hi}$, $(\mathbf{ih})^2 = 1$ and $\mathbb{C}$ is set of complex number. For any two circular-hyperbolic numbers $w_1 = z_1 + z_2\mathbf{h}$ and $w_2 = z_3 + z_4\mathbf{h}$, they write

- $w_1 \pm w_2 = (z_1 \pm z_3) + (z_2 \pm z_4)\mathbf{h}$ (addition and subtraction),
- $w_1 \times w_2 = (z_1z_3 + z_2z_4) + (z_1z_4 + z_2z_3)\mathbf{h}$ (multiplication),
- $\frac{w_1}{w_2} = \frac{(z_1z_3 - z_2z_4)}{z_3^2 - z_4^2} + \frac{(z_2z_3 - z_1z_4)}{z_3^2 - z_4^2}\mathbf{h}$ $(z_3 \neq z_4)$ (division).

If $Re(w_2) \neq 0$,then the division is possible. The circular-hyperbolic numbers are defined by the basis 1, $\mathbf{i}$, $\mathbf{h}$, $\mathbf{ih}$. The base elements of the circular-hyperbolic numbers satisfy the following multiplication scheme (Table 1).

x	1	$\mathbf{i}$	$\mathbf{h}$	$\mathbf{ih}$
1	1	$\mathbf{i}$	$\mathbf{h}$	$\mathbf{ih}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{ih}$	$-\mathbf{h}$
$\mathbf{h}$	$\mathbf{h}$	$-\mathbf{ih}$	1	$-\mathbf{i}$
$\mathbf{ih}$	$\mathbf{ih}$	$\mathbf{h}$	$\mathbf{i}$	1

Table 1. Multiplication of the circular-hyperbolic numbers ([2], Table 1)

The circular-hyperbolic numbers, just like quaternions, are a generalization of complex hyperbolic numbers by means of entities specified by four-component numbers. But hyperbolic and dual-hyperbolic numbers are commutative, whereas, circular-hyperbolic numbers are non-commutative. Moreover, the multiplication of these numbers gives the circular-hyperbolic numbers [2, 5].
On the other hand, for $n \geq 2$, the Fibonacci and Lucas numbers are defined as [16]

$$F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1 \tag{1.1}$$

and

$$L_n = L_{n-1} + L_{n-2},\ L_0 = 2,\ L_1 = 1. \tag{1.2}$$

In recently years, Fibonacci, Lucas quaternions and hyperbolic numbers cover a wide range of interest in modern mathematics as they appear in the comprehensive works of [2, 4, 8, 9, 11, 12, 13, 14, 15, 18, 19, 21]. For example in [2], the $\mathbb{CH}FLQ$ are defined as

$$\mathbb{CH}F_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{h} + F_{n+3}\mathbf{ih} \tag{1.3}$$

and

$$\mathbb{CH}L_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\mathbf{h} + L_{n+3}\mathbf{ih}, \tag{1.4}$$

where $n \in \mathbb{N}$. Also, the author found identity and Binet formulas of these quaternions as follows

$$\mathbb{CH}F_{n+1} + \mathbb{CH}F_{n-1} = \mathbb{CH}L_n, \tag{1.5}$$

$$\mathbb{CH}F_n = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta} \tag{1.6}$$

and

$$\mathbb{CH}L_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n, \tag{1.7}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\hat{\alpha} = (1 + \alpha\mathbf{i} + \alpha^2\mathbf{h} + \alpha^3\mathbf{ih})$, $\hat{\beta} = (1 + \beta\mathbf{i} + \beta^2\mathbf{h} + \beta^3\mathbf{ih})$.
It is known that, the matrices for the Fibonacci numbers are

$$U^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.8}$$

In this study, we obtain a new matrix Q_n similar to the above matrix U^n for the $\mathbb{CH}FLQ$. Also, we define the negative subscripts of these quaternions. We give several properties and different sums for the $\mathbb{CH}FLQ$.

2. Main results

Firstly, we define the $\mathbb{CH}FLQ$ with negative subscripts for $n \in \mathbb{N}$.

