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# Some New Traveling Wave Solutions of Nonlinear Fluid Models via the MSE Method

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**Abstract**

In this study, some new exact wave solutions of nonlinear partial differential equations are investigated by the modified simple equation method. This method is applied to the $(2+1)$-dimensional Calogero-Bogoyavlenskii-Schiff equation and the $(3+1)$-dimensional Jimbo-Miwa equation. Our applications reveal how to use the proposed method to solve nonlinear partial differential equations with the balance number equal to two. Consequently, some new exact traveling wave solutions of these equations are achieved, and types of waves are determined. To verify our results and draw the graphs of the solutions, we use the Mathematica package program.

## 1. Introduction

Nonlinear partial differential equations (NPDEs) have proved to be precious instruments for the modelling of physical phenomena, and have been the focus of many researchers due to their extensive use in several areas such as mathematical physics, biology, nonlinear optics, fluid mechanics, ocean engineering, chemical physics, plasma physics etc. [1]-[6]. Thus, it has gained great importance in the literature to examine the solutions of these equations to explain the nonlinear complex processes in nature. However, exact solutions of equations in the nonlinear form are not always obtained by classical methods. In recent times, many useful methods and techniques such as the modified simple equation (MSE) method [7], the improved $\tan(\varphi/2)$-expansion method [8], the extended rational sine-cosine method [9], the $(G'/G,1/G)$-expansion method [10], the improved $F$-expansion method [11], the modified exp $(-\phi(\varepsilon))$-expansion method [12], the first integral method [13], the $(G'/G)$-expansion method [14] etc. have been enhanced to find traveling wave solutions. In this paper, we propose the MSE method, which is a remarkable and useful method for finding various solutions of NPDEs. This method converts NPDEs into nonlinear ordinary differential equations (NODEs) with wave transformation. Also, the advantage of the proposed method is that the general solution form is defined as the sum of the finite series and an unknown function in this solution form is determined according to the solution of a system of algebraic equations obtained from the main equation. Compared to other methods in the literature such as $(G'/G,1/G)$-expansion method, the sine-cosine method, the improved $F$-expansion method, the $(G'/G)$-expansion method, etc., the MSE method does not require symbolic computational software programs to solve algebraic equation systems. In addition, the unknown function in this method is not depend on a pre-defined function or a solution of the ODE, and the obtained exact solutions have arbitrary coefficients. Thus, the traveling wave solutions can be obtained in a new and extensible form. We observe that this method is highly systematic, understandable and applicable. We perform the MSE method to NPDEs, namely, the $(2+1)$-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation [15] and the $(3+1)$-dimensional Jimbo-Miwa equation [16]. The CBS equation is a frequently used model in fluid dynamics that describes and explains situations such as fusion, annihilation and fission of complex waves [17]. The Jimbo-Miwa equation is

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used in fluid mechanics to define some specific $(3+1)$-dimensional nonlinear waves, and this equation is the second equation in the notable Kadomtsev–Petviashvili hierarchy of integrable systems [18]. As a result, new exact solutions of the equations are obtained and their graphs are drawn to observe the physical behaviors of these solutions. The article is concerted in the following: In Sec. 2, we summarize the illustration of the MSE method. In Sec. 3, applications of the MSE method are given. In Sec. 4, we draw graphs of wave solutions and physical explanations. Sec. 5 includes the conclusion.

