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A Comparative Study on the Generalized Spherical Fuzzy Einstein Aggregation Operators

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Highlights

- A multi-criteria group decision-making method is given based on the Einstein aggregation operators.
- A numerical example is shown to illustrate the validity of the proposed technique.
- A comparison of the results under these Einstein aggregation operators is given.

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Abstract

Generalized spherical fuzzy set theory is a powerful and useful tool that is capable to process uncertainty and vagueness. In this study, we investigate some induced aggregation operators under the generalized spherical fuzzy environment with the help of Einstein norms operations to merge the generalized spherical fuzzy information into a single one in the decision-making process. After we observe some properties of the presented aggregation operators, we establish an algorithm to use in the solution of the multiple criteria group decision-making problems by using these aggregation operators and also we give an illustrative example. Then, we compare the results under all defined generalized spherical fuzzy Einstein aggregation operators used within the decision-making process.

1. INTRODUCTION

Aggregation operators which combine data taken from multiple sources into a single value play a key role in computational science. So, aggregation operators have become a significant area of research in recent studies. Attention to the importance of this topic is given by the fact that the necessity of merging the information contained in a collection of pieces of information into one, especially in applied sciences. A sample where aggregation functions have been successfully applied is provided by the databases in which uncertain information can be managed. Also, another important motivation is that aggregation operators are one of the most significant tools in multi-criteria decision-making (MCDM) procedures to aggregate information in evaluation. In the multi-criteria group decision-making (MCGDM) process, decision-makers (DMs), participated the process, construct the decision matrices (DeMs) by evaluating the alternatives according to the attribute in the given problem. Aggregation functions are used to merge the DeMs constructed by DMs into one in this process. In both situations mentioned above, this theory has led to a growing interest in studying numeric functions which allow aggregation.

Zadeh [1] initiated the fuzzy set (FS) theory, in 1965, to manage pragmatic events including imprecision and vagueness in the real-life by describing the uncertainty of an object or event via a degree of membership (MemD) with a value in the interval [1]. FS theory has been applied successfully in nearly every field of science such as engineering, artificial intelligence, computer science, economics, social sciences and etc. At the time, the studies and developments related to FS theory were progressing, Atanassov [2] observed that there is some inadequacies in the FS theory and proposed the structure of intuitionistic fuzzy set (IFS) as an extension of FS. Because each element is given by MemD and a non-membership degree (NMemD)

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in the IFS theory, IFS theory has been a more beneficial and affective tool to handle vagueness than the theory of FS. After, Yager [3] generalized the theory of IFS to the theory of Pythagorean fuzzy set (PyFS) by relaxing the condition on the MemD and NMemD. In spite of the IFS theory and PyFS theory having been extensively applied to a lot of areas, these theories could not capable to handle the situations that we face opinions including different kinds of answers such as “yes”, “abstain”, “no”, and “refusal”. We can consider a voting problem in a democratic election to explain such an issue. In the voting problem, the voters can be separated into 4 groups those who vote for, abstain, refuse the vote, and vote against. With this motivation, Cuong [4] suggested the notion of the picture fuzzy set (PFS) as an extension of IFS where the elements are described with the neutral membership degree (NeuMemD) addition to the MemD and NMemD. Hence, the PFS theory solved successfully the voting problem. However there were still some cases in that the theory of PFS can not be handled in some unstable and uncertain information. For instance, if one denoted their ideas about the situation in terms of “yes” is .8, “no” is .3, and “abstained” is .4, then we have $.8+.3+.4\nleq 1$. Thus, PFS theory was not capable to handle such kinds of cases. To solve these kinds of cases, Kahraman and Gündoğdu [5] proposed the theory of the spherical fuzzy set (SFS) as a generalization of FS, IFS and PFS. Still there existed some cases which could not be handled with the SFSs. For instance, if one takes MemD=.8, NeuMemD=.3 and NMemD=.6, then the sum of the squares of these numbers exceeds one. With this consideration, Mahmood et al. [6] presented the concept of the T-spherical fuzzy set (T-SFS) where the sum of the n-th power of the MemD, NeuMemD and NMemD is ≤ 1. After, Haque et al. [7] introduced the concept of generalized spherical fuzzy set (GSFS) as a generalization of the SFS where the sum of the squares of the MemD, NeuMemD and NMemD is ≤ 3.

In the light of the exploration of set theories mentioned above, decision-making theory has been developed by processing the information of fuzzy and its extensions in two ways such as traditional methods (AHP, TOPSIS, VIKOR, COPRAS, WASPAS, ELECTRE, MULTIMOORA, CRITIC, TODIM, etc.) and methods based on aggregation operators. We now mention the decision-making techniques depend on aggregation operators. In literature, aggregation operators have been constructed on either algebraic operational laws of the related set theory or t-norms and t-conorms families such as Dombi, Hamacher, Einstein, etc. For instance, the weighted averaging (WA) operator (see [8]) and the ordered weighted averaging (OWA) operator (see [9]) have been based on the algebraic sum and algebraic product in the crisp manner. Xu [10] developed the intuitionistic fuzzy (IF) WA operator, IF OWA operator, and IF hybrid averaging operator for aggregating IF information. Wang and Liu [11] presented some IF aggregation operators such as the IF Einstein WA operator and the IF Einstein ordered WA operator which extend the WA operator and the OWA operator in the IF environment. Yager [12] proposed some kinds of aggregation operators such as Pythagorean fuzzy (PyF) WA operator, PyF weighted geometric average (WGA) operator, PyF weighted power average operator, and PyF weighted power geometric operator to use them in the process of solving MCDM problems with PyF information. Garg [13] presented the PyF Einstein WA operator, PyF Einstein ordered WA (OWA) operator, generalized PyF Einstein WA operator, and generalized PyF Einstein OWA operator by investigating some desirable properties and applied to decision-making problems where experts provide their preferences in the PyF environment. Wei [14] introduced the picture fuzzy (PF) WA operator, PF WGA operator, PF OWA operator, PF ordered WGA operator, PF hybrid average operator, and PF hybrid geometric operator to develop some approaches for solving the PF MCDM problems. Khan et al. [15] and Munir et al. [16] described the Einstein aggregation operators (E-aggregation operators) for PF and T-spherical fuzzy information, respectively. They also give different methods to solce MCGDM problems by giving illustrative examples for real-life problems.

The motivation for this study can be explained as: GSFS theory is more capable to understand and process of vagueness and impreciseness when comparing the other fuzzy set theories. Especially, in decision-making problems this theory allows to accomplish more information related to alternatives and criteria. To defuzzify the generalized spherical fuzzy (GSF) value in decision-making process, Haque et al. [7] defined a score function and an accuracy function in the GSF environment. They also introduced GSF weighted exponential averaging operator to develop a MCGDM method in the GSF environment. Then, Güner and Aygün [17, 18] presented the E-aggregation operators (GSF Einstein weighted averaging (GSEWA) operator and GSF Einstein weighted geometric (GSEWG) operator) and Hamacher aggregation operators for GSF information and established MCGDM methods based on these operators. They [19] give the well-known TOPSIS method for GSF environment. Moreover, Güner and Aygün [20] studied the GSF topological spaces with applications to the MCDM problems. As a recent study, Haque et al. [21] initiated the linguistic GSFS by combining the idea of GSFS and linguistic fuzzy set. In addition, they presented various types of aggregation operators by applying to MCDM methods and also they solved the problem of most effective COVID-19 virus protector selection

The shortcoming of the existing works and the motivation for the presented aggregation operators in this study can be listed as:

- As seen above only a few works have been presented in the GSF environment.

- An appropriate option to the algebraic product is the Einstein product (E-product), which typically gives the same smooth approximations as the algebraic product. Hence, Einstein based t-norm and Einstein based t-conorm have the best approximation for the sum and product of the generalized spherical fuzzy numbers (GSFNs) as the alternative to algebraic sum and algebraic product.

- In the literature, it seems that there is a little investigation into aggregation techniques in GSF environments. Hence, there aren't different kinds of MCGDM approaches based on aggregation operators.

Also, the main objectives of this study are listed below:

- GSF Einstein ordered weighted averaging (GSEOWA), GSF Einstein hybrid weighted averaging (GSEHWA), GSF Einstein ordered weighted geometric (GSEOWG) and GSF Einstein hybrid weighted geometric (GSEHWG) operators are described.

- A model depend on the defined aggregation operators to use of the solution of the multiple attribute group decision-making problems consist of GSF information is established.

- An illustrative application to explain the proposed algorithm step by step is shown.

- The results under all defined GSF E-aggregation operators within the decision-making process are compared.

This paper consists of five sections: Section 2 includes some basic and relevant definitions that are needed in the following parts. In section 3, we give GSEOWA operator, GSEHWA operator, GSEOWG operator and GSEHWG operator based on the E-operations (E-sum, E-product and E-scalar multiplication) for GSFSs. We also study some basic properties of the presented aggregation operators. Then, we establish a technique for solving the MCGDM problems in GSF aspect. After, we provide a problem related to the medical treatment selection as an application which shows that the constructed technique is suitable and affective for the decision-making procedure. We compare the results under all defined E-aggregation operators in GSF environment by considering the given problem in section 4. We mention a brief conclusion for future works in section 5.

2. MATERIAL METHOD

In this part of the study, we recollect some notions that will be needed in the next parts. On the whole paper, U will refer to the set of discourse of the universe and I denotes the interval [.1].

Definition 2.1. [2, 3] Let $\mu: U \to I$ and $\nu: U \to I$ be any two mappings. A set $\mathcal{J} = \{(x, \mu(x), \nu(x)) | x \in U\}$ is said to be a/an

(i) IFS if the inequality $0 \leq \mu(x) + \nu(x) \leq 1, \forall x \in U,$ is fulfilled.

(ii) PyFS if the inequality $0 \leq \mu^2(x) + \nu^2(x) \leq 1, \forall x \in U,$ is fulfilled.

The values $\mu(x), \nu(x) \in I$ describe the MemD and NMemD of x to $\mathcal{J}$, respectively.
The pair $\mathcal{J} = (\mu, \nu)$ where $\mu, \nu \in I$ and $\mu + \nu \leq 1$ ($\mu^2 + \nu^2 \leq 1$), is said to be an IF number (IFN) (a PyF number (PyFN)).

Note 2.2. [3] The collection of IFNs is subset of the collection of PyFNs.

Definition 2.3. [4, 5, 7] Let $\mu: U \to I, \iota: U \to I$ and $\nu: U \to I$ be three mappings. A set $G = \{(x, \mu(x), \iota(x), \nu(x)) | x \in U\}$ is said to be a

(i) PFS if the inequality $0 \leq \mu(x) + \iota(x) + \nu(x) \leq 1, \forall x \in U$, is fulfilled.

(ii) SFS if the inequality $0 \leq \mu^2(x) + \iota^2(x) + \nu^2(x) \leq 1, \forall x \in U$, is fulfilled.

(iii) GSFS if the inequality $0 \leq \mu^2(x) + \iota^2(x) + \nu^2(x) \leq 3, \forall x \in U$, is fulfilled.

