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Family of Surfaces with a Common Special Involute and Evolute Curves

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ABSTRACT

In this paper we define the necessary and sufficient conditions for both the involute and evolute of a given curve to be geodesic, asymptotic and curvature line on a parametric surface. Then, the first and second fundamental forms of these surfaces are calculated. By using the Gaussian and mean curvatures, the developability and minimality assumptions are drawn, as well. Moreover we extended the idea to the ruled surfaces. Finally, we provide a set of examples to illustrate the corresponding surfaces.

Keywords: *Geodesic curve, involue-evolute curves, asymptoticity, curvature line, ruled surfaces.*

AMS Subject Classification (2020): *Primary: 53Axx ; Secondary: 53A04; 53A05.*

1. Introduction and Preliminaries

In differential geometry, there are numerous studies covering the curve theory. Especially, researchers put forth new aspects on curves by establishing some connections with the Frenet frames of the corresponding points. These aspects are discussed with different frames in different spaces, as well such as Minkowski, Galileo, Heisenberg and Dual space. Involute-evolute curves are the ones constituted by this manner and have a great potential of use in industrial area especially for gear part designs. Up to now, much of work have been done in the literature about these curves (see, [6], [7], [8], [9]).
The theory of surfaces, on the other hand, is another important subject in differential geometry. Researchers before are always focused on some special curves such as geodesic or asymptotic on a given surface (see, [12], [14], [17],[18]), however, for the first time Wang et al. (2004) approached the problem in reverse and proposed the construction of a surface that has a given curve as a geodesic [16]. This approach appealed Li et al. (2011) and they characterized the parametric forms of surfaces having any given curve as a line of curvature [10]. It was Bayram et al. (2012) who followed the same idea and constructed the parametric form of surfaces with a common asymptotic curve [4]. There have been other studies characterizing surfaces on which a given specific curve lies on as geodesic, asymptotic and line of curvature ([1], [2], [3], [5]). Motivated by these, we present the necessary and sufficient conditions to formulate a family of surfaces having both the involute and evolute curves as of each geodesic, asymptotic and curvature line. We established ruled surfaces, as well and examined the developability and minimality conditions for these surfaces.

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Given a curve α in Euclidean space, E^3, the Frenet vectors and curvatures at the point $\alpha(s)$ are defined as [11]:

$$T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|}, \ B(s) = \frac{\alpha'(s) \times \alpha''(s)}{\|\alpha'(s) \times \alpha''(s)\|}, \ N(s) = B(s) \times T(s), \tag{1.1}$$

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{\|\alpha' \times \alpha''\|^2}, \tag{1.2}$$

$$T' = \kappa\nu N, \ N' = -\kappa\nu T + \tau\nu B, \ B' = -\tau\nu N \ , \ \|\alpha'\| = \nu. \tag{1.3}$$

Let α and β are defined to be two curves sharing the same domain. If the tangent of α at the point $\alpha(s)$ is passing through the point $\beta(s)$ and is perpendicular to the tangent of β at this point, then we name β as the involute of α [13]. Now denote $\{T, N, B, \kappa, \tau\}$ and $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ as the Frenet apparatus of α and β, respectively. Hence we have the following relations:

$$\beta(s) = \alpha(s) + \lambda(s)T(s), \ \lambda(s) = k - s, \ k \in \mathbb{R}, \tag{1.4}$$

$$T_1 = N, \qquad N_1 = -\frac{\kappa T + \tau B}{\sqrt{\kappa^2 + \tau^2}}, \quad B_1 = \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}}, \tag{1.5}$$

$$\kappa_1 = \frac{\sqrt{(\kappa^2 + \tau^2)(s)}}{(k - s)\kappa(s)}, \quad \tau_1 = \frac{(\kappa\tau' - \kappa'\tau)(s)}{(k - s)\kappa(s)(\kappa^2 + \tau^2)(s)}. \tag{1.6}$$

On the other hand, the curve γ is said to be the evolute of α, if it is traced out with the points of centers of curvatures of α [13]. When denoted $\{T_2, N_2, B_2, \kappa_2, \tau_2\}$ as the Frenet apparatus of γ, this time we have the following relations

$$\gamma(s) = \alpha(s) + \rho(s)N(s) - \rho(s) \tan(\varphi(s) + c) B(s), \tag{1.7}$$

$$T_2 = \cos(\varphi + c)N - \sin(\varphi + c)B, \qquad N_2 = -T, \qquad B_2 = \sin(\varphi + c)N + \cos(\varphi + c)B, \tag{1.8}$$

$$\kappa_2 = \frac{\kappa^3 \cos^3(\varphi + c)}{\kappa\tau \sin(\varphi + c) - \kappa' \cos(\varphi + c)}, \qquad \tau_2 = \frac{-\kappa^3 \sin(\varphi + c)\cos^2(\varphi + c)}{\kappa\tau \sin(\varphi + c) - \kappa' \cos(\varphi + c)}, \tag{1.9}$$

where $\varphi(s) + c = \int\limits_0^s \tau(u)du, \quad c \in \mathbb{R}.$

Let α be a regular curve on the surface, $\xi = \xi(s, v)$. If the vector α' is asymptotic vector of the surface, then α is called as asymptotic curve, if it is always in a principal curvature direction then it is called as line of curvature. Moreover, if the normal vector of the surface, n is parallel to the vector α'', then α is known to be geodesic on the surface [6]. On the other hand, for any constant number v_0, if $\alpha(s) = \xi(s, v_0)$, then this time α is called as isoparametric. Now, if a curve is both geodesic (or asymptotic) and isoparametric, then it is named as isogeodesic (or isoasymptotic) [16], [10], [4].

The first and the second fundamental forms of a surface are calculated by

$$\begin{aligned} I &= Eds^2 + 2Fdsdv + Gdv^2, \\ II &= Lds^2 + Mdsdv + Ndv^2 \end{aligned} \tag{1.10}$$

and the Gaussian and the mean curvatures are formulated as

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2} \tag{1.11}$$

where

$$\begin{aligned} E &= \langle \xi_s, \xi_s \rangle, \mathrm{F} = \langle \xi_s, \xi_v \rangle, \mathrm{G} = \langle \xi_v, \xi_v \rangle, \\ L &= \langle \xi_{ss}, n \rangle, \mathrm{M} = \langle \xi_{sv}, n \rangle, \mathrm{N} = \langle \xi_{vv}, n \rangle. \end{aligned} \tag{1.12}$$

If $\xi(s,v)$ is a ruled surface then the parametric form can be written as

$$\xi(s,v) = \alpha(s) + vx(s) \tag{1.13}$$

where α is known as generator and x as the director [15]. A ruled surface is developable iff $\det(\alpha', x, x') = 0$ [15].

2. Family of surfaces with a common involute-evolute curves

2.1. Family of surfaces with a common involute curve as geodesic, asymptotic and curvature line

Let $\xi(s,v)$ be a surface on which the involute curve, β lies. The parametric form of the surface can be written as

$$\xi(s,v) = \beta(s) + x(s,v)T_1(s) + y(s,v)N_1(s) + z(s,v)B_1(s) \tag{2.1}$$

where $x(s,v)$, $y(s,v)$, $z(s,v)$ are differentiable marching scale functions.