## 2. The modified simple equation method

In this section, we present the major steps of the MSE method [7]:
Consider the NPDE in the following:

$$G(u,u_t,u_x,u_y,u_{tt},u_{xx},u_{yy},...)=0, \tag{2.1}$$

where $G$ is a polynomial of $u(x,y,t)$ and its several partial derivatives.
**Step 1.** We use the traveling wave transformation

$$u(x,y,t)=u(\Upsilon),\ \ \Upsilon=x+y-\Theta t, \tag{2.2}$$

to reduce (2.1) into the succeeding NODE:

$$R\left(u,u',u'',u''',...\right)=0, \tag{2.3}$$

where $R$ is a polynomial in $u(\Upsilon)$ and its all derivatives with respect to $\Upsilon$.
**Step 2.** Suppose that the solution of (2.3) can be expressed in the form,

$$u(\Upsilon)=\sum_{k=0}^{N}A_k\left[\frac{\phi'(\Upsilon)}{\phi(\Upsilon)}\right]^k, \tag{2.4}$$

where $A_k$ are arbitrary constants $(A_N\neq 0)$ and $\phi(\Upsilon)$ is an unknown function to be calculated.
**Step 3.** We determine balancing number $N$ in (2.4) by considering the homogeneous balance between the highest order nonlinear terms and the highest order derivatives occurred in (2.3).
**Step 4.** We replace (2.4) and its derivatives into (2.3). Hereby, we have a polynomial of $\phi(\Upsilon)$. Then, we equalize all the coefficients of $\phi^{-i}(\Upsilon)$ $(i=0,1,2...)$ to zero in this polynomial. This operation gives a system of equations to obtain $A_k$ and $\phi(\Upsilon)$. Thus, we achieve the exact solution of (2.1).

## 3. Applications

In this section, the MSE method is applied to nonlinear equations which express some special physical phenomena and wave solutions of these equations are obtained.

### 3.1. $(2+1)$-dimensional Calogera-Bogoyavlenskii-Schiff (CBS) equation

This equation was examined by Schiff and Bogoyavlenskii in varied ways. Bogoyavlenskii used the modified Lax formalism, while Schiff obtained the similar equation by reducing the self-dual Yang-Mills equation [19]. This equation has various forms for different coefficients. Also, many studies in the literature obtain different solution types of this equation [17], [20]-[23]. The $(2+1)$-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation is as follows [15]:

$$u_{xxxy}+2u_yu_{xx}+4u_xu_{xy}+u_{xt}=0, \tag{3.1}$$

where $x$, $y$ represent the position of the wave and $t$ represents the time. Applying the wave transformation in (2.2) to (3.1), integrating once respect to $\Upsilon$ and considering the integration constant as zero, we attain nonlinear ODE in the following form:

$$u'''+3\left(u'\right)^2-\Theta u'=0. \tag{3.2}$$

Now, using the transformation $u'(\Upsilon)=v(\Upsilon)$, (3.2) reduces to

$$v''+3v^2-\Theta v=0. \tag{3.3}$$

Balancing $v''$ and $v^2$ in (3.3), we find $N=2$. Consequently, (2.4) turns into the following form:

$$v(\Upsilon)=A_0+A_1\left(\frac{\phi'(\Upsilon)}{\phi(\Upsilon)}\right)+A_2\left(\frac{\phi'(\Upsilon)}{\phi(\Upsilon)}\right)^2. \tag{3.4}$$

Substituting (3.4) and its derivatives into (3.3), and setting all the coefficients with the same power of $\phi^{-i}(\Upsilon)$, we attain a system as follows:

$$
\begin{aligned}
(\phi)^0 &: 3A_0^2-\Theta A_0=0,\\
(\phi)^{-1} &: 6A_0A_1\phi'(\Upsilon)-\Theta A_1\phi'(\Upsilon)+A_1\phi'''(\Upsilon)=0, &&(3.5)\\
(\phi)^{-2} &: 3A_1^2\left(\phi'(\Upsilon)\right)^2+6A_0A_2\left(\phi'(\Upsilon)\right)^2-\Theta A_2\left(\phi'(\Upsilon)\right)^2-3A_1\phi'(\Upsilon)\phi''(\Upsilon)\\
&\quad +2A_2\phi'''(\Upsilon)\phi'(\Upsilon)+2A_2\left(\phi''(\Upsilon)\right)^2=0, &&(3.6)\\
(\phi)^{-3} &: 6A_1A_2\left(\phi'(\Upsilon)\right)^3+2A_1\left(\phi'(\Upsilon)\right)^3-10A_2\left(\phi'(\Upsilon)\right)^2\phi''(\Upsilon)=0, &&(3.7)\\
(\phi)^{-4} &: 3A_2^2\left(\phi'(\Upsilon)\right)^4+6A_2\left(\phi'(\Upsilon)\right)^4=0.
\end{aligned}
$$