The values $\mu(x), \iota(x), \nu(x) \in I$ denote the MemD, NeuMemD and NMemD of x to G, respectively.
The triplet $G = (\mu, \iota, \nu)$ where $\mu, \iota, \nu \in I$ and $\mu^2 + \iota^2 + \nu^2 \leq 3$ ($\mu + \iota + \nu \leq 1$ and $\mu^2 + \iota^2 + \nu^2 \leq 1$, resp.), is said to be a GSFN (PF number (PFN) and SF number (SFN), respectively).

Note 2.4. [7] (1) The collection of SFNs is subset of the collection of GSFNs and the collection of PFNs is subset of the collection of SFNs.

(2) In the theory of PFN, because the sum of the MemD, NeuMemD and NMemD is ≤ 1, this summation is considered as linearly and this expresses a plane in $\mathbb{R}^3$. However, in theory of SFN and theory of GSFN, we take nonlinear form of the MemD, NeuMemD and NMemD which expresses a sphere in $\mathbb{R}^3$.

Definition 2.5. [7] Let $a \geq 0$ and $G = (\mu, \iota, \nu)$, $G_1 = (\mu_1, \iota_1, \nu_1)$, $G_2 = (\mu_2, \iota_2, \nu_2)$ be three GSFNs. Then the algebraic operations on GSFNs are given as follows:

(i) $G^c = (\nu, \iota, \mu)$,

(ii) $G_1 \leq G_2$ iff $\mu_1 \leq \mu_2, \iota_1 \geq \iota_2$ and $\nu_1 \geq \nu_2$,

(iii) $G_1 = G_2$ iff $G_1 \leq G_2$ and $G_2 \leq G_1$,

(iv) $G_1 + G_2 = \left(\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \iota_1\iota_2, \nu_1\nu_2\right)$,

(v) $G^a = \left(\mu^a, \iota^a, \sqrt{1 - (1 - \nu^2)^a}\right)$,

(vi) $aG = \left(\sqrt{1 - (1 - \mu^2)^a}, \iota^a, \nu^a\right)$.

Lemma 2.6. [7] Let $a, a_1, a_2 \geq 0$ and $G_1 = (\mu_1, \iota_1, \nu_1)$, $G_2 = (\mu_2, \iota_2, \nu_2)$ be any two GSFNs. Then the following assertions are fulfilled:

(i) $G_1 + G_2 = G_2 + G_1$,

(ii) $a(G_1 + G_2) = aG_1 + aG_2$,

(iii) $a_1G_1 + a_2G_1 = (a_1 + a_2)G_1$,

(vi) $\left(G_1^{a_1}\right)^{a_2} = G_1^{a_1a_2}$.

Definition 2.7. [17] Let $a \geq 0$ and $G = (\mu, \iota, \nu)$, $G_1 = (\mu_1, \iota_1, \nu_1)$, $G_2 = (\mu_2, \iota_2, \nu_2)$ be three GSFNs. Then the E-operations are characterized on the GSFNs as follows:

(i) $G_1 \oplus_E G_2 = \left(\sqrt{\frac{\mu_1^2+\mu_2^2}{1+\mu_1^2.\mu_2^2}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{1+(1-\iota_1^2)(1-\iota_2^2)}}, \sqrt{\frac{\nu_1^2.\nu_2^2}{1+(1-\nu_1^2)(1-\nu_2^2)}}\right)$,

(ii) $G_1 \odot_E G_2 = \left(\sqrt{\frac{\mu_1^2.\mu_2^2}{1+(1-\mu_1^2)(1-\mu_2^2)}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{1+(1-\iota_1^2)(1-\iota_2^2)}}, \sqrt{\frac{\nu_1^2+\nu_2^2}{1+\nu_1^2.\nu_2^2}}\right)$,

(iii) $a \cdot_E G = \left(\sqrt{\frac{(1+\mu^2)^a-(1-\mu^2)^a}{(1+\mu^2)^a+(1-\mu^2)^a}}, \sqrt{\frac{2\iota^{2a}}{(2-\iota^2)^a+\iota^{2a}}}, \sqrt{\frac{2\nu^{2a}}{(2-\nu^2)^a+\nu^{2a}}}\right)$,

(iv) $G^{\wedge_E a} = \left(\sqrt{\frac{2\mu^{2a}}{(2-\mu^2)^a+\mu^{2a}}}, \sqrt{\frac{2\iota^{2a}}{(2-\iota^2)^a+\iota^{2a}}}, \sqrt{\frac{(1+\nu^2)^a-(1-\nu^2)^a}{(1+\nu^2)^a+(1-\nu^2)^a}}\right)$.

Lemma 2.8. [17] Let $a, a_1, a_2 \geq 0$ and $G_1 = (\mu_1, \iota_1, \nu_1)$, $G_2 = (\mu_2, \iota_2, \nu_2)$ be any two GSFNs. Then the following assertions are fulfilled:

(i) $G_1 \oplus_E G_2 = G_2 \oplus_E G_1$,

(ii) $a \cdot_E (G_1 \oplus_E G_2) = a \cdot_E G_1 \oplus_E a \cdot_E G_2$,

(iii) $(a_1 + a_2) \cdot_E G_1 = a_1 \cdot_E G_1 \oplus_E a_2 \cdot_E G_1$,

(iv) $G_1 \odot_E G_2 = G_2 \odot_E G_1$,

(v) $(G_1 \odot_E G_2)^{\wedge_E a} = G_1^{\wedge_E a} \odot_E G_2^{\wedge_E a}$,

(vi) $G^{\wedge_E a_1} \odot_E G^{\wedge_E a_2} = G^{\wedge_E a_1 + a_2}$,

(vii) $\left(G_1^{\wedge_E a_1}\right)^{\wedge_E a_2} = G_1^{\wedge_E a_1 a_2}$.

Definition 2.9. [17] Let $\mathcal{A}$ be a family of every GSFNs and $(G_1, \dots, G_n) \in \mathcal{A}^n$ where $G_i = (\mu_i, \iota_i, \nu_i)$ for each $i = \overline{1, n}$ and $\mathfrak{w} = (w_1, \dots, w_n)^T$ denote the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0 \leq w_i$ for each i and $\sum_{i=1}^n w_i = 1$. Then

(i) a mapping $GSEWA_w: \mathcal{A}^n \to \mathcal{A}$ is called a GSEWA operator and is given as
$$GSEWA_w(G_1, \dots, G_n) = w_1 \cdot_E G_1 \oplus_E w_2 \cdot_E G_2 \oplus_E \dots w_n \cdot_E G_n = \oplus_{i=1}^n w_i \cdot_E G_i. \quad (1)$$

(ii) a mapping $GSEWG_w: \mathcal{A}^n \to \mathcal{A}$ is called a GSEWG operator and is given as
$$GSEWG_w(G_1, \dots, G_n) = G_1^{\wedge_E w_1} \odot_E G_2^{\wedge_E w_2} \odot_E \dots \odot_E G_n^{\wedge_E w_n} = \odot_{i=1}^n G_i^{\wedge_E w_i}. \quad (2)$$

Definition 2.1. [7] Let $\mathcal{A}$ be the family of all GSFNs and $G \in \mathcal{A}$ where $G = (\mu, \iota, \nu)$. Then a/an

(i) score function $SF: \mathcal{A} \to [-1,1]$ is given by $SF(G) = \frac{3\mu^2-2\iota^2-\nu^2}{3}$.

(ii) accuracy function $AF: \mathcal{A} \to I$ is given by $AF(G) = \frac{1+3\mu^2-\nu^2}{4}$.

Definition 2.11. [7] Suppose that $G_1 = (\mu_1, \iota_1, \nu_1)$ and $G_2 = (\mu_2, \iota_2, \nu_2)$ are two GSFNs. Then the comparison technique (method of ranking) is considered as:

(i) $SF(G_1) > SF(G_2) \Rightarrow G_1 > G_2$,

(ii) $SF(G_1) < SF(G_2) \Rightarrow G_1 < G_2$,

(iii) If $SF(G_1) = SF(G_2)$, then;

(a) $AF(G_1) < AF(G_2) \Rightarrow G_1 < G_2$,

(b) $AF(G_1) > AF(G_2) \Rightarrow G_1 > G_2$,

(c) $AF(G_1) = AF(G_2) \Rightarrow G_1 = G_2$.

3. THE RESEARCH FINDINGS AND DISCUSSION

In this part, we define the GSEOWA operator, GSEOWG operator, GSEHWA operator and GSEHWG operator based on the E-operations by investigating some basic properties of the presented aggregation operators. Then, we give a MCGDM technique based on these operators and show an illustrative application.

3.1. Generalized Spherical Fuzzy Einstein Ordered Aggragation Operators

Definition 3.1. Suppose that $\mathcal{A}$ is a family of each GSFNs and $(G_1, \dots, G_n) \in \mathcal{A}^n$ where $G_i = (\mu_i, \iota_i, \nu_i)$, $\forall i = \overline{1, n}$ and $\mathfrak{w} = (w_1, \dots, w_n)^T$ denotes the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0 \le w_i$ for each i and $\sum_{i=1}^n w_i = 1$. Then a mapping $GSEOWA_w: \mathcal{A}^n \to \mathcal{A}$ is called a GSEOWA operator and is given by

$$GSEOWA_w(G_1, \dots, G_n) = w_1 \cdot_E G_{\delta(1)} \oplus_E w_2 \cdot_E G_{\delta(2)} \oplus_E \dots \oplus_E w_n \cdot_E G_{\delta(n)} = \bigoplus_{i=1}^n w_i \cdot_E G_{\delta(i)} \quad (3)$$

where $\delta(i)$ $\left(i = \overline{1, n}\right)$ is the permutation wrt the score value (SV) satisfying $SF\left(G_{\delta(i-1)}\right) \ge SF\left(G_{\delta(i)}\right)$ for each $i = \overline{2, n}$.

Remark 3.2. If $w = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then $GSEOWA_w(G_1, \dots, G_n) = GSEWA_w(G_1, \dots, G_n)$, $\forall (G_1, \dots, G_n) \in \mathcal{A}^n$.