Theorem 2.1. *Let β be the involute of α. β is isogeodesic on the surface $\xi(s,v)$ iff*

$$x(s,v_0) = y(s,v_0) = z(s,v_0) = \left.\frac{\partial y(s,v)}{\partial v}\right|_{v=v_0} = 0, \quad \left.\frac{\partial z(s,v)}{\partial v}\right|_{v=v_0} \neq 0,$$

and is asymptotic iff

$$x(s,v_0) = y(s,v_0) = z(s,v_0) = \left.\frac{\partial z(s,v)}{\partial v}\right|_{v=v_0} = 0\,, \quad \left.\frac{\partial y(s,v)}{\partial v}\right|_{v=v_0} \neq 0.$$

Proof. Since β is parametric on the surface $\xi(s,v)$, for a constant $v = v_0$, we have

$$x\,(s,v_0) = y\,(s,v_0) = z\,(s,v_0) = 0. \tag{2.2}$$

On the other hand, we calculate the normal of the surface, n_1 as

$$n_1(s,v_0) = \frac{\partial \xi(s,v)}{\partial s} \times \frac{\partial \xi(s,v)}{\partial v} = -\left.\frac{\partial z(s,v)}{\partial v}\right|_{v=v_0} N_1(s) + \left.\frac{\partial y(s,v)}{\partial v}\right|_{v=v_0} B_1(s). \tag{2.3}$$

Now, if the curve β is defined to be geodesic on the surface, then from the geodesicity condition we have

$$\left.\frac{\partial y(s,v)}{\partial v}\right|_{v=v_0} = 0, \quad \left.\frac{\partial z(s,v)}{\partial v}\right|_{v=v_0} \neq 0.$$

Similarly, if β is characterized as asymptotic on the surface this time we write

$$\left.\frac{\partial z(s,v)}{\partial v}\right|_{v=v_0} = 0, \qquad \left.\frac{\partial y(s,v)}{\partial v}\right|_{v=v_0} \neq 0,$$

which clearly completes the proof. □

Theorem 2.2. *Let alpha be a unit speed space curve and β be its involute. The curve β is a line of curvature on the surface iff*

$$x(s,v_0) = y(s,v_0) = z(s,v_0) = 0, \quad \theta_1(s) = -\int \tau_1 ds,, \quad \mu_1(s) \neq 0$$
$$\left.\frac{\partial y(s,v)}{\partial v}\right|_{v=v_0} = \mu_1(s)\sin\theta_1(s), \qquad \left.\frac{\partial z(s,v)}{\partial v}\right|_{v=v_0} = -\mu_1(s)\cos\theta_1(s).$$

Proof. Let η_1 is defined to be the orthogonal vector field of the surface. Therefore we can write

$$\eta_1(s) = \cos\theta_1(s)N_1(s) + \sin\theta_1(s)B_1(s).$$

where $\theta_1 = \measuredangle(N_1, \eta_1)$. The necessary and sufficient condition for β to be a curvature line on the surface, $\xi = \xi(s,v)$ is two fold. First that $\eta_1(s) \parallel n_1(s, v_0)$ and second the surface defined as

$$\Phi(s,v) = \beta(s) + t\eta_1(s)$$

must be developable. For $\eta_1(s) \parallel n_1(s, v_0)$ we write,

$$\frac{\partial y(s,v)}{\partial v}\bigg|_{v=v_0} = \mu_1(s)\sin\theta_1(s), \qquad \frac{\partial z(s,v)}{\partial v}\bigg|_{v=v_0} = -\mu_1(s)\cos\theta_1(s).$$

On the other hand, in order for $\Phi = \Phi(s,v)$ to be developable, $\theta_1(s) = -\int \tau_1(s)ds$. □

Theorem 2.3. *The first and second fundamental forms and the Gaussian and mean curvatures of the surface $\xi = \xi(s,v)$ having the involute curve β are given as*

$$\begin{aligned}
I &= ds^2 + 2\frac{\partial x}{\partial v}dsdv + \left(\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right)dv^2,\\
II &= -\kappa_1\frac{\partial z}{\partial v}ds^2 + \left(-\kappa_1\frac{\partial x}{\partial v}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial^2 y}{\partial s\partial v} + \tau_1\left(\frac{\partial y}{\partial v}\right)^2 + \tau_1\left(\frac{\partial z}{\partial v}\right)^2 + \frac{\partial y}{\partial v}\frac{\partial^2 z}{\partial s\partial v}\right)dsdv\\
&\quad + \left(-\frac{\partial z}{\partial v}\frac{\partial^2 y}{\partial^2 v} + \frac{\partial y}{\partial v}\frac{\partial^2 z}{\partial^2 v}\right)dv^2,\\
K &= \frac{\kappa_1\left(\frac{\partial z}{\partial v}\right)^2\frac{\partial^2 y}{\partial^2 v} - \kappa_1\frac{\partial y}{\partial v}\frac{\partial z}{\partial v}\frac{\partial^2 z}{\partial^2 v} - \left(-\kappa_1\frac{\partial x}{\partial v}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial^2 y}{\partial s\partial v} + \tau_1\left(\frac{\partial y}{\partial v}\right)^2 + \tau_1\left(\frac{\partial z}{\partial v}\right)^2 + \frac{\partial y}{\partial v}\frac{\partial^2 z}{\partial s\partial v}\right)^2}{\left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2},\\
H &= \frac{\begin{array}{l}-\frac{\partial z}{\partial v}\frac{\partial^2 y}{\partial^2 v} + \frac{\partial y}{\partial v}\frac{\partial^2 z}{\partial^2 v} - 2\frac{\partial x}{\partial v}\left(-\kappa_1\frac{\partial x}{\partial v}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial^2 y}{\partial s\partial v} + \tau_1\left(\frac{\partial y}{\partial v}\right)^2 + \tau_1\left(\frac{\partial z}{\partial v}\right)^2 + \frac{\partial y}{\partial v}\frac{\partial^2 z}{\partial s\partial v}\right)\\ -\kappa_1\frac{\partial z}{\partial v}\left(\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right)\end{array}}{2\left(\left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right)}.
\end{aligned}$$

Proof. From the equations given in (1.12) we write,

$$\begin{aligned}
E &= \langle\xi_s, \xi_s\rangle = 1, \quad G = \langle\xi_v, \xi_v\rangle = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2, \quad F = \langle\xi_s, \xi_v\rangle = \frac{\partial x}{\partial v},\\
L &= \langle\xi_{ss}, \xi_s \times \xi_v\rangle = -\kappa_1\frac{\partial z}{\partial v}, \quad N = \langle\xi_{vv}, \xi_s \times \xi_v\rangle = -\frac{\partial z}{\partial v}\frac{\partial^2 y}{\partial^2 v} + \frac{\partial y}{\partial v}\frac{\partial^2 z}{\partial^2 v},\\
M &= \langle\xi_{sv}, \xi_s \times \xi_v\rangle = -\kappa_1\frac{\partial x}{\partial v}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial^2 y}{\partial s\partial v} + \tau_1\left(\frac{\partial y}{\partial v}\right)^2 + \tau_1\left(\frac{\partial z}{\partial v}\right)^2 + \frac{\partial y}{\partial v}\frac{\partial^2 z}{\partial s\partial v}.
\end{aligned}$$

When substituted these relations in (1.10) and (1.11) we complete the proof. □

Corollary 2.1.