**Case 1:** $A_0=0$, $A_1\neq 0$, $A_2=-2$ and $\phi'(\Upsilon)\neq 0$. In this case, by using (3.5) and (3.7), we obtain $\phi'(\Upsilon)=2\frac{c_1}{A_1}e^{\frac{2\Theta}{A_1}\Upsilon}$ and $\phi(\Upsilon)=\frac{c_1}{\Theta}e^{\frac{2\Theta}{A_1}\Upsilon}+c_2$. Here and throughout the paper, $c_1$ and $c_2$ are arbitrary constants of integration. Then, we use these equations and (3.6), we achieve $A_1=\pm 2\sqrt{\Theta}$. Inserting $A_0$, $A_1$, $A_2$, $\phi(\Upsilon)$ and $\phi'(\Upsilon)$ into (3.4), we deduce the exact solution of (3.1) as follows:

$$
v(\Upsilon)=\pm 2\sqrt{\Theta}\left(\frac{\pm\frac{c_1}{\sqrt{\Theta}}e^{\pm\sqrt{\Theta}\Upsilon}}{\frac{c_1}{\Theta}e^{\pm\sqrt{\Theta}\Upsilon}+c_2}\right)-2\left(\frac{\pm\frac{c_1}{\sqrt{\Theta}}e^{\pm\sqrt{\Theta}\Upsilon}}{\frac{c_1}{\Theta}e^{\pm\sqrt{\Theta}\Upsilon}+c_2}\right)^2,
$$

where $\Upsilon=x+y-\Theta t$.
Now, by using hyperbolic function features, we obtain the wave solutions when $c_1=\Theta$ and $c_2=1$ as:

$$
v_{1,2}(x,y,t)=\Theta\left(1+\tanh\left(\frac{\pm\sqrt{\Theta}}{2}(x+y-\Theta t)\right)\right)-\frac{\Theta}{2}\left(1+\tanh\left(\frac{\pm\sqrt{\Theta}}{2}(x+y-\Theta t)\right)\right)^2,
$$

$$
u_{1,2}(x,y,t)=\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}(x+y-\Theta t)\right). \tag{3.8}
$$

When $c_1=\Theta$, $c_2=-1$ as:

$$
v_{3,4}(x,y,t)=\Theta\left(1+\coth\left(\frac{\pm\sqrt{\Theta}}{2}(x+y-\Theta t)\right)\right)-\frac{\Theta}{2}\left(1+\coth\left(\frac{\pm\sqrt{\Theta}}{2}(x+y-\Theta t)\right)\right)^2,
$$

$$
u_{3,4}(x,y,t)=\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}(x+y-\Theta t)\right). \tag{3.9}
$$

**Case 2:** $A_0=\frac{\Theta}{3}$, $A_1\neq 0$, $A_2=-2$ and $\phi'(\Upsilon)\neq 0$. By using (3.5) and (3.7), we obtain $\phi'(\Upsilon)=2\frac{c_1}{A_1}e^{\frac{-2\Theta}{A_1}\Upsilon}$ and $\phi(\Upsilon)=c_2-\frac{c_1}{\Theta}e^{\frac{-2\Theta}{A_1}\Upsilon}$. Considering these equations and (3.6), we have $A_1=\pm 2i\sqrt{\Theta}$. Now, inserting $A_0$, $A_1$, $A_2$, $\phi(\Upsilon)$ and $\phi'(\Upsilon)$ into (3.4), the exact solution of (3.1) follows as:

$$
v(\Upsilon)=\frac{\Theta}{3}\pm 2i\sqrt{\Theta}\left(\frac{\frac{\pm\sqrt{c_1}}{i\sqrt{\Theta}}e^{\mp\frac{\sqrt{\Theta}}{i}\Upsilon}}{c_2-\frac{c_1}{\Theta}e^{\mp\frac{\sqrt{\Theta}}{i}\Upsilon}}\right)-2\left(\frac{\frac{\pm\sqrt{c_1}}{i\sqrt{\Theta}}e^{\mp\frac{\sqrt{\Theta}}{i}\Upsilon}}{c_2-\frac{c_1}{\Theta}e^{\mp\frac{\sqrt{\Theta}}{i}\Upsilon}}\right)^2.
$$

where $\Upsilon=x+y-\Theta t$.
Hence, by using hyperbolic function features, we achieve the wave solutions for $c_1=-\Theta$ and $c_2=1$ as:

$$
v_{5,6}(x,y,t)=\frac{\Theta}{3}-\Theta\left(1+\tanh\left(\frac{\mp\sqrt{\Theta}}{2i}(x+y-\Theta t)\right)\right)+\frac{\Theta}{2}\left(1+\tanh\left(\frac{\mp\sqrt{\Theta}}{2i}(x+y-\Theta t)\right)\right)^2,
$$

$$
u_{5,6}(x,y,t)=\frac{\Theta(x+y-\Theta t)}{3}-\sqrt{\Theta}\tan\left(\frac{\sqrt{\Theta}}{2}(x+y-\Theta t)\right). \tag{3.10}
$$

For $c_1=-\Theta$ and $c_2=-1$ as:

$$v_{7,8}(x,y,t)=\frac{\Theta}{3}-\Theta\left(1+\coth\left(\frac{\mp\sqrt{\Theta}}{2i}(x+y-\Theta t)\right)\right)+\frac{\Theta}{2}\left(1+\coth\left(\frac{\mp\sqrt{\Theta}}{2i}(x+y-\Theta t)\right)\right)^2,$$

$$u_{7,8}(x,y,t)=\frac{\Theta(x+y-\Theta t)}{3}+\sqrt{\Theta}\cot\left(\frac{\sqrt{\Theta}}{2}(x+y-\Theta t)\right). \tag{3.11}$$

### 3.2. (3+1)-dimensional Jimbo-Miwa equation

This equation appears in many areas of science, such as geochemistry, fluid mechanics, optical fiber, astrophysics, plasma physics, chemical kinematics and solid state physics [24]. Furthermore, there are many studies in the literature investigating the different forms of solutions for this equation [18], [25]-[28].
The $(3+1)$-dimensional Jimbo-Miwa equation is as follows [16]:

$$\begin{aligned} u_{xxxy}+6u_xu_y+3uv_{xx}+3u_{xx}v+3u_{yt}-3u_{zz}&=0,\\ u_y&=v_x, \end{aligned} \tag{3.12}$$

where $x,y,z$ represent the position of the wave and $t$ represents the time. Using the wave transformation in the following:

$$u(x,y,z,t)=u(\Upsilon),\quad v(x,y,z,t)=v(\Upsilon),\quad \Upsilon=x+y+z-\Theta t,$$

and three times integrating with respect to $\Upsilon$, considering the integration constants as zero, (3.12) converts to nonlinear ODE:

$$u''+3u^2-3(\Theta+1)u=0. \tag{3.13}$$

Balancing $u''$ and $u^2$ in (3.13), we get $N=2$. Therefore, (2.4) turns into the following form:

$$u(\Upsilon)=A_0+A_1\left(\frac{\phi'(\Upsilon)}{\phi(\Upsilon)}\right)+A_2\left(\frac{\phi'(\Upsilon)}{\phi(\Upsilon)}\right)^2. \tag{3.14}$$