Definition 1. *The $\mathbb{CH}FLQ$ with negative subscripts are defined as*

$$\mathbb{CH}F_{-n} = F_{-n} + F_{-n+1}\mathbf{i} + F_{-n+2}\mathbf{h} + F_{-n+3}\mathbf{ih} \tag{2.1}$$

and

$$\mathbb{CH}L_{-n} = L_{-n} + L_{-n+1}\mathbf{i} + L_{-n+2}\mathbf{h} + L_{-n+3}\mathbf{ih}. \tag{2.2}$$

From the equalities $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$, we have

$$\mathbb{CH}F_{-n} = \begin{cases} -F_n + F_{n-1}\mathbf{i} - F_{n-2}\mathbf{h} + F_{n-3}\mathbf{ih}, & n \text{ is even} \\ F_n - F_{n-1}\mathbf{i} + F_{n-2}\mathbf{h} - F_{n-3}\mathbf{ih}, & n \text{ is odd} \end{cases}, \tag{2.3}$$

and

$$\mathbb{CH}L_{-n} = \begin{cases} L_n - L_{n-1}\mathbf{i} + L_{n-2}\mathbf{h} - L_{n-3}\mathbf{ih}, & n \text{ is even} \\ -L_n + L_{n-1}\mathbf{i} - L_{n-2}\mathbf{h} + L_{n-3}\mathbf{ih}, & n \text{ is odd} \end{cases}. \tag{2.4}$$

The following theorem gives us the matrix of the $\mathbb{CH}FLQ$.

Theorem 1. *For n integer numbers, we have*

$$Q_n = QU^n = U^n Q, \tag{2.5}$$

where

$$Q_n = \begin{pmatrix} \mathbb{CH}F_{n+1} & \mathbb{CH}F_n \\ \mathbb{CH}F_n & \mathbb{CH}F_{n-1} \end{pmatrix}, Q = \begin{pmatrix} 1+\mathbf{i}+2\mathbf{h}+3\mathbf{ih} & \mathbf{i}+\mathbf{h}+2\mathbf{ih} \\ \mathbf{i}+\mathbf{h}+2\mathbf{ih} & 1+\mathbf{h}+\mathbf{ih} \end{pmatrix} \tag{2.6}$$

and U^n is given by the equation (1.8).

Proof. Using the recurrence relations in (1.1) and (1.3), we success the desired equation. □

Theorem 2. *For $n, m > 0$ integers, the identities related with the $\mathbb{CH}FLQ$ are follow:*

i) $F_{n+1}\mathbb{CH}F_{m+1} + F_n\mathbb{CH}F_m = \mathbb{CH}F_{n+m+1}$,
ii) $F_{n+1}\mathbb{CH}L_{m+1} + F_n\mathbb{CH}L_m = \mathbb{CH}L_{n+m+1}$.

Proof. *i)* Given the matrices $U^{n+m}Q$, U^mQ as the equations (1.8) and in Theorem 1, and considering the first row first column elements of the product $U^n(U^mQ)$, which is equal to the first row first column elements of matrix Q_{n+m} we get the result.

ii) From the well-known identity $F_{m+1} + F_{m-1} = L_m$ and the property $\mathbb{CH}F_{m+1} + \mathbb{CH}F_{m-1} = \mathbb{CH}L_m$ in equation (1.5), by considering the first row first column elements of the product $U^n\left((U^{m+1} + U^{m-1})Q\right)$, which is equal to the first row first column elements of matrix $U^{n+m+1}Q + U^{n+m-1}Q$ we obtain the result.

□

The properties in the following theorem are called Honsberger identity of the $\mathbb{CH}FLQ$. The Honsberger identity in *i)* for the circular-hyperbolic Fibonacci quaternions was given by Aydin ([2], Theorem 2). She used the definition of the circular-hyperbolic Fibonacci quaternions, but we get the Honsberger identity by using the Q matrix.