- *In order for $\xi = \xi(s,v)$ having β as isogeodesic to be developable is that*

$$\kappa_1\frac{\partial^2 y}{\partial^2 v} = \left(\kappa_1\frac{\partial x}{\partial v} - \tau_1\frac{\partial z}{\partial v}\right)^2.$$

- *In order for $\xi = \xi(s,v)$ having β as isoasymptotic to be developable is that*

$$\tau_1 = 0.$$

Corollary 2.2.

- *In order for* $\xi = \xi(s,v)$ *having* β *as isogeodesic to be minimal is that*

$$\frac{\partial^2 y}{\partial^2 v} + 2\tau_1 \frac{\partial x}{\partial v}\frac{\partial z}{\partial v} = \kappa_1 \left(\left(\frac{\partial x}{\partial v}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2 \right).$$

- *In order for* $\xi = \xi(s,v)$ *having* β *as isoasymptotic to be minimal is that*

$$\frac{\partial x}{\partial v} = 0.$$

2.2. Family of surfaces with a common evolute curve as geodesic, asymptotic and curvature line

Let $\psi(s,v)$ be a surface possessing the evolute curve, γ of the curve α. The parametric form of this surface is given as

$$\psi(s,v) = \gamma(s) + \Delta_1(s,v)T_2(s) + \Delta_2(s,v)N_2(s) + \Delta_3(s,v)B_2(s). \tag{2.4}$$

Here we name $\Delta_1(s,v)$, $\Delta_2(s,v)$, $\Delta_3(s,v)$ as differentiable marching scale functions.

Theorem 2.4. *The evolute curve,* γ *of* α *is isogeodesic on the surface* $\psi(s,v)$*, iff*

$$\Delta_1(s,v_0) = \Delta_2(s,v_0) = \Delta_3(s,v_0) = \left.\frac{\partial \Delta_2(s,v)}{\partial v}\right|_{v=v_0} = 0, \quad \left.\frac{\partial \Delta_3(s,v)}{\partial v}\right|_{v=v_0} \neq 0,$$

and is isoasymptotic on the surface $\psi(s,v)$*, iff*

$$\Delta_1(s,v_0) = \Delta_2(s,v_0) = \Delta_3(s,v_0) = \left.\frac{\partial \Delta_3(s,v)}{\partial v}\right|_{v=v_0} = 0, \quad \left.\frac{\partial \Delta_2(s,v)}{\partial v}\right|_{v=v_0} \neq 0\,.$$

Proof. Since γ is parametric on the surface $\psi(s,v)$, for a constant $v = v_0$ we get

$$\Delta_1(s,v_0) = \Delta_2(s,v_0) = \Delta_3(s,v_0) = 0.$$

□

The normal of the surface denoted as n_2 is calculated as

$$n_2(s,v_0) = \frac{\partial \psi(s,v)}{\partial s} \times \frac{\partial \psi(s,v)}{\partial v} = -\left.\frac{\partial \Delta_3(s,v)}{\partial v}\right|_{v=v_0} N_2(s) + \left.\frac{\partial \Delta_2(s,v)}{\partial v}\right|_{v=v_0} B_2(s).$$

Now, if γ is geodesic on the surface we write,

$$\left.\frac{\partial \Delta_2(s,v)}{\partial v}\right|_{v=v_0} = 0, \quad \left.\frac{\partial \Delta_3(s,v)}{\partial v}\right|_{v=v_0} \neq 0.$$

On the other hand, if γ is asymptotic, we get

$$\left.\frac{\partial \Delta_3(s,v)}{\partial v}\right|_{v=v_0} = 0, \quad \left.\frac{\partial \Delta_2(s,v)}{\partial v}\right|_{v=v_0} \neq 0.$$

Theorem 2.5. *Let* γ *be the evolute of the unit speed curve* α*. The curve* γ *is a line of curvature on the surface iff*

$$\Delta_1(s,v_0) = \Delta_2(s,v_0) = \Delta_3(s,v_0) = 0, \quad \theta_2(s) = -\int \tau_2(s)ds, \quad \mu_2(s) \neq 0,$$
$$\left.\frac{\partial \Delta_2(s,v)}{\partial v}\right|_{v=v_0} = \mu_2(s)\sin\theta_2(s), \qquad \left.\frac{\partial \Delta_3(s,v)}{\partial v}\right|_{v=v_0} = -\mu_2(s)\cos\theta_2(s).$$

Proof. Let us define η_2 to be the orthogonal vector field of the surface which γ lies on it. Therefore, for $\theta_2 = \measuredangle(N_2, \eta_2)$, we can write

$$\eta_2(s) = \cos\theta_2(s)N_2(s) + \sin\theta_2(s)B_2(s).$$

□

This time, the necessary and sufficient conditions for γ to be a curvature line on the surface, $\psi = \psi(s, v)$ are that $\eta_2(s) \parallel n_2(s, v_0)$ and that the surface defined as $\Gamma(s, v) = \gamma(s) + t\eta_2(s)$ must be developable. For $\eta_2(s) \parallel n_2(s, v_0)$ we get,

$$\left.\frac{\partial y(s,v)}{\partial v}\right|_{v=v_0} = \mu_1(s)\sin\theta_1(s), \qquad \left.\frac{\partial z(s,v)}{\partial v}\right|_{v=v_0} = -\mu_1(s)\cos\theta_1(s).$$

On the other hand, in order for $\Gamma = \Gamma(s, v)$ to be developable, we have $\theta_2(s) = -\int \tau_2(s)ds$.

Theorem 2.6. *The first, I and second, II fundamental forms and the Gaussian, K and mean, H curvatures of the surface $\psi = \psi(s, v)$ having the involute curve γ are given as*