Theorem 3.3. Let $(G_1, \dots, G_n) \in \mathcal{A}^n$. Then the aggregated value $GSEOWA_w(G_1, \dots, G_n)$ is also a GSFN and is calculated by

$$GSEOWA_w(G_1, \dots, G_n) = \left(\sqrt{\frac{\prod_{i=1}^n \left(1+\mu_{\delta(i)}^2\right)^{w_i} - \prod_{i=1}^n \left(1-\mu_{\delta(i)}^2\right)^{w_i}}{\prod_{i=1}^n \left(1+\mu_{\delta(i)}^2\right)^{w_i} + \prod_{i=1}^n \left(1-\mu_{\delta(i)}^2\right)^{w_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota_{\delta(i)}^{2w_i}}{\prod_{i=1}^n \left(2-\iota_{\delta(i)}^2\right)^{w_i} + \prod_{i=1}^n \iota_{\delta(i)}^{2w_i}}}, \sqrt{\frac{2\prod_{i=1}^n \nu_{\delta(i)}^{2w_i}}{\prod_{i=1}^n \left(2-\nu_{\delta(i)}^2\right)^{w_i} + \prod_{i=1}^n \nu_{\delta(i)}^{2w_i}}} \right) \quad (4)$$

Proof. Equation (4) can be proved by using the mathematical induction method on n in the next way:
If $n = 2$, we get $GSEOWA_w(G_1, G_2) = w_1 \cdot_E G_{\delta(1)} \oplus_E w_2 \cdot_E G_{\delta(2)}$. Since $w_1 \cdot_E G_{\delta(1)}$ and $w_2 \cdot_E G_{\delta(2)}$ are GSFNs, then $w_1 \cdot_E G_{\delta(1)} \oplus_E w_2 \cdot_E G_{\delta(2)}$ is also a GSFN. Then, we obtain

$$GSEOWA_w(G_1, G_2) = w_1 \cdot_E G_{\delta(1)} \oplus_E w_2 \cdot_E G_{\delta(2)}$$

$$= \left(\sqrt{\frac{\left(1+\mu_{\delta(1)}^2\right)^{w_1} - \left(1-\mu_{\delta(1)}^2\right)^{w_1}}{\left(1+\mu_{\delta(1)}^2\right)^{w_1} + \left(1-\mu_{\delta(1)}^2\right)^{w_1}}}, \sqrt{\frac{2\iota_{\delta(1)}^{2w_1}}{\left(2-\iota_{\delta(1)}^2\right)^{w_1} + \iota_{\delta(1)}^{2w_1}}}, \sqrt{\frac{2\nu_{\delta(1)}^{2w_1}}{\left(2-\nu_{\delta(1)}^2\right)^{w_1} + \nu_{\delta(1)}^{2w_1}}} \right)$$
$$\oplus_E \left(\sqrt{\frac{\left(1+\mu_{\delta(2)}^2\right)^{w_2} - \left(1-\mu_{\delta(2)}^2\right)^{w_2}}{\left(1+\mu_{\delta(2)}^2\right)^{w_2} + \left(1-\mu_{\delta(2)}^2\right)^{w_2}}}, \sqrt{\frac{2\iota_{\delta(2)}^{2w_2}}{\left(2-\iota_{\delta(2)}^2\right)^{w_2} + \iota_{\delta(2)}^{2w_2}}}, \sqrt{\frac{2\nu_{\delta(2)}^{2w_2}}{\left(2-\nu_{\delta(2)}^2\right)^{w_2} + \nu_{\delta(2)}^{2w_2}}} \right)$$

$$= \left(\sqrt{\frac{\frac{\left(1+\mu_{\delta(1)}^2\right)^{w_1}-\left(1-\mu_{\delta(1)}^2\right)^{w_1}}{\left(1+\mu_{\delta(1)}^2\right)^{w_1}+\left(1-\mu_{\delta(1)}^2\right)^{w_1}}+\frac{\left(1+\mu_{\delta(2)}^2\right)^{w_2}-\left(1-\mu_{\delta(2)}^2\right)^{w_2}}{\left(1+\mu_{\delta(2)}^2\right)^{w_2}+\left(1-\mu_{\delta(2)}^2\right)^{w_2}}}{1+\left[\frac{\left(1+\mu_{\delta(1)}^2\right)^{w_1}-\left(1-\mu_{\delta(1)}^2\right)^{w_1}}{\left(1+\mu_{\delta(1)}^2\right)^{w_1}+\left(1-\mu_{\delta(1)}^2\right)^{w_1}}\right]\left[\frac{\left(1+\mu_{\delta(2)}^2\right)^{w_2}-\left(1-\mu_{\delta(2)}^2\right)^{w_2}}{\left(1+\mu_{\delta(2)}^2\right)^{w_2}+\left(1-\mu_{\delta(2)}^2\right)^{w_2}}\right]}}, \sqrt{\frac{\left(\frac{2\iota_{\delta(1)}^{2w_1}}{\left(2-\iota_{\delta(1)}^2\right)^{w_1}+\iota_{\delta(1)}^{2w_1}}\right)\left(\frac{2\iota_{\delta(2)}^{2w_2}}{\left(2-\iota_{\delta(2)}^2\right)^{w_2}+\iota_{\delta(2)}^{2w_2}}\right)}{1+\left[1-\frac{2\iota_{\delta(1)}^{2w_1}}{\left(2-\iota_{\delta(1)}^2\right)^{w_1}+\iota_{\delta(1)}^{2w_1}}\right]\left[1-\frac{2\iota_{\delta(2)}^{2w_2}}{\left(2-\iota_{\delta(2)}^2\right)^{w_2}+\iota_{\delta(2)}^{2w_2}}\right]}}, \sqrt{\frac{\left(\left\{\frac{2\nu_{\delta(1)}^{2w_1}}{\left(2-\nu_{\delta(1)}^2\right)^{w_1}+\nu_{\delta(1)}^{2w_1}}\right\}\right)\left(\frac{2\nu_{\delta(2)}^{2w_2}}{\left(2-\nu_{\delta(2)}^2\right)^{w_2}+\nu_{\delta(2)}^{2w_2}}\right)}{1+\left[1-\frac{2\nu_{\delta(1)}^{2w_1}}{\left(2-\nu_{\delta(1)}^2\right)^{w_1}+\nu_{\delta(1)}^{2w_1}}\right]\left[1-\frac{2\nu_{\delta(2)}^{2w_2}}{\left(2-\nu_{\delta(2)}^2\right)^{w_2}+\nu_{\delta(2)}^{2w_2}}\right]}} \right)$$

$$= \left(\sqrt{\frac{\left(1+\mu_{\delta(1)}^2\right)^{w_1}.\left(1+\mu_{\delta(2)}^2\right)^{w_2}-\left(1-\mu_{\delta(1)}^2\right)^{w_1}.\left(1-\mu_{\delta(2)}^2\right)^{w_2}}{\left(1+\mu_{\delta(1)}^2\right)^{w_1}.\left(1+\mu_{\delta(2)}^2\right)^{w_2}+\left(1-\mu_{\delta(1)}^2\right)^{w_1}.\left(1-\mu_{\delta(2)}^2\right)^{w_2}}}, \sqrt{\frac{2\iota_{\delta(1)}^{2w_1}.\iota_{\delta(2)}^{2w_2}}{\left(2-\iota_{\delta(1)}^2\right)^{w_1}.\left(2-\iota_{\delta(2)}^2\right)^{w_2}+\iota_{\delta(1)}^{2w_1}.\iota_{\delta(2)}^{2w_2}}}, \sqrt{\frac{2\nu_{\delta(1)}^{2w_1}.\nu_{\delta(2)}^{2w_2}}{\left(2-\nu_{\delta(1)}^2\right)^{w_1}.\left(2-\nu_{\delta(2)}^2\right)^{w_2}+\nu_{\delta(1)}^{2w_1}.\nu_{\delta(2)}^{2w_2}}} \right)$$

$$= \left(\sqrt{\frac{\prod_{i=1}^2\left(1+\mu_{\delta(i)}^2\right)^{w_i}-\prod_{i=1}^2\left(1-\mu_{\delta(i)}^2\right)^{w_i}}{\prod_{i=1}^2\left(1+\mu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^2\left(1-\mu_{\delta(i)}^2\right)^{w_i}}}, \sqrt{\frac{2\prod_{i=1}^2 \iota_{\delta(i)}^{2w_i}}{\prod_{i=1}^2\left(2-\iota_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^2 \iota_{\delta(i)}^{2w_i}}}, \sqrt{\frac{2\prod_{i=1}^2 \nu_{\delta(i)}^{2w_i}}{\prod_{i=1}^2\left(2-\nu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^2 \nu_{\delta(i)}^{2w_i}}} \right)$$

Hence, the Equation (4) is fulfilled for $n = 2$. Now, we suppose that the Equation (4) is fulfilled for $n = k$:

$$GSEOWA_w(G_1, \dots, G_k) = w_1 \cdot_E G_{\delta(1)} \oplus_E w_2 \cdot_E G_{\delta(2)} \oplus_E \dots \oplus_E w_k \cdot_E G_{\delta(k)}$$

$$= \left(\sqrt{\frac{\prod_{i=1}^k\left(1+\mu_{\delta(i)}^2\right)^{w_i}-\prod_{i=1}^k\left(1-\mu_{\delta(i)}^2\right)^{w_i}}{\prod_{i=1}^k\left(1+\mu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k\left(1-\mu_{\delta(i)}^2\right)^{w_i}}}, \sqrt{\frac{2\prod_{i=1}^k \iota_{\delta(i)}^{2w_i}}{\prod_{i=1}^k\left(2-\iota_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k \iota_{\delta(i)}^{2w_i}}}, \sqrt{\frac{2\prod_{i=1}^k \nu_{\delta(i)}^{2w_i}}{\prod_{i=1}^k\left(2-\nu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k \nu_{\delta(i)}^{2w_i}}} \right).$$

Similarly, we show that the Equation (4) is fulfilled for $n = k + 1$. Then, we have

$$GSEOWA_w(G_1, \dots, G_k, G_{k+1}) = GSEOWA_w(G_1, \dots, G_k) \oplus_E w_{k+1} G_{\delta(k+1)}$$

$$= \left(\sqrt{\frac{\prod_{i=1}^k\left(1+\mu_{\delta(i)}^2\right)^{w_i}-\prod_{i=1}^k\left(1-\mu_{\delta(i)}^2\right)^{w_i}}{\prod_{i=1}^k\left(1+\mu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k\left(1-\mu_{\delta(i)}^2\right)^{w_i}}}, \sqrt{\frac{2\prod_{i=1}^k \iota_{\delta(i)}^{2w_i}}{\prod_{i=1}^k\left(2-\iota_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k \iota_{\delta(i)}^{2w_i}}}, \sqrt{\frac{2\prod_{i=1}^k \nu_{\delta(i)}^{2w_i}}{\prod_{i=1}^k\left(2-\nu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k \nu_{\delta(i)}^{2w_i}}} \right)$$

$$\oplus_E \left(\sqrt{\frac{\left(1+\mu_{\delta(k+1)}^2\right)^{w_{k+1}}-\left(1-\mu_{\delta(k+1)}^2\right)^{w_{k+1}}}{\left(1+\mu_{\delta(k+1)}^2\right)^{w_{k+1}}+\left(1-\mu_{\delta(k+1)}^2\right)^{w_{k+1}}}}, \sqrt{\frac{2\iota_{\delta(k+1)}^{2w_{k+1}}}{\left(2-\iota_{\delta(k+1)}^2\right)^{w_{k+1}}+\iota_{\delta(k+1)}^{2w_{k+1}}}}, \sqrt{\frac{2\nu_{\delta(k+1)}^{2w_1}}{\left(2-\nu_{\delta(k+1)}^2\right)^{w_{k+1}}+\nu_{\delta(k+1)}^{2w_{k+1}}}} \right)$$