Theorem 3. *For $m, n \geq 0$ integers, we have*

i) $\mathbb{CH}F_{n+1}\mathbb{CH}F_{m+1} + \mathbb{CH}F_n\mathbb{CH}F_m = \mathbb{CH}F_{n+m+1} + \mathbb{CH}F_{n+m+2}\mathbf{i} + \mathbb{CH}F_{n+m+3}\mathbf{h} + \mathbb{CH}F_{n+m+4}\mathbf{ih}$,
ii) $\mathbb{CH}F_{n+1}\mathbb{CH}L_{m+1} + \mathbb{CH}F_n\mathbb{CH}L_m = \mathbb{CH}L_{n+m+1} + \mathbb{CH}L_{n+m+2}\mathbf{i} + \mathbb{CH}L_{n+m+3}\mathbf{h} + \mathbb{CH}L_{n+m+4}\mathbf{ih}$.

Proof. *i)* Given the matrices QU^n, U^mQ as the equations (1.8) and in Theorem 1, and considering the first row first column elements of the product $(QU^n)(U^mQ)$, which is equal to the first row first column elements of matrix $Q(U^{n+m}Q)$ we get the result.

ii) From the well-known identity $F_{m+1} + F_{m-1} = L_m$ and the property $\mathbb{CH}F_{m+1} + \mathbb{CH}F_{m-1} = \mathbb{CH}L_m$ in equation (1.5), by considering the first row first column elements of the product $(QU^n)\left((U^{m+1} + U^{m-1})Q\right)$, which is equal to the first row first column elements of matrix $Q\left(U^{n+m+1}Q + U^{n+m-1}Q\right)$ we obtain the result.

□

If we take $m = n - 1$ in Theorem 3, we obtain the following results:

Corollary 1. *For $n \geq 1$ integers, we have*

i) $\mathbb{CH}F_{n+1}\mathbb{CH}F_n + \mathbb{CH}F_n\mathbb{CH}F_{n-1} = \mathbb{CH}F_{2n} + \mathbb{CH}F_{2n+1}\mathbf{i} + \mathbb{CH}F_{2n+2}\mathbf{h} + \mathbb{CH}F_{2n+3}\mathbf{ih}$,
ii) $\mathbb{CH}F_{n+1}\mathbb{CH}L_n + \mathbb{CH}F_n\mathbb{CH}L_{n-1} = \mathbb{CH}L_{2n} + \mathbb{CH}L_{2n+1}\mathbf{i} + \mathbb{CH}L_{2n+2}\mathbf{h} + \mathbb{CH}L_{2n+3}\mathbf{ih}$.

If we take $m = n$ in Theorem 3, we obtain the following results:

Corollary 2. *For $n \geq 0$ integers, we have*
i) $\mathbb{CH}F_{n+1}^2 + \mathbb{CH}F_n^2 = \mathbb{CH}F_{2n+1} + \mathbb{CH}F_{2n+2}\mathbf{i} + \mathbb{CH}F_{2n+3}\mathbf{h} + \mathbb{CH}F_{2n+4}\mathbf{ih}$,
ii) $\mathbb{CH}F_{n+1}\mathbb{CH}L_{n+1} + \mathbb{CH}F_n\mathbb{CH}L_n = \mathbb{CH}L_{2n+1} + \mathbb{CH}L_{2n+2}\mathbf{i} + \mathbb{CH}L_{2n+3}\mathbf{h} + \mathbb{CH}L_{2n+4}\mathbf{ih}$.

The properties in the following theorem are called Vajda identity of the $\mathbb{CH}FLQ$.

Theorem 4. *For n, r, s integers, we have*
i) $\mathbb{CH}F_{n+r}\mathbb{CH}F_{n+s} - \mathbb{CH}F_n\mathbb{CH}F_{n+r+s} = (-1)^n F_r \left[2F_s - 2F_{s-1}\mathbf{i} + 2F_{s+2}\mathbf{h} + (F_s + 2F_{s-2})\,\mathbf{ih}\right]$,
ii) $\mathbb{CH}L_{n+r}\mathbb{CH}L_{n+s} - \mathbb{CH}L_n\mathbb{CH}L_{n+r+s} = 5(-1)^{n+1} F_r \left[2F_s - 2F_{s-1}\mathbf{i} + 2F_{s+2}\mathbf{h} + (F_s + 2F_{s-2})\,\mathbf{ih}\right]$.