$$\begin{aligned}
I &= ds^2 + 2\frac{\partial\Delta_1}{\partial v}dsdv + \left(\left(\frac{\partial\Delta_1}{\partial v}\right)^2 + \left(\frac{\partial\Delta_2}{\partial v}\right)^2 + \left(\frac{\partial\Delta_3}{\partial v}\right)^2\right)dv^2,\\
II &- \kappa_2\frac{\partial\Delta_3}{\partial v}ds^2 + \left(-\kappa_2\frac{\partial\Delta_1}{\partial v}\frac{\partial\Delta_3}{\partial v} - \frac{\partial\Delta_3}{\partial v}\frac{\partial^2\Delta_2}{\partial s\partial v} + \tau_2\left(\frac{\partial\Delta_2}{\partial v}\right)^2 + \tau_2\left(\frac{\partial\Delta_3}{\partial v}\right)^2 + \frac{\partial\Delta_2}{\partial v}\frac{\partial^2\Delta_3}{\partial s\partial v}\right)dsdv\\
&+ \left(-\frac{\partial\Delta_3}{\partial v}\frac{\partial^2\Delta_2}{\partial^2 v} + \frac{\partial\Delta_2}{\partial v}\frac{\partial^2\Delta_3}{\partial^2 v}\right)dv^2,\\
K &= \frac{\kappa_2\left(\frac{\partial\Delta_3}{\partial v}\right)^2\frac{\partial^2\Delta_2}{\partial^2 v} - \kappa_2\frac{\partial\Delta_2}{\partial v}\frac{\partial\Delta_3}{\partial v}\frac{\partial^2\Delta_3}{\partial^2 v} - \left(-\kappa_2\frac{\partial\Delta_1}{\partial v}\frac{\partial\Delta_3}{\partial v} - \frac{\partial\Delta_3}{\partial v}\frac{\partial^2\Delta_2}{\partial s\partial v} + \tau_2\left(\frac{\partial\Delta_2}{\partial v}\right)^2 + \tau_2\left(\frac{\partial\Delta_3}{\partial v}\right)^2 + \frac{\partial\Delta_2}{\partial v}\frac{\partial^2\Delta_3}{\partial s\partial v}\right)^2}{\left(\frac{\partial\Delta_2}{\partial v}\right)^2 + \left(\frac{\partial\Delta_3}{\partial v}\right)^2},\\
H &= \frac{\begin{array}{c}-\frac{\partial\Delta_3}{\partial v}\frac{\partial^2\Delta_2}{\partial^2 v} + \frac{\partial\Delta_2}{\partial v}\frac{\partial^2\Delta_3}{\partial^2 v} - 2\frac{\partial\Delta_1}{\partial v}\left(-\kappa_2\frac{\partial\Delta_1}{\partial v}\frac{\partial\Delta_3}{\partial v} - \frac{\partial\Delta_3}{\partial v}\frac{\partial^2\Delta_2}{\partial s\partial v} + \tau_2\left(\frac{\partial\Delta_2}{\partial v}\right)^2 + \tau_2\left(\frac{\partial\Delta_3}{\partial v}\right)^2 + \frac{\partial\Delta_2}{\partial v}\frac{\partial^2\Delta_3}{\partial s\partial v}\right)\\ -\kappa_2\frac{\partial\Delta_3}{\partial v}\left(\left(\frac{\partial\Delta_1}{\partial v}\right)^2 + \left(\frac{\partial\Delta_2}{\partial v}\right)^2 + \left(\frac{\partial\Delta_3}{\partial v}\right)^2\right)\end{array}}{2\left(\left(\frac{\partial\Delta_2}{\partial v}\right)^2 + \left(\frac{\partial\Delta_3}{\partial v}\right)^2\right)}.
\end{aligned}$$

Proof. By using (1.12) we first calculate the corresponding coefficients as,

$$\begin{aligned}
&E = \langle\xi_s, \xi_s\rangle = 1, \quad G = \langle\xi_v, \xi_v\rangle = \left(\frac{\partial\Delta_1}{\partial v}\right)^2 + \left(\frac{\partial\Delta_2}{\partial v}\right)^2 + \left(\frac{\partial\Delta_3}{\partial v}\right)^2, \quad F = \langle\xi_s, \xi_v\rangle = \frac{\partial\Delta_1}{\partial v},\\
&L = \langle\xi_{ss}, \xi_s \times \xi_v\rangle = -\kappa_2\frac{\partial\Delta_3}{\partial v}, \quad N = \langle\xi_{vv}, \xi_s \times \xi_v\rangle = -\frac{\partial\Delta_3}{\partial v}\frac{\partial^2\Delta_2}{\partial^2 v} + \frac{\partial\Delta_2}{\partial v}\frac{\partial^2\Delta_3}{\partial^2 v},\\
&M = \langle\xi_{sv}, \xi_s \times \xi_v\rangle = -\kappa_2\frac{\partial\Delta_1}{\partial v}\frac{\partial\Delta_3}{\partial v} - \frac{\partial\Delta_3}{\partial v}\frac{\partial^2\Delta_2}{\partial s\partial v} + \tau_2\left(\frac{\partial\Delta_2}{\partial v}\right)^2 + \tau_2\left(\frac{\partial\Delta_3}{\partial v}\right)^2 + \frac{\partial\Delta_2}{\partial v}\frac{\partial^2\Delta_3}{\partial s\partial v}.
\end{aligned}$$

Substituting these coefficients in (1.10) and (1.11) completes the proof. □

Corollary 2.3.

- *In order for $\psi = \psi(s, v)$ possessing γ as isogeodesic to be developable is that*

$$\kappa_2\frac{\partial^2\Delta_2}{\partial^2 v} = \left(\kappa_2\frac{\partial\Delta_1}{\partial v} - \tau_2\frac{\partial\Delta_3}{\partial v}\right)^2.$$

- *In order for $\psi = \psi(s, v)$ possessing γ as isoasymptotic to be developable is that*

$$\tau_2 = 0.$$

Corollary 2.4.

- *In order for $\psi = \psi(s,v)$ possessing γ as isogeodesic to be minimal is that*

$$\frac{\partial^2 \Delta_2}{\partial^2 v} + 2\tau_2 \frac{\partial \Delta_1}{\partial v}\frac{\partial \Delta_3}{\partial v} = \kappa_2 \left(\left(\frac{\partial \Delta_1}{\partial v}\right)^2 - \left(\frac{\partial \Delta_3}{\partial v}\right)^2 \right).$$

- *In order for $\psi = \psi(s,v)$ possessing γ as isoasymptotic to be minimal is that*

$$\frac{\partial \Delta_1}{\partial v} = 0.$$

2.3. Family of ruled surfaces with a common involute curve as geodesic and asymptotic

The parametric form of a ruled surface having the involute curve, β as the basis can be written by following

$$\xi(s,v) = \beta(s) + (v - v_0)R(s). \tag{2.5}$$

By referring (2.1) and (2.5), we have

$$x\,(s,v)\,\mathrm{T}_1(\mathrm{s}) + y(s,v)N_1(\mathrm{s}) + z(s,v)\mathrm{B}_1(v - v_0))\mathrm{R}(\mathrm{s}). \tag{2.6}$$

Now, the inner product of the (2.6) with T_1, N_1 and B_1 results

$$x(s,v) = (v - v_0)\left\langle R, T_1 \right\rangle, \;\; y(s,v) = (v - v_0)\left\langle R, N_1 \right\rangle, \quad z(s,v) = (v - v_0)\left\langle R, B_1 \right\rangle. \tag{2.7}$$

By taking into account the theorem 2.1 in this last relation (2.7), we define the family of ruled surfaces with a common geodesic involute curve as

$$\xi_{igr}(s,v) = \beta(s) + (v - v_0)\left(x(s)T_1(s) + z(s)B_1(s)\right), z(s) \neq 0, \tag{2.8}$$

and with a common asymptotic involute curve as

$$\xi_{iar}(s,v) = \beta(s) + (v - v_0)\left(x(s)T_1(s) + y(s)N_1(s)\right), \tag{2.9}$$

respectively.