$$= \left(\sqrt{\frac{\frac{\prod_{i=1}^k\left(1+\mu_{\delta(k+1)}^2\right)^{w_i}-\prod_{i=1}^k\left(1-\mu_{\delta(k+1)}^2\right)^{w_i}}{\prod_{i=1}^k\left(1+\mu_{\delta(k+1)}^2\right)^{w_i}+\prod_{i=1}^k\left(1-\mu_{\delta(k+1)}^2\right)^{w_i}}+\frac{\left(1+\mu_{\delta(k+1)}^2\right)^{w_{k+1}}-\left(1-\mu_{\delta(k+1)}^2\right)^{w_{k+1}}}{\left(1+\mu_{\delta(k+1)}^2\right)^{w_{k+1}}+\left(1-\mu_{\delta(k+1)}^2\right)^{w_{k+1}}}}{1+\left(\frac{\prod_{i=1}^k\left(1+\mu_{\delta(i)}^2\right)^{w_i}-\prod_{i=1}^k\left(1-\mu_{\delta(i)}^2\right)^{w_i}}{\prod_{i=1}^k\left(1+\mu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k\left(1-\mu_{\delta(i)}^2\right)^{w_i}}\right)\left(\frac{\left(1+\mu_{\delta(k+1)}^2\right)^{w_{k+1}}-\left(1-\mu_{\delta(k+1)}^2\right)^{w_{k+1}}}{\left(1+\mu_{\delta(k+1)}^2\right)^{w_{k+1}}+\left(1-\mu_{\delta(k+1)}^2\right)^{w_{k+1}}}\right)}}, \right.$$
$$\left. \sqrt{\frac{\left(\frac{2\prod_{i=1}^k \iota_{\delta(i)}^{2w_i}}{\prod_{i=1}^k\left(2-\iota_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k \iota_{\delta(i)}^{2w_i}}\right)\left(\frac{2\iota_{\delta(k+1)}^{2w_{k+1}}}{\left(2-\iota_{\delta(k+1)}^2\right)^{w_{k+1}}+\iota_{\delta(k+1)}^{2w_{k+1}}}\right)}{1+\left(1-\frac{2\prod_{i=1}^k \iota_{\delta(i)}^{2w_i}}{\prod_{i=1}^k\left(2-\iota_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k \iota_{\delta(i)}^{2w_i}}\right)\left(1-\frac{2\iota_{\delta(k+1)}^{2w_{k+1}}}{\left(2-\iota_{\delta(k+1)}^2\right)^{w_{k+1}}+\iota_{\delta(k+1)}^{2w_{k+1}}}\right)}}, \sqrt{\frac{\left(\frac{2\prod_{i=1}^k \nu_{\delta(i)}^{2w_i}}{\prod_{i=1}^k\left(2-\nu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k \nu_{\delta(i)}^{2w_i}}\right)\left(\frac{2\nu_{\delta(k+1)}^{2w_{k+1}}}{\left(2-\nu_{\delta(k+1)}^2\right)^{w_{k+1}}+\nu_{\delta(k+1)}^{2w_{k+1}}}\right)}{1+\left(1-\frac{2\prod_{i=1}^k \nu_{\delta(i)}^{2w_i}}{\prod_{i=1}^k\left(2-\nu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^k \nu_{\delta(i)}^{2w_i}}\right)\left(1-\frac{2\nu_{\delta(k+1)}^{2w_{k+1}}}{\left(2-\nu_{\delta(k+1)}^2\right)^{w_{k+1}}+\nu_{\delta(k+1)}^{2w_{k+1}}}\right)}} \right).$$

Thus, the Equation (4) is fulfilled for $n = k + 1$. Thus, by the mathematical induction method the Equation (4) is valid for each $n \in \mathbb{N}$.

Lemma 3.4. (i) (Idempotency of $GSEOWA_w$ operator) If $G_i = G$, $\forall i = \overline{1,n}$ where $G = (\mu, \iota, \nu)$, $G_i = (\mu_i, \iota_i, \nu_i)$ and $\mathfrak{w} = (w_1, \dots, w_n)^T$ denotes the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0 \le w_i$ for each i and $\sum_{i=1}^n w_i = 1$, then $GSEOWA_w(G_1, \dots, G_n) = G$.

(ii) (Boundedness of $GSEOWA_w$ operator) Let $(G_1, \dots, G_n) \in \mathcal{A}^n$. Then,

$$\underline{\min}_{i=\overline{1,n}} G_i \le GSEOWA_w(G_1, \dots, G_n) \le \underline{\max}_{i=\overline{1,n}} G_i.$$

Here, $\underline{\min}_{i=\overline{1,n}} G_i = \left(\underline{\min}_{i=\overline{1,n}} \mu_i, \underline{\max}_{i=\overline{1,n}} \iota_i, \underline{\max}_{i=\overline{1,n}} \nu_i\right)$ and $\underline{\max}_{i=\overline{1,n}} G_i = \left(\underline{\max}_{i=\overline{1,n}} \mu_i, \underline{\min}_{i=\overline{1,n}} \iota_i, \underline{\min}_{i=\overline{1,n}} \nu_i\right)$.

(iii) (Monotonicity of $GSEOWA_w$ operator) Let $(G_1, \dots, G_n), (G_1', \dots, G_n') \in \mathcal{A}^n$. If $G_i \le G_i'$ for each $i = \overline{1,n}$, then $GSEOWA_w(G_1, \dots, G_n) \le GSEOWA_w(G_1', \dots, G_n')$.

Proof. (i) Let $G_i = G, \forall i = \overline{1,n}$ where $G = (\mu, \iota, \nu)$ and $G_i = (\mu_i, \iota_i, \nu_i)$. It follows that $G_{\delta(i)} = G$ for each $i = \overline{1,n}$. Suppose that $\mathfrak{w} = (w_1, \dots, w_n)^T$ denotes the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0 \le w_i$ for each i and $\sum_{i=1}^n w_i = 1$. Since $G_{\delta(i)} = G$, $\forall i = \overline{1,n}$, we have that $\mu_{\delta(i)} = \mu, \iota_{\delta(i)} = \iota$ and $\nu_{\delta(i)} = \nu$ for each $i = \overline{1,n}$. Then

$$GSEOWA_w(G_1, \dots, G_n) = \left(\sqrt{\frac{\prod_{i=1}^n (1+\mu_{\delta(i)}^2)^{w_i} - \prod_{i=1}^n (1-\mu_{\delta(i)}^2)^{w_i}}{\prod_{i=1}^n (1+\mu_{\delta(i)}^2)^{w_i} + \prod_{i=1}^n (1-\mu_{\delta(i)}^2)^{w_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota_{\delta(i)}^{2w_i}}{\prod_{i=1}^n (2-\iota_{\delta(i)}^2)^{w_i} + \prod_{i=1}^n \iota_{\delta(i)}^{2w_i}}}, \sqrt{\frac{2\prod_{i=1}^n \nu_{\delta(i)}^{2w_i}}{\prod_{i=1}^n (2-\nu_{\delta(i)}^2)^{w_i} + \prod_{i=1}^n \nu_{\delta(i)}^{2w_i}}}\right)$$

$$= \left(\sqrt{\frac{\prod_{i=1}^n (1+\mu^2)^{w_i} - \prod_{i=1}^n (1-\mu^2)^{w_i}}{\prod_{i=1}^n (1+\mu^2)^{w_i} + \prod_{i=1}^n (1-\mu^2)^{w_i}}}, \sqrt{\frac{2\prod_{i=1}^n \iota^{2w_i}}{\prod_{i=1}^n (2-\iota^2)^{w_i} + \prod_{i=1}^n \iota^{2w_i}}}, \sqrt{\frac{2\prod_{i=1}^n \nu^{2w_i}}{\prod_{i=1}^n (2-\nu^2)^{w_i} + \prod_{i=1}^n \nu^{2w_i}}}\right)$$

$$= \left(\sqrt{\frac{(1+\mu^2)^{\sum_{i=1}^n w_i} - (1-\mu^2)^{\sum_{i=1}^n w_i}}{(1+\mu^2)^{\sum_{i=1}^n w_i} + (1-\mu^2)^{\sum_{i=1}^n w_i}}}, \sqrt{\frac{2\iota^{\sum_{i=1}^n 2w_i}}{(2-\iota^2)^{\sum_{i=1}^n w_i} + \iota^{\sum_{i=1}^n 2w_i}}}, \sqrt{\frac{2\nu^{\sum_{i=1}^n 2w_i}}{(2-\nu^2)^{\sum_{i=1}^n w_i} + \nu^{\sum_{i=1}^n 2w_i}}}\right) = (\mu, \iota, \nu) = G.$$

(ii) Let $\mathfrak{w} = (w_1, \dots, w_n)^T$ denote the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0 \le w_i$ for each i and $\sum_{i=1}^n w_i = 1$. Take $f_1(x) = \frac{1-x^2}{1+x^2}$ where $x \in I$, then ${f_1}'(x) = \frac{-4x}{(1+x^2)^2} \le$. $\forall x \in I$. Thus f_1 is a nonincreasing function on I. Since $\underline{\min}_{i=\overline{1,n}} \mu_i \le \mu_{\delta(j)} \le \underline{\max}_{i=\overline{1,n}} \mu_i, \forall j = \overline{1,n}$, then $f_1\left(\underline{\max}_{i=\overline{1,n}} \mu_i\right) \le f_1(\mu_{\delta(j)}) \le f_1\left(\underline{\min}_{i=\overline{1,n}} \mu_i\right)$ for each $j = \overline{1,n}$. Hence, we have

$$\left(\frac{1-\left(\underline{\max}_{i=\overline{1,n}} \mu_i\right)^2}{1+\left(\underline{\max}_{i=\overline{1,n}} \mu_i\right)^2}\right)^{w_j} \le \left(\frac{1-{\mu_{\delta(j)}}^2}{1+{\mu_{\delta(j)}}^2}\right)^{w_j} \le \left(\frac{1-\left(\underline{\min}_{i=\overline{1,n}} \mu_i\right)^2}{1+\left(\underline{\min}_{i=\overline{1,n}} \mu_i\right)^2}\right)^{w_j}$$

for every $j = \overline{1,n}$. Thus,

$$\prod_{j=1}^n \left(\frac{1-\left(\underline{\max}_{i=\overline{1,n}} \mu_i\right)^2}{1+\left(\underline{\max}_{i=\overline{1,n}} \mu_i\right)^2}\right)^{w_j} \le \prod_{j=1}^n \left(\frac{1-{\mu_{\delta(j)}}^2}{1+{\mu_{\delta(j)}}^2}\right)^{w_j} \le \prod_{j=1}^n \left(\frac{1-\left(\underline{\min}_{i=\overline{1,n}} \mu_i\right)^2}{1+\left(\underline{\min}_{i=\overline{1,n}} \mu_i\right)^2}\right)^{w_j}$$