Substituting (3.14) and its derivatives into (3.13), and editing all the coefficients with the same power of $\phi^{-i}(\Upsilon)$, we obtain a system as follows:

$$(\phi)^0: 3A_0^2-3(\Theta+1)A_0=0,$$

$$(\phi)^{-1}: 6A_0A_1\phi'(\Upsilon)-3(\Theta+1)A_1\phi'(\Upsilon)+A_1\phi'''(\Upsilon)=0, \tag{3.15}$$

$$\begin{aligned}(\phi)^{-2}: 3A_1^2(\phi'(\Upsilon))^2+6A_0A_2(\phi'(\Upsilon))^2-3(\Theta+1)A_2(\phi'(\Upsilon))^2\\ -3A_1\phi'(\Upsilon)\phi''(\Upsilon)+2A_2\phi'''(\Upsilon)\phi'(\Upsilon)+2A_2(\phi''(\Upsilon))^2=0,\end{aligned} \tag{3.16}$$

$$(\phi)^{-3}: 6A_1A_2(\phi'(\Upsilon))^3+2A_1(\phi'(\Upsilon))^3-10A_2(\phi'(\Upsilon))^2\phi''(\Upsilon)=0, \tag{3.17}$$

$$(\phi)^{-4}: 3A_2^2(\phi'(\Upsilon))^4+6A_2(\phi'(\Upsilon))^4=0.$$

**Case 1:** $A_0=0$, $A_1\neq 0$, $A_2=-2$ and $\phi'(\Upsilon)\neq 0$. From (3.15) and (3.17), we get $\phi'(\Upsilon)=2\frac{c_1}{A_1}e^{\frac{6(\Theta+1)}{A_1}\Upsilon}$ and $\phi(\Upsilon)=\frac{c_1}{3(\Theta+1)}e^{\frac{6(\Theta+1)}{A_1}\Upsilon}+c_2$. Then, by these equations and (3.16), we deduce $A_1=\pm 2\sqrt{3(\Theta+1)}$. Substituting $A_0,A_1,A_2$, $\phi(\Upsilon)$ and $\phi'(\Upsilon)$ into (3.14) we have the exact solution of (3.12) as in the following:

$$u(\Upsilon)=\pm 2\sqrt{3(\Theta+1)}\left(\frac{\pm\frac{c_1}{\sqrt{3(\Theta+1)}}e^{\pm\sqrt{3(\Theta+1)}\Upsilon}}{\frac{c_1}{3(\Theta+1)}e^{\pm\sqrt{3(\Theta+1)}\Upsilon}+c_2}\right)-2\left(\frac{\pm\frac{c_1}{\sqrt{3(\Theta+1)}}e^{\pm\sqrt{3(\Theta+1)}\Upsilon}}{\frac{c_1}{3(\Theta+1)}e^{\pm\sqrt{3(\Theta+1)}\Upsilon}+c_2}\right)^2.$$

where $\Upsilon=x+y+z-\Theta t$.
Hence, by using hyperbolic function properties, we get the wave solutions when $c_1=3(\Theta+1)$ and $c_2=1$ as:

$$\begin{aligned}u_{1,2}(x,y,z,t)=3(\Theta+1)\left(1+\tanh\left(\frac{\pm\sqrt{3(\Theta+1)}}{2}(x+y+z-\Theta t)\right)\right)\\ -\frac{3(\Theta+1)}{2}\left(1+\tanh\left(\frac{\pm\sqrt{3(\Theta+1)}}{2}(x+y+z-\Theta t)\right)\right)^2.\end{aligned} \tag{3.18}$$

When $c_1 = 3(\Theta+1)$ and $c_2 = -1$ as:

$$u_{3,4}(x,y,z,t) = 3(\Theta+1)\left(1+\coth\left(\frac{\pm\sqrt{3(\Theta+1)}}{2}(x+y+z-\Theta t)\right)\right) - \frac{3(\Theta+1)}{2}\left(1+\coth\left(\frac{\pm\sqrt{3(\Theta+1)}}{2}(x+y+z-\Theta t)\right)\right)^2. \tag{3.19}$$