Corollary 2.5. *The conditions for a ruled surface with a common involute curve as "geodesic" and "asymptotic" to be developable are that* $\frac{\tau_1(s)}{\kappa_1(s)} = \frac{x(s)}{z(s)}$ *and* $y(s) = 0$*, respectively.*

Corollary 2.6. *For all ruled surfaces with a common involute curve as asymptotic specify tangent ruled surfaces, and the striction curve of these ruled surfaces is the involute curve,* β*.*

2.4. Family of ruled surfaces with a common evolute curve as geodesic and asymptotic

Similarly, a ruled surface having the evolute curve, γ as the basis can be parameterized by following

$$\psi(s,v) = \gamma(s) + (v - v_0)R(s). \tag{2.10}$$

By referring (2.4) and (2.10), we have

$$\Delta_1\,(s,v)\,\mathrm{T}_2(\mathrm{s}) + \Delta_2(s,v)N_2(\mathrm{s}) + \Delta_3(s,v)\mathrm{B}_2(\mathrm{s}) = (\mathrm{v}{-}\mathrm{v}_0)\mathrm{R}(\mathrm{s}). \tag{2.11}$$

Now, the inner product of the (2.11) with T_2, N_2 and B_2 results

$$\Delta_1(s,v) = (v - v_0)\left\langle R, T_2 \right\rangle, \;\; \Delta_2(s,v) = (v - v_0)\left\langle R, N_2 \right\rangle, \quad \Delta_3(s,v) = (v - v_0)\left\langle R, B_2 \right\rangle. \tag{2.12}$$

By the theorem 2.4 and this last relation (2.12), we form the family of ruled surfaces with a common involute curve as geodesic as

$$\psi_{igr}(s,v) = \gamma(s) + (v - v_0)\left(f(s)T_2(s) + h(s)B_2(s)\right), \mathrm{h}(s) \neq 0, \tag{2.13}$$

and as asymptotic as

$$\psi_{iar}(s,v) = \gamma(s) + (v - v_0)\left(f(s)T_2(s) + g(s)N_2(s)\right), \tag{2.14}$$

respectively.

Corollary 2.7. *The conditions for a ruled surface with a common evolute curve as "geodesic" and "asymptotic" to be developable are that* $\dfrac{\tau_2(s)}{\kappa_2(s)} = \dfrac{f(s)}{h(s)}$ *and* $g(s) = 0$*, respectively.*

Corollary 2.8. *For all ruled surfaces with a common evolute curve as asymptotic specify tangent ruled surfaces, and the striction curve of these ruled surfaces is the evolute curve,* γ*.*

Example 2.1. Let us take the unit speed curve $\alpha(s) = \dfrac{1}{\sqrt{2}}(-\cos s, -\sin s, s)$ lying on the surface $\zeta(s, v) = \dfrac{1}{\sqrt{2}}(-\cos s, -\sin s, s + v)$. For k=c=2, the involute and the evolute curves of α are given as

$$\begin{aligned}
\beta(s) &= \left(-\frac{\sqrt{2}}{2}\cos s + \frac{\sqrt{2}}{2}(2-s)\sin s,\ -\frac{\sqrt{2}}{2}\sin s - \frac{\sqrt{2}}{2}(2-s)\cos s,\ \sqrt{2}\right),\\
\gamma(s) &= \left(\frac{\sqrt{2}}{2}\cos s + \sin s \tan\left(\frac{\sqrt{2}}{2}s + 2\right),\ \frac{\sqrt{2}}{2}\sin s - \cos s \tan\left(\frac{\sqrt{2}}{2}s + 2\right),\right.\\
&\qquad \left.\frac{\sqrt{2}}{2}s - \tan\left(\frac{\sqrt{2}}{2}s + 2\right)\right),
\end{aligned}$$

respectively.
Note that throughout the examples, we have defined different surfaces by manipulating the marching scale functions with respect to the conditions defined in the paper. Moreover, in the following figures, we have fixed the colors for the curves α, β and γ with yellow, red and blue, respectively. For the surfaces, ζ, ξ and ψ we have assigned green, purple and orange in respective order for the sake of simplicity, as well. The range of surface parameters, s and v are also fixed as $-\pi \leq s \leq \pi,\ \ -1 \leq v \leq 1$.

(i) By choosing the marching scale functions as

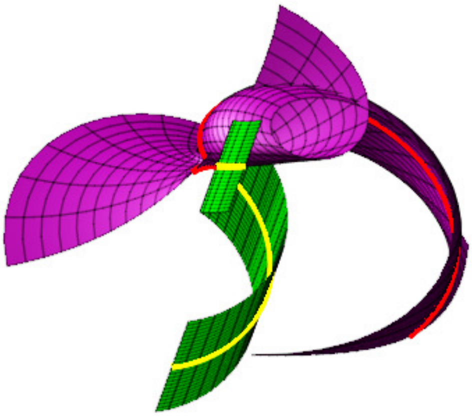
$$x(s,v) = sv = \Delta_1(s,v),\quad y(s,v) = v^2 = \Delta_2(s,v),\quad z(s,v) = \sin v\ = \Delta_3(s,v)\ \ ve\ \ v_0 = 0$$

we calculate the parametric form for the surface with a common geodesic involute curve as

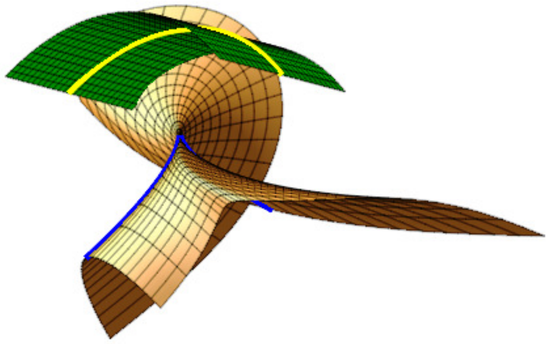
$$\xi_{1g}(s,v) = \begin{pmatrix} -\frac{\sqrt{2}}{2}\left(s\sin s - 2\sin s + \cos s\right) + sv\cos s - v^2\sin s, \\[2ex] \frac{\sqrt{2}}{2}\left(s\cos s - 2\cos s - \sin s\right) + sv\sin s + v^2\cos s, \\[2ex] \sqrt{2} + \sin v \end{pmatrix},$$

and for the surface with a common geodesic evolute curve as

$$\psi_{1g}(s,v) = \begin{pmatrix} \left(\frac{\sqrt{2}}{2}\cos s + \tan\left(\frac{\sqrt{2}}{2}s+2\right)\sin s + sv\left(\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right) + \frac{\sqrt{2}}{2}\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\right)\right. \\[2ex] \left.-\frac{\sqrt{2}}{2}v^2\sin s + \sin v\left(\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right) - \frac{\sqrt{2}}{2}\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right)\right)\right), \\[2ex] \left(\frac{\sqrt{2}}{2}\sin s - \tan\left(\frac{\sqrt{2}}{2}s+2\right)\cos s + sv\left(\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right) - \frac{\sqrt{2}}{2}\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\right)\right. \\[2ex] \left.+\frac{\sqrt{2}}{2}v^2\cos s + \sin v\left(\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right) + \frac{\sqrt{2}}{2}\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right)\right)\right), \\[2ex] \frac{\sqrt{2}}{2}s - \tan\left(\frac{\sqrt{2}}{2}s+2\right) - \frac{\sqrt{2}}{2}sv\sin\left(\frac{\sqrt{2}}{2}s+2\right) - \frac{\sqrt{2}}{2}v^2 + \frac{\sqrt{2}}{2}\sin v\cos\left(\frac{\sqrt{2}}{2}s+2\right) \end{pmatrix}.$$