$$\Leftrightarrow \left(\frac{1-\left(\underline{\max}_{i=1,n}\mu_i\right)^2}{1+\left(\underline{\max}_{i=1,n}\mu_i\right)^2}\right)^{\sum_{j=1}^{n} w_j} \leq \prod_{j=1}^{n}\left(\frac{1-\mu_{\delta(j)}{}^2}{1+\mu_{\delta(j)}{}^2}\right)^{w_j} \leq \left(\frac{1-\left(\underline{\min}_{i=1,n}\mu_i\right)^2}{1+\left(\underline{\min}_{i=1,n}\mu_i\right)^2}\right)^{\sum_{j=1}^{n} w_j}$$

$$\Leftrightarrow \frac{1-\left(\underline{\max}_{i=1,n}\mu_i\right)^2}{1+\left(\underline{\max}_{i=1,n}\mu_i\right)^2} \leq \prod_{j=1}^{n}\left(\frac{1-\mu_{\delta(j)}{}^2}{1+\mu_{\delta(j)}{}^2}\right)^{w_j} \leq \frac{1-\left(\underline{\min}_{i=1,n}\mu_i\right)^2}{1+\left(\underline{\min}_{i=1,n}\mu_i\right)^2}$$

$$\Leftrightarrow \frac{2}{1+\left(\underline{\max}_{i=1,n}\mu_i\right)^2} \leq \prod_{j=1}^{n}\left(\frac{1-\mu_{\delta(j)}{}^2}{1+\mu_{\delta(j)}{}^2}\right)^{w_j} + 1 \leq \frac{2}{1+\left(\underline{\min}_{i=1,n}\mu_i\right)^2}$$

$$\Leftrightarrow \frac{1+\left(\underline{\min}_{i=1,n}\mu_i\right)^2}{2} \leq \frac{1}{1+\prod_{j=1}^{n}\left(\frac{1-\mu_{\delta(j)}{}^2}{1+\mu_{\delta(j)}{}^2}\right)^{w_j}} \leq \frac{1+\left(\underline{\max}_{i=1,n}\mu_i\right)^2}{2} \Leftrightarrow \left(\underline{\min}_{i=1,n}\mu_i\right)^2 \leq \frac{2}{1+\prod_{j=1}^{n}\left(\frac{1-\mu_{\delta(j)}{}^2}{1+\mu_{\delta(j)}{}^2}\right)^{w_j}} - 1 \leq \left(\underline{\max}_{i=1,n}\mu_i\right)^2$$

$$\Leftrightarrow \left(\underline{\min}_{i=1,n}\mu_i\right)^2 \leq \frac{1-\prod_{j=1}^{n}\left(\frac{1-\mu_{\delta(j)}{}^2}{1+\mu_{\delta(j)}{}^2}\right)^{w_j}}{1+\prod_{j=1}^{n}\left(\frac{1-\mu_{\delta(j)}{}^2}{1+\mu_{\delta(j)}{}^2}\right)^{w_j}} \leq \left(\underline{\max}_{i=1,n}\mu_i\right)^2$$

$$\Leftrightarrow \underline{\min}_{i=1,n}\mu_i \leq \sqrt{\frac{\prod_{j=1}^{n}(1+\mu_{\delta(j)}{}^2)-\prod_{j=1}^{n}(1-\mu_{\delta(j)}{}^2)^{w_j}}{\prod_{j=1}^{n}(1+\mu_{\delta(j)}{}^2)+\prod_{j=1}^{n}(1-\mu_{\delta(j)}{}^2)^{w_j}}} \leq \underline{\max}_{i=1,n}\mu_i\,.$$

Let $f_2(x) = \frac{2-x^2}{x^2}$ where $x \in (.1]$, then $f_2{}'(x) = \frac{-4}{x^3} <. \forall x \in (.1]$, Thus, f_2 is a nonincreasing function on the interval $(.1]$. Since we have $\underline{\min}_{i=1,n}\iota_i \leq \iota_{\delta(j)} \leq \underline{\max}_{i=1,n}\iota_i$, $\forall j = \overline{1,n}$, then $f_2\left(\underline{\max}_{i=1,n}\iota_i\right) \leq f_2(\iota_{\delta(j)}) \leq f_2\left(\underline{\min}_{i=1,n}\iota_i\right)$ for each $j = \overline{1,n}$. Hence, we have

$$\left(\frac{2-\left(\underline{\max}_{i=1,n}\iota_i\right)^2}{\left(\underline{\max}_{i=1,n}\iota_i\right)^2}\right)^{w_j} \leq \left(\frac{2-\iota_{\delta(j)}{}^2}{\iota_{\delta(j)}{}^2}\right)^{w_j} \leq \left(\frac{2-\left(\underline{\min}_{i=1,n}\iota_i\right)^2}{\left(\underline{\min}_{i=1,n}\iota_i\right)^2}\right)^{w_j}$$

for all $j = \overline{1,n}$. Thus,

$$\prod_{j=1}^{n}\left(\frac{2-\left(\underline{\max}_{i=1,n}\iota_i\right)^2}{\left(\underline{\max}_{i=1,n}\iota_i\right)^2}\right)^{w_j} \leq \prod_{j=1}^{n}\left(\frac{2-\iota_{\delta(j)}{}^2}{\iota_{\delta(j)}{}^2}\right)^{w_j} \leq \prod_{j=1}^{n}\left(\frac{2-\left(\underline{\min}_{i=1,n}\iota_i\right)^2}{\left(\underline{\min}_{i=1,n}\iota_i\right)^2}\right)^{w_j}$$

$$\Leftrightarrow \left(\frac{2-\left(\underline{\max}_{i=1,n}\iota_i\right)^2}{\left(\underline{\max}_{i=1,n}\iota_i\right)^2}\right)^{\sum_{j=1}^{n} w_j} \leq \prod_{j=1}^{n}\left(\frac{2-\iota_{\delta(j)}{}^2}{\iota_{\delta(j)}{}^2}\right)^{w_j} \leq \left(\frac{2-\left(\underline{\min}_{i=1,n}\iota_i\right)^2}{\left(\underline{\min}_{i=1,n}\iota_i\right)^2}\right)^{\sum_{j=1}^{n} w_j}$$

$$\Leftrightarrow \frac{2-\left(\underline{\max}_{i=1,n}\iota_i\right)^2}{\left(\underline{\max}_{i=1,n}\iota_i\right)^2} \leq \prod_{j=1}^{n}\left(\frac{2-\iota_{\delta(j)}{}^2}{\iota_i{}^2}\right)^{w_{\delta(j)}} \leq \frac{2-\left(\underline{\min}_{i=1,n}\iota_i\right)^2}{\left(\underline{\min}_{i=1,n}\iota_i\right)^2}$$

$$\Leftrightarrow \frac{2}{\left(\underset{i=\overline{1,n}}{max}\,\iota_i\right)^2} \leq \prod_{j=1}^{n}\left(\frac{2-\iota_{\delta(j)}{}^2}{\iota_{\delta(j)}{}^2}\right)^{w_j}+1 \leq \frac{2}{\left(\underset{i=\overline{1,n}}{min}\,\iota_i\right)^2} \Leftrightarrow \frac{\left(\underset{i=\overline{1,n}}{min}\,\iota_i\right)^2}{2} \leq \frac{1}{1+\prod_{j=1}^{n}\left(\frac{2-\iota_{\delta(j)}{}^2}{\iota_{\delta(j)}{}^2}\right)^{w_j}} \leq \frac{\left(\underset{i=\overline{1,n}}{max}\,\iota_i\right)^2}{2}$$

$$\Leftrightarrow \left(\underset{i=\overline{1,n}}{min}\,\iota_i\right)^2 \leq \frac{2}{1+\prod_{j=1}^{n}\left(\frac{2-\iota_{\delta(j)}{}^2}{\iota_i{}^2}\right)^{w_j}} \leq \left(\underset{i=\overline{1,n}}{max}\,\iota_i\right)^2 \Leftrightarrow \underset{i=\overline{1,n}}{min}\,\iota_i \leq \sqrt{\frac{2\prod_{j=1}^{n}\iota_{\delta(j)}{}^{2w_j}}{\prod_{j=1}^{n}\left(2-\iota_{\delta(j)}{}^2\right)^{w_j}+\prod_{j=1}^{n}\iota_{\delta(j)}{}^{2w_j}}} \leq \underset{i=\overline{1,n}}{max}\,\iota_i \quad (5)$$

Additionally, it is clear that (5) is fulfilled even if $\underset{i=\overline{1,n}}{max}\,\iota_i\ =$. Also, we have with the similar consideration that

$$\underset{i=\overline{1,n}}{min}\,\nu_i \leq \sqrt{\frac{2\prod_{j=1}^{n}\nu_{\delta(j)}{}^{2w_j}}{\prod_{j=1}^{n}\left(2-\nu_{\delta(j)}{}^2\right)^{w_j}+\prod_{j=1}^{n}\nu_{\delta(j)}{}^{2w_j}}} \leq \underset{i=\overline{1,n}}{max}\,\nu_i.$$

By the Definition 3.1, we obtain that $\underset{i=\overline{1,n}}{min}\,G_{\delta(i)} \leq GSEOWA_w(G_1,\dots,G_n) \leq \underset{i=\overline{1,n}}{max}\,G_{\delta(i)}$.

(iii) This assertion can be easily proved similar to (i).

Definition 3.5. Suppose that $\mathcal{G}$ is a family of each GSFNs and $(G_1,\dots,G_n)\in\mathcal{A}^n$ where $G_i=(\mu_i,\iota_i,\nu_i)$ for each $i=\overline{1,n}$ and $\mathfrak{w}=(w_1,\dots,w_n)^T$ denotes the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0\leq w_i$ for each i and $\sum_{i=1}^n w_i=1$. Then a mapping $GSEOWG_w\colon\mathcal{A}^n\to\mathcal{A}$ is said to be a GSEOWG operator and is given by

$$GSEOWG_w(G_1,\dots,G_n)\ =G_{\delta(1)}^{\wedge_E w_1}\odot_E G_{\delta(2)}^{\wedge_E w_2}\odot_E\dots\odot_E G_{\delta(n)}^{\wedge_E w_n}=\odot_{i=1}^n G_{\delta(i)}^{\wedge_E w_i} \quad (6)$$

where $\delta(i)\left(i=\overline{1,n}\right)$ is the permutation wrt the SV satisfying $SF\left(G_{\delta(i-1)}\right)\geq SF\left(G_{\delta(i)}\right)$, $\forall i=\overline{2,n}$.

Remark 3.6. If $w=\left(\frac{1}{n},\dots,\frac{1}{n}\right)^T$, then $GSEOWG_w(G_1,\dots,G_n)=GSEWG_w(G_1,\dots,G_n)$, $\forall(G_1,\dots,G_n)\in\mathcal{A}^n$.