**Case 2:** $A_0 = \Theta+1$, $A_1 \neq 0$, $A_2 = -2$ and $\phi'(\Upsilon) \neq 0$. Taking (3.15) and (3.17) into account, we get $\phi'(\Upsilon) = 2\frac{c_1}{A_1}e^{\frac{-6(\Theta+1)}{A_1}\Upsilon}$ and $\phi(\Upsilon) = c_2 - \frac{c_1}{3(\Theta+1)}e^{\frac{-6(\Theta+1)}{A_1}\Upsilon}$. From these equations and (3.16), we have $A_1 = \pm 2i\sqrt{3(\Theta+1)}$. Substituting $A_0$, $A_1$, $A_2$, $\phi(\Upsilon)$ and $\phi'(\Upsilon)$ into (3.14), we get the exact solutions of (3.12) as follows:

$$u(\Upsilon) = (\Theta+1) \pm 2i\sqrt{3(\Theta+1)}\left(\frac{\frac{\pm c_1}{i\sqrt{3(\Theta+1)}}e^{\frac{\mp 3\sqrt{\Theta+1}}{i\sqrt{3}}\Upsilon}}{\frac{-c_1}{3(\Theta+1)}e^{\frac{\mp 3\sqrt{\Theta+1}}{i\sqrt{3}}\Upsilon}+c_2}\right) - 2\left(\frac{\frac{\pm c_1}{i\sqrt{3(\Theta+1)}}e^{\frac{\mp 3\sqrt{\Theta+1}}{i\sqrt{3}}\Upsilon}}{\frac{-c_1}{3(\Theta+1)}e^{\frac{\mp 3\sqrt{\Theta+1}}{i\sqrt{3}}\Upsilon}+c_2}\right)^2.$$

where $\Upsilon = x+y+z-\Theta t$.
Then, by using hyperbolic function properties, the wave solutions are obtained for $c_1 = -3(\Theta+1)$ and $c_2 = 1$ as:

$$u_{5,6}(x,y,z,t) = (\Theta+1) - 3(\Theta+1)\left(1+\tanh\left(\frac{\mp\sqrt{3(\Theta+1)}}{2i}(x+y+z-\Theta t)\right)\right) - \frac{3(\Theta+1)}{2}\left(1+\tanh\left(\frac{\mp\sqrt{3(\Theta+1)}}{2i}(x+y+z-\Theta t)\right)\right)^2. \tag{3.20}$$

For $c_1 = -3(\Theta+1)$ and $c_2 = -1$ as:

$$u_{7,8}(x,y,z,t) = (\Theta+1) - 3(\Theta+1)\left(1+\coth\left(\frac{\mp\sqrt{3(\Theta+1)}}{2i}(x+y+z-\Theta t)\right)\right) - \frac{3(\Theta+1)}{2}\left(1+\coth\left(\frac{\mp\sqrt{3(\Theta+1)}}{2i}(x+y+z-\Theta t)\right)\right)^2. \tag{3.21}$$

Moreover, the values of $v(x,y,z,t)$ can be easily calculated according to the $u_y = v_x$.

Consequently, the set of exact solutions for the CBS and the Jimbo-Miwa equations can be expanded by selecting more varied arbitrary constants $c_1$ and $c_2$.

## 4. Physical explanation and graphs

This part shows physical behaviour of the achieved exact wave solutions of the CBS and the Jimbo-Miwa equations. The MSE method is implemented to both equations and the new traveling wave solutions are obtained in (3.8), (3.9), (3.10), (3.11) and (3.18), (3.19), (3.20), (3.21), respectively. These results are drawn with proper values in different types of graphs and intervals such as 3D $(-8 \le x,t \le 8)$, 2D $(-8 \le x \le 8)$ and contour graph $(0 \le x,t \le 10)$. Other independent variables $y$ and $z$ are used with appropriate values in the solution graphs.