Proof. *i)* By using the equation (1.6), we acquire

$$\begin{aligned} &\mathbb{CH}F_{n+r}\mathbb{CH}F_{n+s} - \mathbb{CH}F_n\mathbb{CH}F_{n+r+s} = \\ &\left(\frac{\hat{\alpha}\alpha^{n+r} - \hat{\beta}\beta^{n+r}}{\alpha-\beta}\right)\left(\frac{\hat{\alpha}\alpha^{n+s} - \hat{\beta}\beta^{n+s}}{\alpha-\beta}\right) - \left(\frac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}\right)\left(\frac{\hat{\alpha}\alpha^{n+r+s} - \hat{\beta}\beta^{n+r+s}}{\alpha-\beta}\right) \\ &= (-1)^n F_r \left(\frac{\hat{\beta}\hat{\alpha}\alpha^s - \hat{\alpha}\hat{\beta}\beta^s}{\alpha-\beta}\right). \end{aligned}$$

By taking into account the equalities $\alpha\beta = -1$, $\hat{\alpha}\hat{\beta} = 2 + 2\alpha\mathbf{i} + 2\beta^2\mathbf{h} + \left(2\alpha^2 + 1\right)\mathbf{ih}$ and $\hat{\beta}\hat{\alpha} = 2 + 2\beta\mathbf{i} + 2\alpha^2\mathbf{h} + \left(2\beta^2 + 1\right)\mathbf{ih}$, we get

$$\mathbb{CH}F_{n+r}\mathbb{CH}F_{n+s} - \mathbb{CH}F_n\mathbb{CH}F_{n+r+s} = (-1)^n F_r \left[2F_s - 2F_{s-1}\mathbf{i} + 2F_{s+2}\mathbf{h} + (F_s + 2F_{s-2})\,\mathbf{ih}\right].$$

ii) The proof is done similar to *i)*.

□

If we take $s = -r$ in Theorem 4, we obtain the following results:

Corollary 3. *For n, r integers, we have*
i) $\mathbb{CH}F_{n+r}\mathbb{CH}F_{n-r} - \mathbb{CH}F_n^2 = (-1)^{n+r+1} F_r \left[2F_r + 2F_{r+1}\mathbf{i} + 2F_{r-2}\mathbf{h} + (F_r + 2F_{r+2})\,\mathbf{ih}\right]$,
ii) $\mathbb{CH}L_{n+r}\mathbb{CH}L_{n-r} - \mathbb{CH}L_n^2 = 5(-1)^{n+r} F_r \left[2F_r + 2F_{r+1}\mathbf{i} + 2F_{r-2}\mathbf{h} + (F_r + 2F_{r+2})\,\mathbf{ih}\right]$.

If we take $-s = r = 1$ in Theorem 4, we obtain the following results:

Corollary 4. *For n integers, we have*
i) $\mathbb{CH}F_{n+1}\mathbb{CH}F_{n-1} - \mathbb{CH}F_n^2 = (-1)^n\,(2 + 2\mathbf{i} + 2\mathbf{h} + 5\mathbf{ih})$,
ii) $\mathbb{CH}L_{n+1}\mathbb{CH}L_{n-1} - \mathbb{CH}L_n^2 = 5(-1)^{n+1}\,(2 + 2\mathbf{i} + 2\mathbf{h} + 5\mathbf{ih})$.

If we take $s = 1$, $r = m - n$ in Theorem 6, we obtain the following results:

Corollary 5. *For n, m integers, we have*
i) $\mathbb{CH}F_m\mathbb{CH}F_{n+1} - \mathbb{CH}F_n\mathbb{CH}F_{m+1} = (-1)^n F_{m-n}\,(2 + 4\mathbf{h} + 3\mathbf{ih})$,
ii) $\mathbb{CH}L_m\mathbb{CH}L_{n+1} - \mathbb{CH}L_n\mathbb{CH}L_{m+1} = 5(-1)^{n+1} F_{m-n}\,(2 + 4\mathbf{h} + 3\mathbf{ih})$.