Theorem 3.7. Let $(G_1,\dots,G_n)\in\mathcal{G}^n$. Then the aggregated value $GSEOWG_w(G_1,\dots,G_n)$ is also a GSFN and is calculated by

$GSEOWG_w(G_1,\dots,G_n)=$

$$\left(\sqrt{\frac{2\prod_{i=1}^n\mu_{\delta(i)}^{2w_i}}{\prod_{i=1}^n\left(2-\mu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^n\mu_{\delta(i)}^{2w_i}}},\sqrt{\frac{2\prod_{i=1}^n\iota_{\delta(i)}^{2w_i}}{\prod_{i=1}^n\left(2-\iota_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^n\iota_{\delta(i)}^{2w_i}}},\sqrt{\frac{\prod_{i=1}^n\left(1+\nu_{\delta(i)}^2\right)^{w_i}-\prod_{i=1}^n\left(1-\nu_{\delta(i)}^2\right)^{w_i}}{\prod_{i=1}^n\left(1+\nu_{\delta(i)}^2\right)^{w_i}+\prod_{i=1}^n\left(1-\nu_{\delta(i)}^2\right)^{w_i}}}\right) \quad (7)$$

Proof. This assertion can be shown with a similar process to Theorem 3.3.

Lemma 3.8. (i) (Idempotency of $GSEOWG_w$ operator) If $G_i=G$, $\forall i=\overline{1,n}$ where $G=(\mu,\iota,\nu)$, $G_i=(\mu_i,\iota_i,\nu_i)$ and $\mathfrak{w}=(w_1,\dots,w_n)^T$ denotes the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0\leq w_i$ for each i and $\sum_{i=1}^n w_i=1$, then $GSEOWG_w(G_1,\dots,G_n)=G$.

(ii) (Boundedness of $GSEOWG_w$ operator) Let $(G_1,\dots,G_n)\in\mathcal{A}^n$. Then,

$$\underset{i=\overline{1,n}}{min}\,G_i\leq GSEOWG_w(G_1,\dots,G_n)\leq\underset{i=\overline{1,n}}{max}\,G_i.$$

Here, $\underset{i=\overline{1,n}}{min} G_i = \left(\underset{i=\overline{1,n}}{min}\, \mu_i, \underset{i=\overline{1,n}}{max}\, \iota_i, \underset{i=\overline{1,n}}{max}\, \nu_i\right)$ and $\underset{i}{max}\, G_i = \left(\underset{i=\overline{1,n}}{max}\, \mu_i, \underset{i=\overline{1,n}}{min}\, \iota_i, \underset{i=\overline{1,n}}{min}\, \nu_i\right)$.
(iii) (Monotonicity of $GSEOWG_w$ operator) Let $(G_1, \dots, G_n), (G_1', \dots, G_n') \in \mathcal{A}^n$. If $G_i \leq G_i'$, $\forall i = \overline{1,n}$, then $GSEOWG_w(G_1, \dots, G_n) \leq GSEOWG_w(G_1', \dots, G_n')$.

Proof. These assertions can be easily proved with a similar process to Lemma 3.4.

3.2. Generalized Spherical Fuzzy Einstein Hybrid Aggragation Operators

Definition 3.9. Suppose that $\mathcal{A}$ is a family of each GSFNs and $(G_1, \dots, G_n) \in \mathcal{A}^n$ where $G_i = (\mu_i, \iota_i, \nu_i)$, $\forall i = \overline{1,n}$ and $\mathfrak{w} = (w_1, \dots, w_n)^T$ denotes the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0 \leq w_i$ for each i and $\sum_{i=1}^n w_i = 1$. Then a mapping $GSEHWA_w: \mathcal{A}^n \to \mathcal{A}$ is called a GSEHWA operator and is given by

$$GSEHWA_w(G_1, \dots, G_n) = w_1 \cdot_E G_{\delta(1)}^* \oplus_E w_2 \cdot_E G_{\delta(2)}^* \oplus_E \dots w_n \cdot_E G_{\delta(n)}^* = \bigoplus_{i=1}^n w_i \cdot_E G_{\delta(i)}^* \tag{8}$$

where $\delta(i)\left(i = \overline{1,n}\right)$ is the permutation wrt the SV satisfying $SF\left(G_{\delta(i-1)}\right) \geq SF\left(G_{\delta(i)}\right)$, $\forall i = \overline{2,n}$ and $G_{\delta(i)}^* = nw_i G_{\delta(i)}$, $\forall i = \overline{1,n}$.

Remark 3.1. If $w = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then $GSEHWA_w(G_1, \dots, G_n) = GSEOWA_w(G_1, \dots, G_n)$, $\forall (G_1, \dots, G_n) \in \mathcal{A}^n$.

Theorem 3.11. Let $(G_1, \dots, G_n) \in \mathcal{A}^n$. Then the aggregated value $GSEHWA_w(G_1, \dots, G_n)$ is also a GSFN and is calculated by

$$GSEHWA_w(G_1, \dots, G_n) = \left(\begin{array}{c} \sqrt{\dfrac{\prod_{i=1}^n \left(1+\left(\mu_{\delta(i)}^*\right)^2\right)^{w_i} - \prod_{i=1}^n \left(1-\left(\mu_{\delta(i)}^*\right)^2\right)^{w_i}}{\prod_{i=1}^n \left(1+\left(\mu_{\delta(i)}^*\right)^2\right)^{w_i} + \prod_{i=1}^n \left(1-\left(\mu_{\delta(i)}^*\right)^2\right)^{w_i}}}, \\ \sqrt{\dfrac{2\prod_{i=1}^n \left(\iota_{\delta(i)}^*\right)^{2w_i}}{\prod_{i=1}^n \left(2-\left(\iota_{\delta(i)}^*\right)^2\right)^{w_i} + \prod_{i=1}^n \left(\iota_{\delta(i)}^*\right)^{2w_i}}}, \sqrt{\dfrac{2\prod_{i=1}^n \left(\nu_{\delta(i)}^*\right)^{2w_i}}{\prod_{i=1}^n \left(2-\left(\nu_{\delta(i)}^*\right)^2\right)^{w_i} + \prod_{i=1}^n \left(\nu_{\delta(i)}^*\right)^{2w_i}}} \end{array} \right).$$

Proof. This assertion can be shown with a similar process to Theorem 3.3.

Lemma 3.12. (i) (Idempotency of $GSEHWA_w$ operator) If $G_i = G$, $\forall i = \overline{1,n}$ where $G = (\mu, \iota, \nu)$, $G_i = (\mu_i, \iota_i, \nu_i)$ and $\mathfrak{w} = (w_1, \dots, w_n)^T$ denotes the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0 \leq w_i$ for each i, and $\sum_{i=1}^n w_i = 1$, then $GSEHWA_w(G_1, \dots, G_n) = G$.

(ii) (Boundedness of $GSEHWA_w$ operator) Let $(G_1, \dots, G_n) \in \mathcal{A}^n$. Then,

$$\underset{i=\overline{1,n}}{min} G_i \leq GSEHWA_w(G_1, \dots, G_n) \leq \underset{i=\overline{1,n}}{max} G_i.$$

Here, $\underset{i=\overline{1,n}}{min} G_i = \left(\underset{i=\overline{1,n}}{min}\, \mu_i, \underset{i=\overline{1,n}}{max}\, \iota_i, \underset{i=\overline{1,n}}{max}\, \nu_i\right)$ and $\underset{i=\overline{1,n}}{max} G_i = \left(\underset{i=\overline{1,n}}{max}\, \mu_i, \underset{i=\overline{1,n}}{min}\, \iota_i, \underset{i=\overline{1,n}}{min}\, \nu_i\right)$.

(iii) (Monotonicity of $GSEHWA_w$ operator) Let $(G_1, \dots, G_n), (G_1', \dots, G_n') \in \mathcal{A}^n$. If $G_i \leq G_i'$, $\forall i = \overline{1,n}$, then $GSEHWA_w(G_1, \dots, G_n) \leq GSEHWA_w(G_1', \dots, G_n')$.

Proof. These assertions can be easily proved with a similar process to Lemma 3.4.

Definition 3.13. Suppose that $\mathcal{A}$ is a collection of each GSFNs and $(G_1, \dots, G_n) \in \mathcal{A}^n$ where $G_i = (\mu_i, \iota_i, \nu_i)$, $\forall i = \overline{1, n}$ and $\mathfrak{w} = (w_1, \dots, w_n)^T$ denotes the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0 \leq w_i$ for each i and $\sum_{i=1}^n w_i = 1$. Then a mapping $GSEHWG_w \colon \mathcal{A}^n \to \mathcal{A}$ is said to be a GSEHWG operator and is given as

$$GSEHWG_w(G_1, \dots, G_n) = \left(G^*_{\delta(1)}\right)^{\wedge_E w_1} \odot_E \left(G^*_{\delta(2)}\right)^{\wedge_E w_2} \odot_E \dots \odot_E \left(G^*_{\delta(n)}\right)^{\wedge_E w_n} = \odot_{i=1}^n \left(G^*_{\delta(i)}\right)^{\wedge_E w_i} \qquad (9)$$

where $\delta(i)$ $\left(i = \overline{1, n}\right)$ is the permutation wrt the SV satisfying $SF\left(G_{\delta(i-1)}\right) \geq SF\left(G_{\delta(i)}\right)$, $\forall i = \overline{2, n}$ and $G^*_{\delta(i)} = G^{n w_i}_{\delta(i)}$, $\forall i = \overline{1, n}$.

Remark 3.14. If $w = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then $GSEHWG_w(G_1, \dots, G_n) = GSEOWG_w(G_1, \dots, G_n)$, $\forall (G_1, \dots, G_n) \in \mathcal{A}^n$.

Theorem 3.15. Let $(G_1, \dots, G_n) \in \mathcal{A}^n$. Then the aggregated value $GSEHWG_w(G_1, \dots, G_n)$ is also a GSFN and is calculated by using

$$GSEHWG_w(G_1, \dots, G_n) = \left(\sqrt{\frac{2\prod_{i=1}^n \left(\mu^*_{\delta(i)}\right)^{2w_i}}{\prod_{i=1}^n \left(2 - \left(\mu^*_{\delta(i)}\right)^2\right)^{w_i} + \prod_{i=1}^n \left(\mu^*_{\delta(i)}\right)^{2w_i}}} \sqrt{\frac{2\prod_{i=1}^n \left(\iota^*_{\delta(i)}\right)^{2w_i}}{\prod_{i=1}^n \left(2 - \left(\iota^*_{\delta(i)}\right)^2\right)^{w_i} + \prod_{i=1}^n \left(\iota^*_{\delta(i)}\right)^{2w_i}}}, \sqrt{\frac{\prod_{i=1}^n \left(1 + \left(\nu^*_{\delta(i)}\right)^2\right)^{w_i} - \prod_{i=1}^n \left(1 - \left(\nu^*_{\delta(i)}\right)^2\right)^{w_i}}{\prod_{i=1}^n \left(1 + \left(\nu^*_{\delta(i)}\right)^2\right)^{w_i} + \prod_{i=1}^n \left(1 - \left(\nu^*_{\delta(i)}\right)^2\right)^{w_i}}}\right).$$

Proof. This assertion can be shown with a similar process to Theorem 3.3.