(a) Surface with a common isogeodesic involute curve, ξ_{1g}



(b) Surface with a common isogeodesic evolute curve, ψ_{1g}

Figure 1. Surfaces having the curves as isogeodesic

(ii) For ruled surfaces, when chosen the marching scale functions as

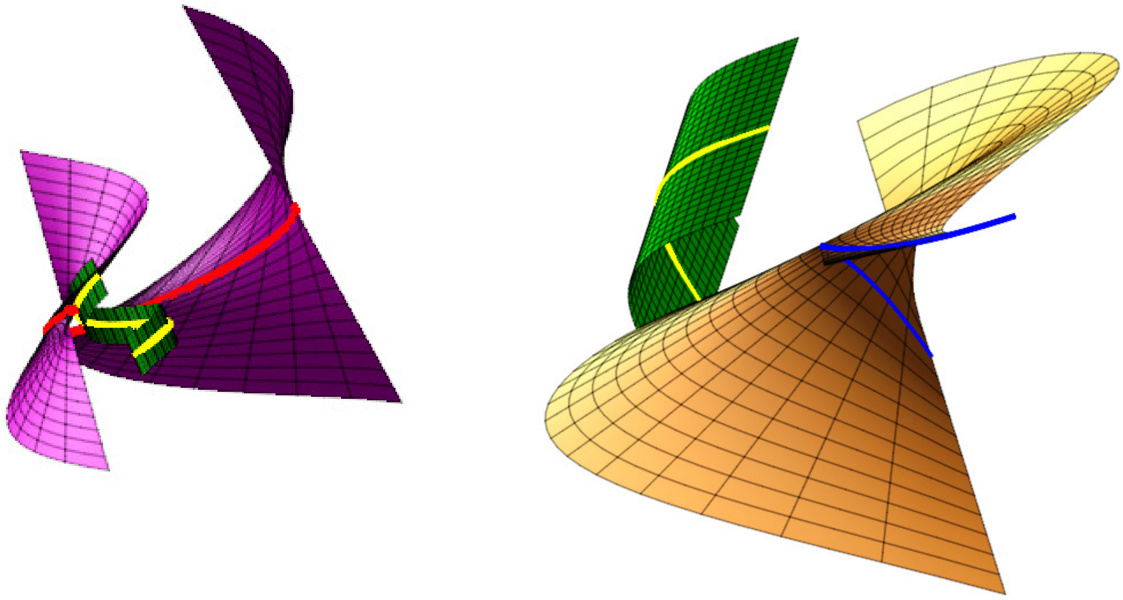
$$x(s) = s = f(s),\ \ z(s) = s^2 = h(s)\ ve\ \ v_0 = 0,$$

the fallowing parametric forms are obtained for the surfaces with a common isogeodesic involute and isogeodesic evolute curves

$$\xi_{1gr}(s,v) = \begin{pmatrix} -\frac{\sqrt{2}}{2} s\sin s + sv\cos s + \sqrt{2}\sin s - \frac{\sqrt{2}}{2}\cos s, \\ \frac{\sqrt{2}}{2} s\cos s + sv\sin s - \sqrt{2}\cos s - \frac{\sqrt{2}}{2}\sin s,\ \sqrt{2} + vs^2 \end{pmatrix},$$

$$\psi_{1gr}(s,v) = \frac{1}{2cos\left(\frac{\sqrt{2}}{2}s+2\right)} \begin{pmatrix} -vs^2\sqrt{2}\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right)^2 + vs\sqrt{2}\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) \\ +2vs^2\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) + 2vs\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right)^2 \\ +\sqrt{2}\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right) + 2\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right), \\ vs^2\sqrt{2}\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right)^2 - vs\sqrt{2}\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) \\ +2vs^2\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) + 2vs\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right)^2 \\ +\sqrt{2}\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right) - 2\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right), \\ vs^2\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}s+2\right)^2 - vs\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) \\ +\sqrt{2}s\cos\left(\frac{\sqrt{2}}{2}s+2\right) - 2\sin\left(\frac{\sqrt{2}}{2}s+2\right) \end{pmatrix},$$

respectively.



(a) Ruled surface with a common isogeodesic involute curve, ξ_{1gr} (b) Ruled surface with a common isogeodesic evolute curve, ψ_{1gr}

Figure 2. Ruled surfaces having the curves as isogeodesic

(iii) The following set of functions given by

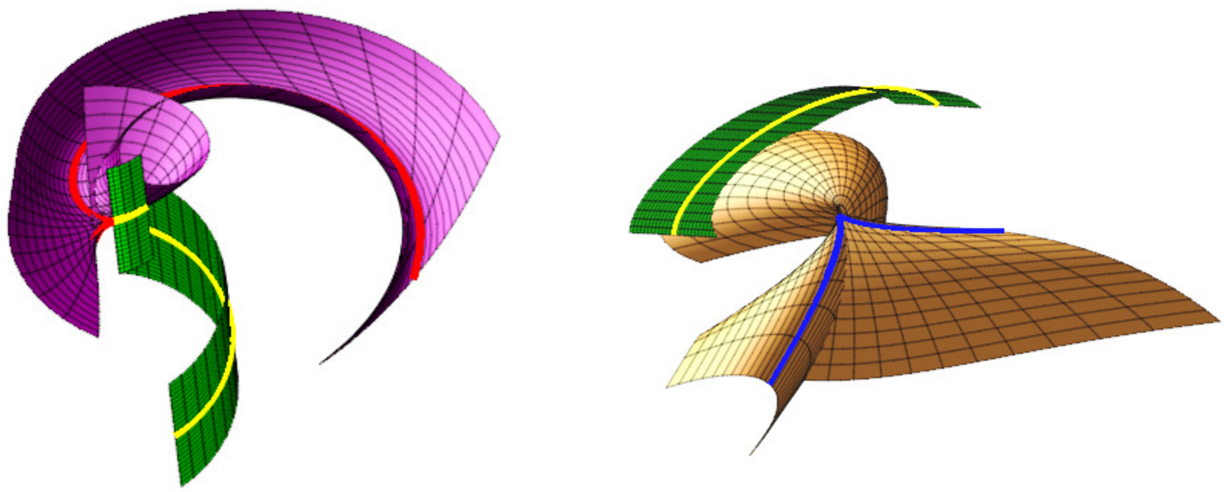
$$x(s,v) = sv = \Delta_1(s,v),$$
$$y(s,v) = \sin v = \Delta_2(s,v),$$
$$z(s,v) = v^2 = \Delta_3(s,v) \text{ and } v_0 = 0,$$

provide the parametric forms of surfaces with asymptotic involute and evolute curves as

$$\xi_{1a}(s,v) = \begin{pmatrix} -\frac{\sqrt{2}}{2}\left(s\sin s - 2\sin s + \cos s\right) + sv\cos s - \sin v \sin s, \\ \frac{\sqrt{2}}{2}\left(s\cos s - 2\cos s - \sin s\right) + sv\sin s + \sin v\cos s, \sqrt{2} + v^2 \end{pmatrix}$$

and

$$\psi_{1a}(s,v) = \begin{pmatrix} \frac{\sqrt{2}}{2}\cos s + \sin s\tan\left(\frac{\sqrt{2}}{2}s+2\right) + sv\left(\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right) + \frac{\sqrt{2}}{2}\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\right) \\ -\frac{\sqrt{2}}{2}\sin v\sin s + v^2\left(\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right) - \frac{\sqrt{2}}{2}\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right)\right), \\ \frac{\sqrt{2}}{2}\sin s - \cos s\tan\left(\frac{\sqrt{2}}{2}s+2\right) + sv\left(\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right) - \frac{\sqrt{2}}{2}\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\right) \\ +\frac{\sqrt{2}}{2}\sin v\cos s + v^2\left(\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right) + \frac{\sqrt{2}}{2}\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right)\right), \\ \frac{\sqrt{2}}{2}s - \tan\left(\frac{\sqrt{2}}{2}s+2\right) - \frac{\sqrt{2}}{2}sv\sin\left(\frac{\sqrt{2}}{2}s+2\right) - \frac{\sqrt{2}}{2}\sin v + \frac{\sqrt{2}}{2}v^2\cos\left(\frac{\sqrt{2}}{2}s+2\right) \end{pmatrix}.$$