### 4.1. Graphs of solutions for the CBS equation:

Fig.4.1-(a), (b), (c), (d) demonstrate (3.8) $u_{1,2}(x,y,t)$, (3.9) $u_{3,4}(x,y,t)$ for $\Theta = 1.39$, and (3.10) $u_{5,6}(x,y,t)$, (3.11) $u_{7,8}(x,y,t)$ for $\Theta = 1.5$, respectively. Fig.4.2-(a)-(b) represent (3.8) $u_{1,2}(x,y,t)$ and (3.9) $u_{3,4}(x,y,t)$ for $\Theta = 1.39$, $t = 1$ and $y = 0$. Also, Fig.4.2-(c)-(d) show (3.10) $u_{5,6}(x,y,t)$ and (3.11) $u_{7,8}(x,y,t)$ for $\Theta = 1.5$, $t = 1$ and $y = 0$.

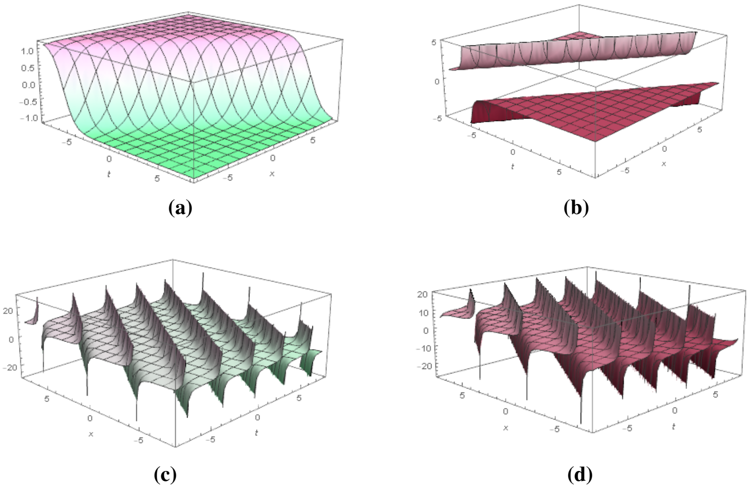



**Figure 4.1:** 3D-graphs.

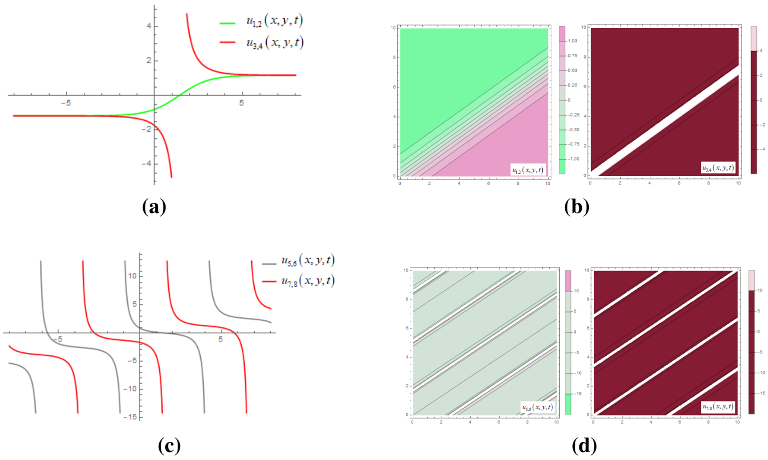



**Figure 4.2:** (a)-(c) 2D-graphs. (b)-(d) Contour graphs.

## 4.2. Graphs of solutions for the Jimbo-Miwa equation:

Fig.4.3-(a), (b), (c), (d) indicate (3.18) $u_1(x,y,z,t)$, (3.19) $u_3(x,y,z,t)$ for $\Theta = 1.2$, and (3.20) $u_5(x,y,z,t)$, (3.21) $u_7(x,y,z,t)$ for $\Theta = 1.5$, respectively. Fig.4.4-(a)-(b) express (3.18) $u_1(x,y,z,t)$ and (3.19) $u_3(x,y,z,t)$ for $\Theta = 1.2$, $t = 1$, $y = 0$ and $z = 0$. Further, Fig.4.4-(c)-(d) represent (3.20) $u_5(x,y,z,t)$ and (3.21) $u_7(x,y,z,t)$ for $\Theta = 1.5$, $t = 1$, $y = 0$ and $z = 0$.