Lemma 3.16. (i) (Idempotency of $GSEHWG_w$ operator) If $G_i = G$, $\forall i = \overline{1, n}$ where $G = (\mu, \iota, \nu)$, $G_i = (\mu_i, \iota_i, \nu_i)$ and $\mathfrak{w} = (w_1, \dots, w_n)^T$ denotes the vector of weight corresponding to $(G_i)_{i=1}^n$ satisfying $0 \leq w_i$, $\forall i$ and $\sum_{i=1}^n w_i = 1$, then $GSEHWG_w(G_1, \dots, G_n) = G$.

(ii) (Boundedness of $GSEHWG_w$ operator) Let $(G_1, \dots, G_n) \in \mathcal{A}^n$. Then,

$$\underset{i=\overline{1,n}}{\min} G_i \leq GSEHWG_w(G_1, \dots, G_n) \leq \underset{i=\overline{1,n}}{\max} G_i.$$

Here, $\underset{i=\overline{1,n}}{\min} G_i = \left(\underset{i=\overline{1,n}}{\min} \mu_i, \underset{i=\overline{1,n}}{\max} \iota_i, \underset{i=\overline{1,n}}{\max} \nu_i\right)$ and $\underset{i=\overline{1,n}}{\max} G_i = \left(\underset{i=\overline{1,n}}{\max} \mu_i, \underset{i=\overline{1,n}}{\min} \iota_i, \underset{i=\overline{1,n}}{\min} \nu_i\right)$.

(iii) (Monotonicity of $GSEHWG_w$ operator) Let $(G_1, \dots, G_n), (G'_1, \dots, G'_n) \in \mathcal{A}^n$. If $G_i \leq G_i{}'$, $\forall i = \overline{1, n}$, then $GSEOWG_w(G_1, \dots, G_n) \leq GSEHWG_w(G'_1, \dots, G'_n)$.

Proof. These assertions can be easily proved in a similar way as the proof of Lemma 3.4.

3.3. A Method to Solve the MCGDM Problem with Induced Generalized Spherical Fuzzy Einstein Aggregation Operators

Suppose that $A = \{A_1, \dots, A_m\}$ denotes the set of m different options and $E = \{E_1, \dots, E_n\}$ is the set of n different attributes. Assume that $\mathfrak{w} = (w_1, \dots, w_n)$ denotes the vector of weight of the criteria E_i $\left(i = \overline{1, n}\right)$ where $0 \leq w_i$ for each $i = \overline{1, n}$ and $\sum_{i=1}^n w_i = 1$. Assume that $D = \{D_1, \dots, D_k\}$ demonstrates the set of k different DMs with the choices whose vector of weight is given by $\mathfrak{d} = (\delta_1, \dots, \delta_k)$ where $0 \leq \delta_i$ for each $i = \overline{1, k}$ and $\sum_{i=1}^k \delta_i = 1$. The vector of weight $\mathfrak{d}$ has been considered via the education, age, experience, knowledge power and thinking ability of the DM. In fact, firstly, DeMs combined with alternatives to criteria values are established by evaluating the opinions of the DMs. However now, we take the entity of the DeMs as GSFNs and are shown by $B^r_{ij} = \left(\mu^r_{ij}, \iota^r_{ij}, \nu^r_{ij}\right)$, $(i = \overline{1, m})$, $(r = \overline{1, k})$, $\left(j = \overline{1, n}\right)$ and the combined DeM is written in Table 1.

Table 1. *DeM D_r*

	E_1	E_2	…	E_n
A_1	B_{11}^r	B_{12}^r	…	B_{1n}^r
A_2	B_{21}^r	B_{22}^r	…	B_{2n}^r
…	…	…	…	…
A_m	B_{m1}^r	B_{m2}^r	…	B_{mn}^r

Now, we establish the MCGDM process under the GSF environment based on the next steps:

Step I: Consider the either GSEOWA, GSEOWG, GSEHWA or GSEHWG operators on all DeM D_r with the vector of weight $\mathfrak{w} = (w_1, \dots, w_n)$ to get the following matrix.

Table 2. *Aggregated values*

	$F_{m\times1}^1$	$F_{m\times1}^2$	…	$F_{m\times1}^n$
	E_1	E_2	…	E_n
A_1	C_{11}^1	C_{11}^2	…	C_{11}^n
A_2	C_{21}^1	C_{21}^2	…	C_{21}^n
…	…	…	…	…
A_m	C_{m1}^1	C_{m1}^2	…	C_{m1}^n

Step II: Calculate the scores of all attributes of each alternatives given in Table 2 by using the score function $SF(G) = \frac{3\mu^2 - 2\iota^2 - \nu^2}{3}$.

Table 3. *Scores of all attributes of each alternatives*

	E_1	E_2	…	E_n
A_1	$SF(C_{11}^1)$	$SF(C_{11}^2)$	…	$SF(C_{11}^n)$
A_2	$SF(C_{21}^1)$	$SF(C_{21}^2)$	…	$SF(C_{21}^n)$
…	…	…	…	…
A_m	$SF(C_{m1}^1)$	$SF(C_{m1}^2)$	…	$SF(C_{m1}^n)$

Step III: Order the values of Table 2 basing on above score analysis given in Table 3 to get the Table 4.

Table 4. *Reordered aggregated values*

	$F_{m\times1}^1$	$F_{m\times1}^2$	…	$F_{m\times1}^n$
	E_1	E_2	…	E_n
$A_{\delta(1)}$	${}^*C_{11}^1$	${}^*C_{11}^2$	…	${}^*C_{11}^n$
$A_{\delta(2)}$	${}^*C_{21}^1$	${}^*C_{21}^2$	…	${}^*C_{21}^1$
…	…	…	…	…
$A_{\delta(m)}$	${}^*C_{m1}^1$	${}^*C_{m1}^2$	…	${}^*C_{m1}^n$

Step IV: Apply the DM's vector of weight ($\mathfrak{d}$) under the E-operations of GSFNs to evalute the result matrix D. This matrix is calculated as follows:

$$D = \begin{cases} \sum_{i=1}^{k} \delta_i \, F_{m\times1}^i, & when\ GSEOWA\ (or\ GSEHWA)\ operator\ is\ used \\ \sum_{i=1}^{k} \left(F_{m\times1}^i\right)^{\delta_i}, & when\ GSEOWG\ (or\ GSEHWG)\ operator\ is\ used \end{cases}$$

where $\left(\mathrm{F}^{\mathrm{i}}_{\mathrm{m}\times 1}\right)^{\delta_{\mathrm{i}}} = \begin{pmatrix} (\mathrm{c}^{\mathrm{i}}_{11})^{\delta_{\mathrm{i}}} \\ (\mathrm{c}^{\mathrm{i}}_{21})^{\delta_{\mathrm{i}}} \\ \vdots \\ (\mathrm{c}^{\mathrm{i}}_{\mathrm{m}1})^{\delta_{\mathrm{i}}} \end{pmatrix}$. Denote this matrix as given in Table 5.

Table 5. *Final matrix*

$$\mathrm{D} = \begin{pmatrix} \widetilde{\mathrm{A}}_{\delta(1)} \\ \widetilde{\mathrm{A}}_{\delta(2)} \\ \vdots \\ \widetilde{\mathrm{A}}_{\delta(\mathrm{m})} \end{pmatrix}$$

Step V: Calculate the SVs $\mathrm{SF}\left(\mathrm{A}_{\delta(\mathrm{i})}\right)$ $(\mathrm{i} = \overline{1, m})$ of final matrix given in Table 5 and rank the orders to find the best option.

3.4. An Illustrative Example

There is a three-shareholder company in which the rates of share are effective at the decisions to be made by shareholders and the sharing of the earnings. Let the shareholders be denoted by D_1, D_2, D_3. The shareholder D_1 has 35% share rate, the shareholder D_2 has 45% share rate and the shareholder D_3 has 20% share rates. This company is planning to make an investment in an area where the alternatives are A_1: Development of small business, A_2: Information Technology, A_3: Tourism, A_4: Transportation. They are taking into consideration the degree of risk, volume of income and investment recovery period when making an investment in these areas. Let the degree of risk, volume of income and investment recovery period be denoted by E_1, E_2, E_3, respectively. A prioritization relation through the criteria E_i $(i = \overline{1,3})$ which fulfills $E_3 < E_1 < E_2$ was determined by means of the shareholder's opinions. Hence,suppose that $\mathfrak{w} = (.3, .45, .25)$ denotes the vector of weight of the criteria $\{E_1, E_2, E_3\}$. To choose the optimum investment, the shareholder's D_1, D_2, D_3 with the DMs vector of weight $\mathfrak{d} = (.35, .45, .2)$ evaluate 4 investment alternatives depend on these attribute considering the induced GSF E-aggregation operators. The constructed DeMs are shown in Table 6-Table 8 as follows:

Table 6. *DeM D_1*

	E_1	E_2	E_3
A_1	(.6, .8, .2)	(.4, .3, .7)	(.2, .7, .4)
A_2	(.55, .2, .8)	(.8, .75, .65)	(.9, .8, .2)
A_3	(.7, .4, .4)	(.55, .2, .45)	(.5, .7, .8)
A_4	(.35, .6, .5)	(.7, .8, .55)	(.8, .6, .5)

Table 7. *DeM D_2*

	E_1	E_2	E_3
A_1	(.85, .7, .8)	(.4, .75, .8)	(.6, .8, .5)
A_2	(.3, .4, .4)	(.8, .2, .45)	(.5, .6, .8)
A_3	(.9, . 8, .2)	(.4, .8, .7)	(.8, .7, .4)
A_4	(.75, .3, .5)	(.8, .5, .45)	(.5, .6, .8)

Table 8. *DeM D_3*

	E_1	E_2	E_3
A_1	(.75, .4, .5)	(.8, .8, .45)	(.8, .6, .8)
A_2	(.9, . 6, . 4)	(.4, .6, .9)	(.2, .7, .4)
A_3	(.55, .5, .8)	(.8, .75, .85)	(.6, .8, .2)
A_4	(.75, .4, .8)	(.4, .8, .45)	(.8, .6, .6)

Step I: Use the GSEOWA operator to merge the DeMs D_1, D_2, D_3 with the vector of weight $\mathfrak{w} = (.3, .45, .25)$ to get the Table 9 which is the aggregated values.

***Table 9**. Aggregated values*

	$F^1_{4\times1}$	$F^2_{4\times1}$	$F^3_{4\times1}$
	E_1	E_2	E_3
A_1	(.4396, .5141, .4299)	(.6436, .7473, .7196)	(.7861, .6162, .5431)
A_2	(.7841, .5376, .5345)	(.638. .3287, .5093)	(.6305, .6243, .6000)
A_3	(.5914, .3442, .5093)	(.733. .7748, .4299)	(.6962, .6819, .6151)
A_4	(.6606, .6874, .5221)	(.7309, .4525, .5431)	(.6515, .6162, .5826)

Step II: Calculate the scores of all attributes of all alternatives using the function $SF(G) = \frac{3\mu^2 - 2\iota^2 - \nu^2}{3}$ where G is an GSFN to get the Table 1.