We obtain the binomial summations of the circular-hyperbolic Fibonacci quaternions in the following theorem.

Theorem 5. *For $n \in \mathbb{N}$, the identities are hold:*
i) $\sum_{k=0}^{n}\binom{n}{k}\mathbb{CH}F_k = \mathbb{CH}F_{2n}$,
ii) $\sum_{k=0}^{n}\binom{n}{k}\mathbb{CH}F_{k+1} = \mathbb{CH}F_{2n+1}$,
iii) $\sum_{k=0}^{n}\binom{n}{k}(-1)^k\mathbb{CH}F_k = (-1)^n\mathbb{CH}F_{-n}$,
iv) $\sum_{k=0}^{n}\binom{n}{k}\mathbb{CH}F_{4k} = 3^n\mathbb{CH}F_{2n}$,
v) $\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\mathbb{CH}F_{5k} = 5^n\mathbb{CH}F_{2n}$,
vi) $\sum_{k=0}^{n}\binom{n}{k}3^{n-k}\mathbb{CH}F_{6k} = 8^n\mathbb{CH}F_{2n}$,
vii) $\sum_{k=0}^{n}\binom{n}{k}(-2)^k\mathbb{CH}F_{2k} = (-1)^n\mathbb{CH}F_{3n}$,
viii) $\sum_{k=0}^{n}\binom{n}{k}(-2)^k\mathbb{CH}F_{5k} = (-5)^n\mathbb{CH}F_{3n}$.

Proof. *i)* From the equation (1.6), we write

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\mathbb{CH}F_k &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{\hat{\alpha}\alpha^k-\hat{\beta}\beta^k}{\alpha-\beta}\right)\\ &= \frac{\hat{\alpha}}{\alpha-\beta}(1+\alpha)^n-\frac{\hat{\beta}}{\alpha-\beta}(1+\beta)^n.\end{aligned}$$

By considering the well-known equalities $1+\alpha=\alpha^2$, $1+\beta=\beta^2$ and again the equation (1.6), we obtain claimed result.

iii) By considering the equation (1.3) and the Binet formula of Fibonacci numbers, we get

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\mathbb{CH}F_k &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(F_k+F_{k+1}\mathbf{i}+F_{k+2}\mathbf{h}+F_{k+3}\mathbf{ih}\right)\\ &= \frac{1}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\left((-\alpha)^k-(-\beta)^k\right)\\ &\quad+\frac{\mathbf{i}}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\left(\alpha(-\alpha)^k-\beta(-\beta)^k\right)\\ &\quad+\frac{\mathbf{h}}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\left(\alpha^2(-\alpha)^k-\beta^2(-\beta)^k\right)\\ &\quad+\frac{\mathbf{ih}}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\left(\alpha^3(-\alpha)^k-\beta^3(-\beta)^k\right).\end{aligned}$$

By taking account the equalities $\alpha+\beta=1$, $\alpha\beta=-1$ and Definition 1, we obtain

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\mathbb{CH}F_k &= -F_n+F_{n-1}\mathbf{i}-F_{n-2}\mathbf{h}+F_{n-3}\mathbf{ih}\\ &= (-1)^n\mathbb{CH}F_{-n}.\end{aligned}$$

In the same way, the other parts of the theorem can be proved. □

We present the binomial summations of the circular-hyperbolic Lucas quaternions in the following proposition. Because the proof of these summations are similar to the circular-hyperbolic Fibonacci quaternions in above theorem, we omit the proof.