(a) Surface with a common isoasymptotic involute curve, ξ_{1a} (b) Surface with a common isoasymptotic evolute curve, ψ_{1a}

Figure 3. Surfaces having the curves as isoasymptotic

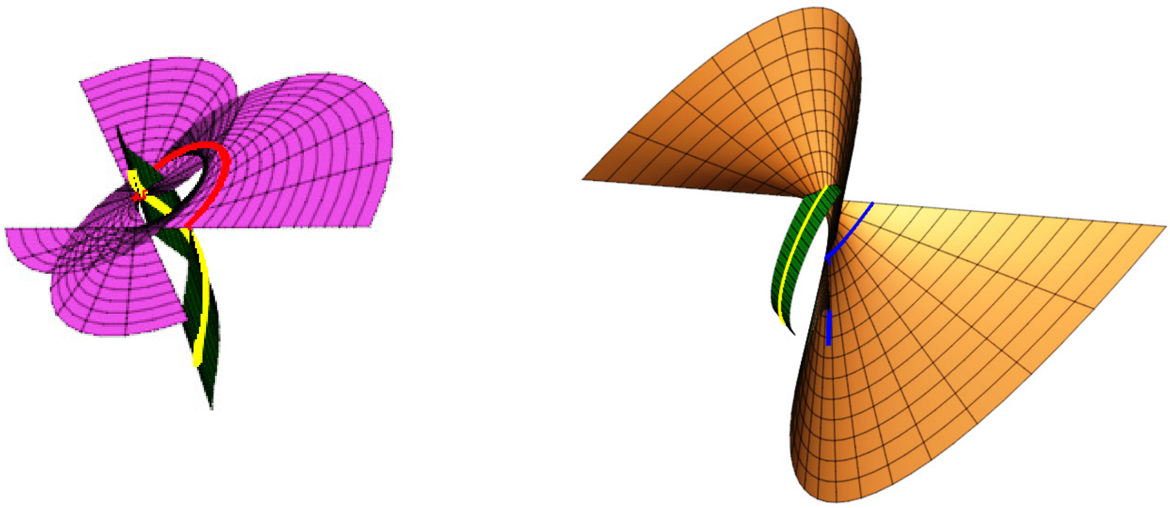
(iv) This time, by choosing the scale functions as $x(s) = s = f(s),\ \ y(s) = s^2 = g(s)\, ve\ \ v_0 = 0$ we can form the ruled surfaces with common isoasymptotic involute curve and isoasymptotic evolute curve as

$$\xi_{1ar}(s,v) = \begin{pmatrix} -\frac{\sqrt{2}}{2}s\sin s + sv\cos s + \sqrt{2}\sin s - \frac{\sqrt{2}}{2}\cos s - s^2 v\sin s, \\ \frac{\sqrt{2}}{2}s\cos s + sv\sin s - \sqrt{2}\cos s - \frac{\sqrt{2}}{2}\sin s + s^2 v\cos s,\ \sqrt{2} \end{pmatrix}$$

and

$$\psi_{1ar}(s,v) = \frac{1}{2cos\left(\frac{\sqrt{2}}{2}s+2\right)}\begin{pmatrix} vs\sqrt{2}\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) - vs^2\sqrt{2}\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right) \\ +2vs\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right)^2 + \sqrt{2}\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right) + 2\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right), \\ -vs\sqrt{2}\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) + vs^2\sqrt{2}\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right) \\ -2vs\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right)^2 + \sqrt{2}\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right) - 2\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right), \\ -vs\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) - vs^2\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}s+2\right) \\ +\sqrt{2}s\cos\left(\frac{\sqrt{2}}{2}s+2\right) - 2\sin\left(\frac{\sqrt{2}}{2}s+2\right) \end{pmatrix},$$

respectively.



(a) Ruled surface with a common isoasymptotic involute curve, ξ_{1ar} (b) Ruled surface with a common isoasymptotic evolute curve, ψ_{1ar}

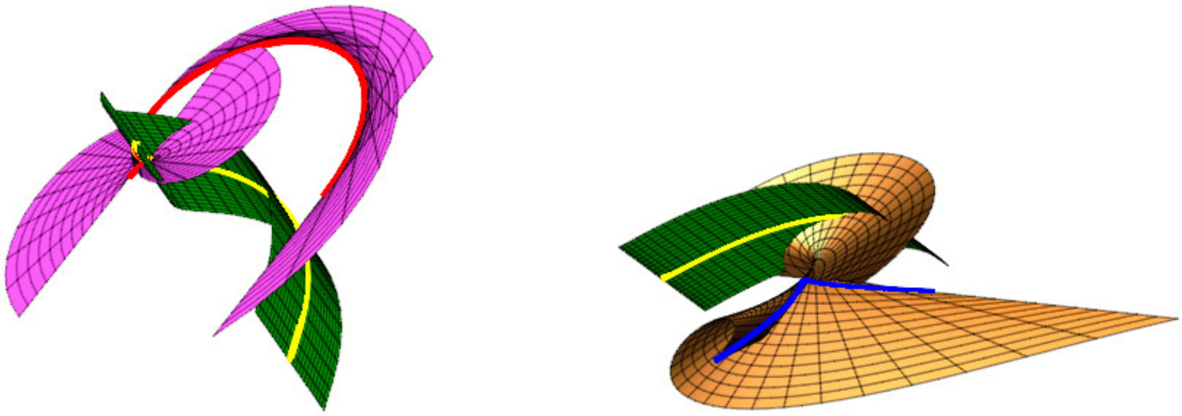
Figure 4. Ruled surfaces having the curves as isoasymptotic