***Table 10.** Scores of attribute of alternatives*

	E_1	E_2	E_3
A_1	.04454	-.1307	.2665
A_2	.3269	.2486	.0177
A_3	.1844	.0755	.0486
A_4	.0304	.2994	.0582

Step III: Order the values of Table 9 basing on the score analysis given in Table 10 to get the Table 11.

***Table 11.** Reordered aggregated values*

	$F^1_{4\times1}$	$F^2_{4\times1}$	$F^3_{4\times1}$
	E_1	E_2	E_3
$A_{\delta(1)}$	(.7861, .6162, .5431)	(.4396, .5141, .4299)	(.6436, .7473, .7196)
$A_{\delta(2)}$	(.7841, .5376, .5345)	(.638, .3287, .5093)	(.6305, .6243, .6000)
$A_{\delta(3)}$	(.5914, .3442, .5093)	(.733, .7748, .4299)	(.6962, .6819, .6151)
$A_{\delta(4)}$	(.7309, .4525, .5431)	(.6515, .6162, .5826)	(.6606, .6874, .5221)

Step IV: Apply the DM's vector of weight $\mathfrak{d} = (.35, .45, .2)$ to get the Table 12 which is the final matrix.

***Table 12.** Final matrix*

$$D = \begin{array}{c} \tilde{A}_{\delta(1)} \\ \tilde{A}_{\delta(2)} \\ \tilde{A}_{\delta(3)} \\ \tilde{A}_{\delta(4)} \end{array} \begin{pmatrix} (.6532, .6014, .5306) \\ (.6987, .4584, .5396) \\ (.6816, .5649, .4989) \\ (.6843, .5684, .5531) \end{pmatrix}$$

Step V: Calculate the SVs $SF(\tilde{A}_{\delta(i)})$ $(i = 1,2,3,4)$ of final matrix given in Table 12 and rank the orders wrt the SVs: $SF(\tilde{A}_{\delta(1)}) = .0917$, $SF(\tilde{A}_{\delta(2)}) = .251$. $SF(\tilde{A}_{\delta(3)}) = .1688$, $F(\tilde{A}_{\delta(4)}) = .151$.

A_2 is choosable for this problem since the SV of A_2 is the highest.
In the following, we check the validity of the obtained result by using GSEHWA operator.

Step I: Use the GSEHWA operator on all DeM D_1, D_2, D_3 with the vector of weight $\mathfrak{w} = (.3, .45, .25)$ to get the Table 13 which is the aggregated values.

Table 13. *Aggregated values under GSEHWA*

	$F^1_{4\times 1}$	$F^2_{4\times 1}$	$F^3_{4\times 1}$
	E_1	E_2	E_3
A_1	(.4512, .4481, .4463)	(.6302, .7337, .7254)	(.8008, .6264, .5019)
A_2	(.7948, .5421, .5487)	(.6853, .2751, .4726)	(.6258, .6017, .6333)
A_3	(.6015, .2857, .4726)	(.7104, .7696, .4463)	(.7301, .6737, .6527)
A_4	(.6755, .6922, .5076)	(.7605, .4353, .5019)	(.6301, .6264, .5418)

Step II: Calculate the scores of all attribute of all alternatives using the function $SF(G) = \frac{3\mu^2-2\iota^2-\nu^2}{3}$ where G is an GSFN to get the Table 14.

Table 14. *Scores of all attribute of all alternatives*

	E_1	E_2	E_3
A_1	.0033	-.1371	.2957
A_2	.3354	.3448	.0166
A_3	.2330	.0434	.0884
A_4	.0510	.3681	.0375

Step III: Order the values of Table 13 basing on the above score analysis given in Table 14 to get the Table 15.

Table 15. *Reordered aggregated values*

	$F^1_{4\times 1}$	$F^2_{4\times 1}$	$F^3_{4\times 1}$
	E_1	E_2	E_3
$A_{\delta(1)}$	(.8008, .6264, .5019)	(.4512, .4481, .4463)	(.6302, .7337, .7254)
$A_{\delta(2)}$	(.6853, .2751, .4726)	(.7948, .5421, .5487)	(.6258, .6017, .6333)
$A_{\delta(3)}$	(.6015, .2857, .4726)	(.7301, .6737, .6530)	(.7104, .7696, .4463)
$A_{\delta(4)}$	(.7605, .4353, .5019)	(.6755, .6922, .5076)	(.6301, .6264, .5418)

Step IV: Apply the DM's vector of weight $\mathfrak{d} = (.35, .45, .2)$ to get the Table 16 which is the final matrix.

Table 16. *Final matrix*

$$D = \begin{array}{c} \widetilde{A}_{\delta(1)} \\ \widetilde{A}_{\delta(2)} \\ \widetilde{A}_{\delta(3)} \\ \widetilde{A}_{\delta(4)} \end{array} \begin{pmatrix} (.6332,.5982,.5685) \\ (.7212,.4079,.5279) \\ (.682..5297,.4913) \\ (.7106,.5507,.5116) \end{pmatrix}$$

Step V: Calculate the SVs $SF(\widetilde{A}_{\delta(i)})$ $(i = 1,2,3,4)$ of final matrix given in Table 16 and rank the orders wrt the SVs: $SF(\widetilde{A}_{\delta(1)}) = .0546$, $SF(\widetilde{A}_{\delta(2)}) = .3163$, $SF(\widetilde{A}_{\delta(3)}) = .1975$, $SF(\widetilde{A}_{\delta(4)}) = .2155$.

As obtained same in the above technique, A_2 is choosable for this problem since the SV of A_2 is the highest.

4. RESULTS AND DISCUSSION

In this part, we give an analysis to compare the reliability and effectiveness of the proposed technique to merge the GSF information. Güner and Aygün [17] proposed the GSF E-aggregation operators to merge the GSF information. Now, we present a comparison between the presented and novel GSF E-aggregation operators. Using the given data in the above decision-making problem, aggregated values under all defined

E-aggregation operators GSEWA, GSEWG, GSEOWA, GSEOWG, GSEHWA and GSEHWG have been shown in the Table 17.

Table 17. *Aggregated values under all aggregation operators*

	GSEWA	GSEWG	GSEOWA	GSEOWG	GSEHWA	GSEHWG
A_1	(.6312, .6308, .5680)	(.991. .6308, .05263)	(.6532, .6014, .5306)	(.9909, .5903, .3054)	(.6332, .5982, .5685)	(.9914, .5561, .0505)
A_2	(.6989, .4439, .5352)	(.9893, .4439, .0619)	(.6987, .4584, .5396)	(.9893, .4439, .0619)	(.7212, .4079, .5279)	(.9898, .4079, .0583)
A_3	(.6837, .5685, .4900)	(.992. .5685, .0589)	(.6816, .5649, .4989)	(.9918, .5506, .0571)	(.682. .5297, .4913)	(.9924, .5124, .0525)
A_4	(.6932, .5572, .5432)	(.9922, .5572, .0319)	(.6843, .5684, .5531)	(.9923, .5653, .0317)	(.7106, .5507, .5116)	(.9927, .5626, .0317)

The scores of the aggregated values under all aggregation operators obtained in Table 17 are shown in Table 18.

Table 18. *SVs*

	GSEWA	GSEWG	GSEOWA	GSEOWG	GSEHWA	GSEHWG
A_1	.03	.7159	.0917	.7184	.0546	.7758
A_2	.26	.8460	.2510	.8460	.3163	.8676
A_3	.1719	.7674	.1688	.7805	.1975	.8088
A_4	.1752	.7772	.1510	.7714	.2155	.7742

The demonstration of the ranking results observed in Table 18 are shown in Table 19.

Table 19. *Ranking orders*

Aggregation operators	Ranking
GSEWA [12]	$A_1 < A_3 < A_4 < A_2$
GSEWG [12]	$A_1 < A_3 < A_4 < A_2$
GSEOWA	$A_1 < A_4 < A_3 < A_2$
GSEOWG	$A_1 < A_4 < A_3 < A_2$
GSEHWA	$A_1 < A_3 < A_4 < A_2$
GSEHWG	$A_4 < A_1 < A_3 < A_2$

As seen in Table 19, the best alternative is A_2 under all E-aggregation operators. That is, transportation is the best investment area for the company. Therefore, it has been deduced from following results stated in the Table 19 that the proposed decision-making method can be suitable to solve the MCGDM problem.

5. CONCLUSION

The theory of soft set (SS), suggested by Molodtsov [22], is another idea for an extension of the FS theory to manage uncertainties by parameterization. This theory has attracted many authors since it has a wide range of applications in different fields of science and allows it to hybridize with mathematical models and also different set theories such as fuzzy soft set (FSS) [23], intuitionistic fuzzy soft set (IFSS) [24], Phythogerean fuzzy soft set (PyFSS) [25], picture fuzzy soft set (PFSS) [26], spherical fuzzy soft set (SFSS) [27], T-spherical fuzzy soft set (T-SFSS) [28]. All these mentioned set theories are highly proficient and skilled to carry ambiguous information, however their abilities are limited to handling one-dimensional data. A lot of complex MCDM problems comprise two-dimensional data but the existing MCDM strategies are incompetent to handle the two-dimensional information. To overcome such phenomena, one more extension of the FS theory, in literature, has been a complex fuzzy set (CFS) (given in [28]) where the MemD of an element is in the complex plane. This theory has provided a mathematical framework for describing membership in a set in terms of a complex number, so containing of phase and amplitude terms. The phase term is the pivotal part of the CFS that makes it the sole tool to capture both sides of the two-dimensional information at that time. With the similar consideration on MemD of an element to a set as explained for fuzzy set theory extensions, some generalizations of CFSs have been initiated such as complex FS [29], complex IFS [30], complex PyFS [31], complex PFS [32], complex SFS [33], complex T-SFS [34] with their soft combinations such as complex IFSS [35], complex PyFSS [36], complex PFSS [37], complex SFSS [38] and complex T-SFSS [39]. In literature, notable MCGDM methods have been constructed by considering the information taken above set theories. Especially, in recent studies, most researchers have worked on MCGDM approaches on complex spherical fuzzy (soft) sets. Akram et al. [33, 40] presented the VIKOR method for the complex spherical fuzzy environment and complex spherical fuzzy soft environment and applied these methods to the field of business to rank the objectives of an advertisement on Facebook and selection of firm for the Saudi oil refinery project in Pakistan, respectively. Then, Akram et al. [41] established the ELECTRE I method by using complex spherical fuzzy information and solved the selection of the best network monitoring software for military purposes. Also, Aydoğdu et al. [42] proposed the TOPSIS method based on entropy for complex spherical fuzzy data. Besides these MCGDM approaches, in literature, there are some different techniques based on similarity measure [43], dissimilarity measure [44], correlation measure [45], divergence measure [46], and knowledge measure [47] to solve MCGDM problems. Considering these detailed explanations and the lack of studies related to MCGDM methods in GSF environment, we propose to develop some traditional MCGDM methods such as VIKOR, ELECTRE, COPRAS, VIKOR, etc. based on the entropy or the mentioned measures in this environment. Furthermore, one can introduce the notions of complex GSFSs and GSF soft sets which also will allow us to explore the different kinds of MCGDM techniques.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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