Proposition 1. *For $n\in\mathbb{N}$, the equalities are satisfied:*

i) $\sum_{k=0}^{n}\binom{n}{k}\mathbb{CH}L_k=\mathbb{CH}L_{2n}$,
ii) $\sum_{k=0}^{n}\binom{n}{k}\mathbb{CH}L_{k+1}=\mathbb{CH}L_{2n+1}$,
iii) $\sum_{k=0}^{n}\binom{n}{k}(-1)^k\mathbb{CH}L_k=(-1)^n\mathbb{CH}L_{-n}$,
iv) $\sum_{k=0}^{n}\binom{n}{k}\mathbb{CH}L_{4k}=3^n\mathbb{CH}L_{2n}$,
v) $\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\mathbb{CH}L_{5k}=5^n\mathbb{CH}L_{2n}$,
vi) $\sum_{k=0}^{n}\binom{n}{k}3^{n-k}\mathbb{CH}L_{6k}=8^n\mathbb{CH}L_{2n}$,
vii) $\sum_{k=0}^{n}\binom{n}{k}(-2)^k\mathbb{CH}L_{2k}=(-1)^n\mathbb{CH}L_{3n}$,
viii) $\sum_{k=0}^{n}\binom{n}{k}(-2)^k\mathbb{CH}L_{5k}=(-5)^n\mathbb{CH}L_{3n}$.

We obtain the generalized summations of the $\mathbb{CH}FLQ$ in the following theorem.

Theorem 6. *For $n,m\geq 1$ and $j\geq 0$ integers, the identities are hold:*

i) $\sum_{i=0}^{n-1}\mathbb{CH}F_{mi+j}=\frac{(-1)^m\mathbb{CH}F_{mn+j-m}-\mathbb{CH}F_{mn+j}-(-1)^m\mathbb{CH}F_{j-m}+\mathbb{CH}F_j}{(-1)^m-L_m+1}$,
ii) $\sum_{i=0}^{n-1}\mathbb{CH}L_{mi+j}=\frac{(-1)^m\mathbb{CH}L_{mn+j-m}-\mathbb{CH}L_{mn+j}-(-1)^m\mathbb{CH}L_{j-m}+\mathbb{CH}L_j}{(-1)^m-L_m+1}$.

Proof. We omit Fibonacci case since the proof is quite similar. From the equation (1.7), we write

$$\begin{aligned}\sum_{i=0}^{n-1}\mathbb{CH}L_{mi+j} &= \sum_{i=0}^{n-1}\left(\hat{\alpha}\alpha^{mi+j}+\hat{\beta}\beta^{mi+j}\right)\\ &= \hat{\alpha}\alpha^j\frac{\alpha^{mn}-1}{\alpha^m-1}+\hat{\beta}\beta^j\frac{\beta^{mn}-1}{\beta^m-1}.\end{aligned}$$

By considering the well-known equality $\alpha\beta = -1$ and the Binet formula of Lucas numbers, we get

$$\sum_{i=0}^{n-1} \mathbb{CH}L_{mi+j} = \frac{\hat{\alpha}\left((-1)^m \alpha^{mn+j-m} - \alpha^{mn+j} - \alpha^j\beta^m + \alpha^j\right) + \hat{\beta}\left((-1)^m \beta^{mn+j-m} - \beta^{mn+j} - \alpha^m\beta^j + \beta^j\right)}{(-1)^m - L_m + 1}.$$

From again the equation (1.7), we obtain

$$\sum_{i=0}^{n-1} \mathbb{CH}L_{mi+j} = \frac{(-1)^m \mathbb{CH}L_{mn+j-m} - \mathbb{CH}L_{mn+j} - (-1)^m \mathbb{CH}L_{j-m} + \mathbb{CH}L_j}{(-1)^m - L_m + 1}.$$

□

If we take $m = 1, j = 0$ in Theorem 6, we obtain the following results:

Corollary 6. *The identities are hold:*

i) $\sum_{i=0}^{n-1} \mathbb{CH}F_i = \mathbb{CH}F_{n+1} - \mathbb{CH}F_1,$

ii) $\sum_{i=0}^{n-1} \mathbb{CH}L_i = \mathbb{CH}L_{n+1} - \mathbb{CH}L_1.$

If we take $m = 2, j = 1$ in Theorem 6, we get the following results:

Corollary 7. *The identities are hold:*

i) $\sum_{i=0}^{n-1} \mathbb{CH}F_{2i+1} = \mathbb{CH}F_{2n} - \mathbb{CH}F_0,$

ii) $\sum_{i=0}^{n-1} \mathbb{CH}L_{2i+1} = \mathbb{CH}L_{2n} - \mathbb{CH}L_0.$

If we take $m = 2, j = 0$ in Theorem 6, we obtain the following results:

Corollary 8. *The identities are hold:*

i) $\sum_{i=0}^{n-1} \mathbb{CH}F_{2i} = \mathbb{CH}F_{2n-1} - \mathbb{CH}F_{-1},$

ii) $\sum_{i=0}^{n-1} \mathbb{CH}L_{2i} = \mathbb{CH}L_{2n-1} - \mathbb{CH}L_{-1}.$

Here we acquire the exponential and Poisson generating functions for the $\mathbb{CH}FLQ$.

Theorem 7. *i) The exponential generating function for the circular-hyperbolic Fibonacci quaternions is*

$$\sum_{n=0}^{\infty} \mathbb{CH}F_n \frac{t^n}{n!} = \frac{\hat{\alpha}e^{\alpha t} - \hat{\beta}e^{\beta t}}{\alpha - \beta},$$

ii) The exponential generating function for the circular-hyperbolic Lucas quaternions is

$$\sum_{n=0}^{\infty} \mathbb{CH}L_n \frac{t^n}{n!} = \hat{\alpha}e^{\alpha t} + \hat{\beta}e^{\beta t},$$

iii) The Poisson generating function for the circular-hyperbolic Fibonacci quaternions is

$$\sum_{n=0}^{\infty} \mathbb{CH}F_n \frac{e^{-t}t^n}{n!} = \frac{\hat{\alpha}e^{(\alpha-1)t} - \hat{\beta}e^{(\beta-1)t}}{\alpha - \beta},$$

iv) The Poisson generating function for the circular-hyperbolic Lucas quaternions is

$$\sum_{n=0}^{\infty} \mathbb{CH}L_n \frac{e^{-t}t^n}{n!} = \hat{\alpha}e^{(\alpha-1)t} + \hat{\beta}e^{(\beta-1)t}.$$

Proof. *i)* From the equation (1.6) and the MacLaurin expansion for the exponential function, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{CH}F_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(\frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta}\right) \frac{t^n}{n!} \\
&= \frac{\hat{\alpha}}{\alpha - \beta} \sum_{n=0}^{\infty} \frac{\alpha^n t^n}{n!} - \frac{\hat{\beta}}{\alpha - \beta} \sum_{n=0}^{\infty} \frac{\beta^n t^n}{n!} \\
&= \frac{\hat{\alpha}e^{\alpha t} - \hat{\beta}e^{\beta t}}{\alpha - \beta}.
\end{aligned}$$

ii) By using the equation (1.7) and the MacLaurin expansion for the exponential function, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{CH}L_n\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\hat{\alpha}\alpha^n+\hat{\beta}\beta^n\right)\frac{t^n}{n!}\\ &= \hat{\alpha}\sum_{n=0}^{\infty}\frac{\alpha^n t^n}{n!}+\hat{\beta}\sum_{n=0}^{\infty}\frac{\beta^n t^n}{n!}\\ &= \hat{\alpha}e^{\alpha t}+\hat{\beta}e^{\beta t}.\end{aligned}$$

iii) From the equation (1.6) and the MacLaurin expansion for the exponential function, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{CH}F_n\frac{e^{-t}t^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{\hat{\alpha}\alpha^n-\hat{\beta}\beta^n}{\alpha-\beta}\right)\frac{e^{-t}t^n}{n!}\\ &= \frac{\hat{\alpha}e^{(\alpha-1)t}-\hat{\beta}e^{(\beta-1)t}}{\alpha-\beta}.\end{aligned}$$

iv) By using the equation (1.7) and the MacLaurin expansion for the exponential function, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{CH}L_n\frac{e^{-t}t^n}{n!} &= \sum_{n=0}^{\infty}\left(\hat{\alpha}\alpha^n+\hat{\beta}\beta^n\right)\frac{e^{-t}t^n}{n!}\\ &= \hat{\alpha}e^{(\alpha-1)t}+\hat{\beta}e^{(\beta-1)t}.\end{aligned}$$

□

Conclusion

In this paper, the $\mathbb{CH}FLQ$ have been investigated. Many of the properties of these quaternions are proved by the fundamental algebraic operations and simple matrix algebra. Actually, the results presented here have the potential to motivate further studies of the subject of the circular-hyperbolic Horadam quaternions including the $\mathbb{CH}FLQ$.