(v) When we pick the marching scale functions as $x(s) = s = f(s),\ \ y(s) = 0 = g(s)\ ve\ \ v_0 = 0$ we can form developable ruled surfaces with a common isoasymptotic involute and evolute curves by following:

$$\xi_{dar}(s,v) = \begin{pmatrix} -\frac{\sqrt{2}}{2} s\sin s + sv\cos s + \sqrt{2}\sin s - \frac{\sqrt{2}}{2}\cos s + \sqrt{2}\sin s, \\ \frac{\sqrt{2}}{2} s\cos s + sv\sin s - \sqrt{2}\cos s - \frac{\sqrt{2}}{2}\sin s,\ \sqrt{2} \end{pmatrix}$$

and

$$\psi_{dar}(s,v) = \frac{1}{2cos\left(\frac{\sqrt{2}}{2}s+2\right)} \begin{pmatrix} vs\sqrt{2}\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) + 2sv\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right)^2 \\ +\sqrt{2}\cos s\cos\left(\frac{\sqrt{2}}{2}s+2\right) + 2\sin s\sin\left(\frac{\sqrt{2}}{2}s+2\right), \\ -vs\sqrt{2}\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) + 2sv\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right)^2 \\ +\sqrt{2}\sin s\cos\left(\frac{\sqrt{2}}{2}s+2\right) - 2\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right), \\ vs\sqrt{2}\cos s\sin\left(\frac{\sqrt{2}}{2}s+2\right)\cos\left(\frac{\sqrt{2}}{2}s+2\right) \\ +\sqrt{2}s\cos\left(\frac{\sqrt{2}}{2}s+2\right) - 2\sin\left(\frac{\sqrt{2}}{2}s+2\right) \end{pmatrix}.$$



(a) Developable ruled surface with a common isoasymptotic involute curve, ξ_{dar}

(b) Developable ruled surface with a common isoasymptotic evolute curve, ψ_{dar}

Figure 5. Developable ruled surfaces having the curves as isoasymptotic

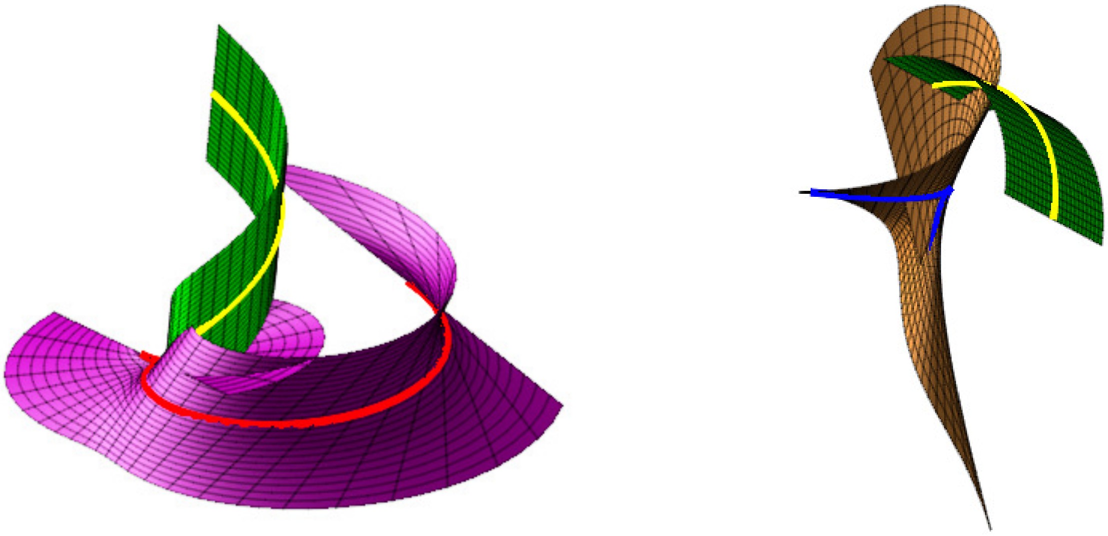
(vi) With the following marching scale functions

$$x(s,v)=sv,\qquad y(s,v)=v\sin\left(\tfrac{1}{\sqrt{2}}\right)=,\qquad z(s,v)=-v\cos\left(\tfrac{1}{\sqrt{2}}\right),$$
$$\Delta_1(s,v)=sv,\qquad \Delta_2(s,v)=v\sin\left(-\tfrac{1}{\sqrt{2}}s\right),\qquad \Delta_3(s,v)=-v\cos\left(-\tfrac{1}{\sqrt{2}}s\right),$$
$$\mu_1=1,\qquad \theta_1(s)=\tfrac{1}{\sqrt{2}},\qquad \mu_2=1,\qquad \theta_2(s)=-\tfrac{1}{\sqrt{2}}s,\qquad v_0=0,$$

we derive the parametric form of surfaces with a common involute and evolute curve as of each curvature line like below

$$\xi_{cl}(s,v)=\begin{pmatrix} -\frac{1}{2}\sqrt{2}\left(\sin(s)\,s-2\sin(s)+\cos(s)\right)+sv\cos(s)-v\sin\left(\frac{1}{2}\sqrt{2}\right)\sin(s), \\ -\frac{1}{2}\sqrt{2}\left(-\cos(s)\,s+\sin(s)+2\cos(s)\right)+sv\sin(s)+v\sin\left(\frac{1}{2}\sqrt{2}\right)\cos(s), \\ \sqrt{2}-v\cos\left(\frac{1}{2}\sqrt{2}\right) \end{pmatrix},$$

$$\psi_{cl}(s,v)=\begin{pmatrix} \frac{\sqrt{2}\cos(s)}{2}+\tan\left(\frac{\sqrt{2}s}{2}+2\right)\sin(s)+sv\left(\cos(s)\cos\left(\frac{\sqrt{2}s}{2}+2\right)+\frac{\sin\left(\frac{\sqrt{2}s}{2}+2\right)\sqrt{2}\sin(s)}{2}\right) \\ -\frac{v\sin\left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\sin(s)}{2}-v\cos\left(\frac{\sqrt{2}}{2}\right)\left(\sin\left(\frac{\sqrt{2}s}{2}+2\right)\cos(s)-\frac{\cos\left(\frac{\sqrt{2}s}{2}+2\right)\sqrt{2}\sin(s)}{2}\right), \\ \frac{\sqrt{2}\sin(s)}{2}-\tan\left(\frac{\sqrt{2}s}{2}+2\right)\cos(s)+sv\left(\sin(s)\cos\left(\frac{\sqrt{2}s}{2}+2\right)-\frac{\sin\left(\frac{\sqrt{2}s}{2}+2\right)\sqrt{2}\cos(s)}{2}\right) \\ +\frac{v\sin\left(\frac{\sqrt{2}}{2}\right)\sqrt{2}\cos(s)}{2}-v\cos\left(\frac{\sqrt{2}}{2}\right)\left(\sin\left(\frac{\sqrt{2}s}{2}+2\right)\sin(s)+\frac{\cos\left(\frac{\sqrt{2}s}{2}+2\right)\sqrt{2}\cos(s)}{2}\right), \\ \frac{\sqrt{2}s}{2}-\tan\left(\frac{\sqrt{2}s}{2}+2\right)-\frac{sv\sin\left(\frac{\sqrt{2}s}{2}+2\right)\sqrt{2}}{2}-\frac{v\sin\left(\frac{\sqrt{2}}{2}\right)\sqrt{2}}{2}-\frac{v\cos\left(\frac{\sqrt{2}}{2}\right)\cos\left(\frac{\sqrt{2}s}{2}+2\right)\sqrt{2}}{2} \end{pmatrix}.$$



(a) Surface with a common involute curve ascurvature line, ξ_{cl}

(b) Surface with a common evolute curve ascurvature line, ψ_{cl}

Figure 6. Surfaces having the curves as curvature line

Conflict of interest statement:

The authors declare no conflict of interest.

Data availability statement:

No data were used to support this study.

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