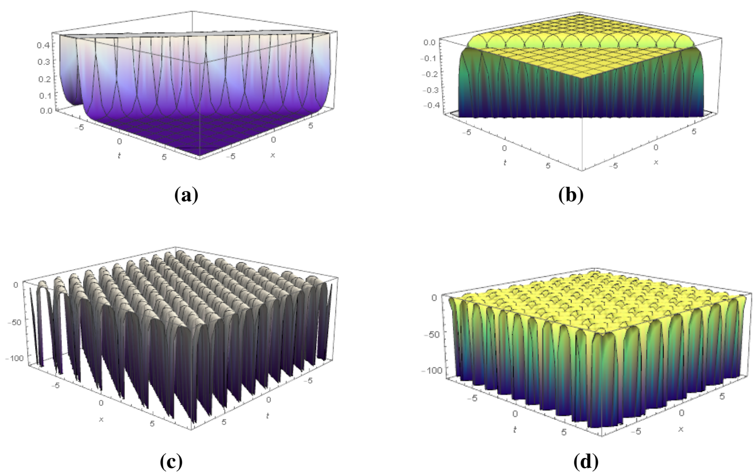
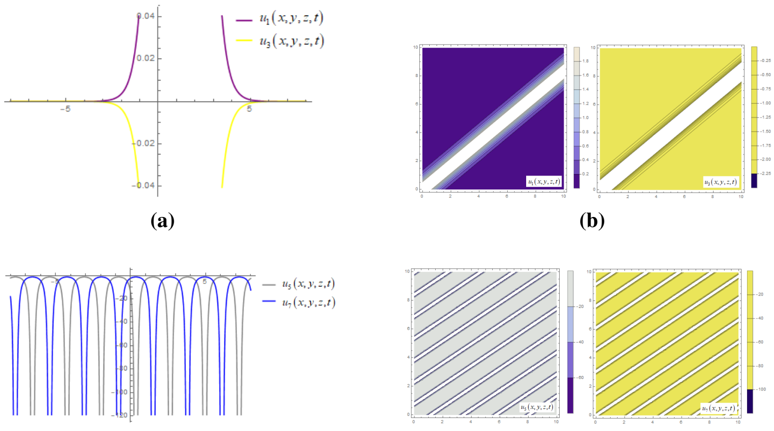



**Figure 4.3:** 3D-graphs.




**(a)** **(b)** **(c)** **(d)**

**Figure 4.4:** (a)-(c) 2D-graphs. (b)-(d) Contour graphs.

As a consequence, we have achieved some new wave solutions of equations (3.1) and (3.12) in hyperbolic and trigonometric forms. The graphs show that the resulting solitary wave solutions have several shapes, such as periodic and kink forms with respect to the wave speed $\Theta$.

## 5. Conclusion

We have implemented the MSE method to attain some new exact solutions of the $(2+1)$-dimensional CBS equation and the $(3+1)$-dimensional Jimbo-Miwa equation. The correctness of the solutions has been demonstrated using the Mathematica package program. The graphics of the solutions have been plotted according to the appropriate values. The features of the MSE method allow us to obtain new traveling wave solutions to explain some complex physical phenomena. Consequently, our results show that the proposed method is practical, straightforward and effective for finding solutions to physics and engineering models. In our future studies, this effective and useful method will be applied to some other nonlinear equations involving integer and fractional derivatives expressing different complex phenomena.

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Not applicable.

## Competing interests

The authors declare that they have no competing interests.

## Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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