References

[1] Adler S.L., Quaternionic quantum mechanics and quantum fields, New York: Oxford University Press, (1994).

[2] Aydın F.T., Circular-hyperbolic Fibonacci quaternions, Notes on Number Theory and Discrete Mathematics, 26(2), 167-176, (2020).

[3] Catoni F., Boccaletti D., Cannata R., Catoni V., Zampetti P., Hyperbolic Numbers in Geometry of Minkowski Space-Time, Springer, Heidelberg, pp.3-23, (2011).

[4] Cihan A., Azak A.Z., Güngör M.A., Tosun M., A study of Dual Hyperbolic Fibonacci and Lucas numbers, An. St. Univ. Ovidius Constanta, 27(1), 35–48, (2019).

[5] Dattoli G., Licciardi S., Pidatella R.M., Sabia E., Hybrid complex numbers: The matrix version, Adv. Appl. Clifford Algebras, 28(3), 58, (2018).

[6] Dixon G.M., Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics, Kluvwer Academic Publishers, ISBN 0-7923-2890-6, (1994).

[7] Gargoubi H., Kossentini S., $f-$algebra structure on hyperbolic numbers, Adv. Appl. Clifford Algebras, 26(4), 1211–1233, (2016).

[8] Güngör M.A. , Azak A.Z., Investigation of dual complex Fibonacci, dual complex Lucas numbers and their properties, Advances in Applied Clifford Algebras, 27(4), 3083–3096, (2017).

[9] Halıcı S., On Fibonacci quaternions, Advances in Applied Clifford Algebras, 22, 321-327, (2012).

[10] Hamilton W.R., Elements of Quaternions, Longmans, Green and Co., London, (1866).

[11] Horadam A.F., Complex Fibonacci Numbers and Fibonacci Quaternions, Amer. Math. Monthly, 70, 289-291, (1963).

[12] Horadam A.F., Quaternion recurrence relations, Ulam Quarterly, 2, 22-33, (1993).

[13] Irmak N., More identities for Fibonacci and Lucas quaternions, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 69(1), 369-375, (2020).

[14] Iyer M.R., A note on Fibonacci quaternions, Fibonacci Quarterly, 3, 225-229, (1969).

[15] Iyer M.R., Some results on Fibonacci quaternions, Fibonacci Quarterly, 7, 201-210, (1969).

[16] Koshy T., Fibonacci and Lucas Numbers with Applications, John Wiley and Sons Inc., NY, (2001).

[17] Motter A.E., Rosa A.F., Hyperbolic calculus, Adv. Appl. Clifford Algebras, 8(1), 109–128, (1998).

[18] Nurkan S.K., Güven I.A., Dual Fibonacci Quaternions, Adv. Appl. Clifford Algebras, 25(2), 403–414, (2015).

[19] Nurkan S.K., Güven I.A., A New Approach to Fibonacci, Lucas numbers and dual vectors, Adv. Appl. Clifford Algebras, 25(3), 577–590, (2015).

[20] Ollerton R.L., Shannon A.G., An extension of circular and hyperbolic functions, International Journal of Mathematical Education in Science and Technology, 23(4), 611–635, (1992).

[21] Şentürk T.D., Bilgici G., Daşdemir A., Ünal Z., A Study on Horadam Hybrid Numbers, Turkish Journal of Mathematics, 44(4), 1212-1221, (2020).

[22] Ward J.P., Quaternions and Cayley Numbers: Algebra and Applications, Springer Science and Business Media, (1997).